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**SEMI-GROUPS OF CLASS (C_0) IN L_p DETERMINED BY
PARABOLIC DIFFERENTIAL EQUATIONS**

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1. Introduction. This paper treats mixed Cauchy problems for the parabolic partial differential equation in one space variable;

$$(1.1) \quad u = p(x)u_{xx} + q(x)u_x + r(x)u .$$

Our results are for non-singular equations, that is, the variable x is restricted to a finite interval $[a, b]$, and the function p is real-valued with $p(x) > 0$ on $[a, b]$. The functions q and r may be complex-valued. We require that p, q and r be in $L_\infty[a, b]$ and that p, p' and q be absolutely continuous with p', p'' and q' in $L_\infty[a, b]$.

We impose usual boundary conditions $\pi(u) = 0$ by

$$(1.2) \quad M_{i1}u(a) + N_{i1}u(b) + M_{i2}u'(a) + N_{i2}u'(b) = 0, \quad i = 1, 2 .$$

The constants M_{ij}, N_{ij} are real or complex and the matrix $(M_{ij}; N_{ij})$ has rank two.

With Equation (1.1) is associated the ordinary differential operator

$$(1.3) \quad A = p(x)D^2 + q(x)D + r(x)I, \quad D = \frac{d}{dx} .$$

With the above restrictions on the coefficients, A is defined in $L_p[a, b]^1$, $1 \leq p < \infty$, as a closed operator with dense domain, $D(A)$, given by

$$(1.4) \quad D(A) = \{u \in L_p \mid u \text{ and } u' \text{ are absolutely continuous} \\ \text{and } u, u', u'' \in L_p\} .$$

The boundary conditions define restrictions A_π of A to subdomains,

$$(1.5) \quad D(A_\pi) = \{u \in L_p \mid u \text{ and } u' \text{ are absolutely continuous,} \\ \pi[u] = 0, \text{ and } u, u', u'' \in L_p\} .$$

Our problem is to determine those A_π which generate *semi-groups of class (C_0)* in $L_p[a, b]$ (see Hille and Phillips [1], p. 320). Our main result is

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¹ We denote by $L_p[a, b]$, $1 \leq p < \infty$ the complex Lebesgue space defined by Lebesgue measure on $[a, b]$. Any Lebesgue space defined by a different measure μ will be denoted by $([a, b], \mu)$.

THEOREM 4. *If π is regular², the operator A_π is the infinitesimal generator of a semi-group of class (C_0) in $L_p[a, b]$, $1 \leq p < \infty$.*

The theory of adjoint semi-groups (Hille and Phillips [10], p. 426) can be used to extend the results of Theorem 4 to the Banach space $L_\infty[a, b]$. However, these results apply only in proper closed subspaces of L_∞ , and for brevity we do not include them.

In § 6 we investigate the necessity of regularity of π for the generation of a semi-group of class (C_0) by the special operators $\Omega_\pi = D^2$ in $L_p[0, 1]$. We have the partial result

THEOREM 5. *Let π and π^+ be adjoint boundary conditions relative to the operator D^2 . If both Ω_π and Ω_{π^+} generate semi-groups of class (C_0) in any $L_p[0, 1]$, $1 < p < \infty$, then π and π^+ are regular.*

We also show that for certain non-regular π the operator $\Omega_\pi = D^2$ can be defined either in $L_1([0, 1], dx^2)$ or in $L_1([0, 1], d(1 - x)^2)$ as the generator of a semi-group of class (C_0) . These operators can be shown to be equivalent to singular operators in $L_1[0, 1]$.

We give, what seems to be, the first application of the Feller-Phillips-Miyadera Theorem (Hille and Phillips [10], p. 360); other applications have been of its corollary, the Hille-Yosida Theorem. Probably Theorem 2, where this theorem is applied, can also be proved by an appropriate use of spectral resolutions of the operators $\Omega_\pi = D^2$ in L_1 and L_2 , however, we use spectral resolutions in only one instance. In any case, the eigenfunctions of the A_π can be used to give analytic representations of the semi-groups. In essence, we simply establish in L_p a certain type of behavior near $t = 0$ of solutions to the heat equation with a variety of boundary conditions.

Extensive application of semi-group theory to parabolic differential equations have been made by W. Feller ([4], [6], [7], [8]) and E. Hille [9]. Their papers contain our results for those real differential equation and real boundary conditions which determine positivity preserving semi-groups in L_1 and in L_2 .

We plan in a later paper to present a study which we have made of the hyperbolic equation

$$(1.6) \quad u_{tt} + a(x)u_t = p(x)u_{xx} + q(x)u_x + r(x)u .$$

2. Equivalent semi-group. We make considerable use of the following notions. If $\{T_t\}$ is a semi-group of class (C_0) on a Banach space U and if H is a linear homeomorphism of U onto another Banach space V , then it is easily shown that $\{S_t\}$ defined by

$$(2.1) \quad S_t = HT_tH^{-1}$$

² See G. D. Birkhoff [1], p. 383; J. D. Tamarkin [12]; or Coddington and Levinson [2], pp. 299-305.

is a semi-group of class (C_0) on V . We say that $\{T_t\}$ and $\{S_t\}$ are *homeomorphically equivalent*.

If ω is a constant and α a real positive constant, and if $\{T_t\}$ is a semi-group of class (C_0) , then $\{S_t\}$ defined by

$$(2.2) \quad S_t = e^{\omega t} T_{\alpha t}$$

is a semi-group of class (C_0) .¹

We make the following

DEFINITION 1. Let $\{T_t\}$ and $\{S_t\}$ be semi-groups of class (C_0) defined respectively on Banach spaces U and V . Then $\{T_t\}$ and $\{S_t\}$ are said to be *equivalent* provided there exist constants ω and α , α real and $\alpha < 0$, such that $\{T_t\}$ and $e^{\omega t} S_{\alpha t}$ are homeomorphically equivalent.

For our applications we need the following theorem, which is easily verified.²

THEOREM 1. Let $\{T_t\}$ and $\{S_t\}$ be equivalent semi-groups of class (C_0) defined respectively in Banach spaces U and V , i.e.

$$(2.3) \quad S_t = H(e^{\omega t} T_{\alpha t})H^{-1}.$$

The infinitesimal generators A and B are related by

$$(2.4) \quad B = (\omega I + \alpha HAH^{-1}), \quad D(B) = HD(A).$$

The resolvents of A and B are related by

$$(2.5) \quad R(\lambda; B) = HR(\lambda - \omega; \alpha A)H^{-1}.$$

We make now

DEFINITION 2. Let A and B be closed operators defined respectively in Banach spaces U and V with dense domains. Then A and B are said to be *equivalent* provided there exists a linear homeomorphism H of U onto V such that (i) $D(B) = HD(A)$ and (ii) $B = (\omega I + \alpha HAH^{-1})$ for some constants ω and α , α real and $\alpha > 0$.

3. Boundary conditions. The linear forms in (1.2) define a two dimensional sub-space of a four dimensional complex vector space. It is convenient for our discussion to specify such subspaces by Grassman coordinates, which are defined by

^{1,2} See Hille and Phillips [10], Theorem 12.2.2 and Theorem 13.6.1.

$$(3.1) \quad \begin{aligned} A &= \begin{vmatrix} M_{11} & N_{11} \\ M_{21} & N_{21} \end{vmatrix} & B &= \begin{vmatrix} N_{11} & M_{12} \\ N_{21} & M_{22} \end{vmatrix} & C &= \begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix} \\ D &= \begin{vmatrix} M_{11} & N_{12} \\ M_{21} & N_{22} \end{vmatrix} & E &= \begin{vmatrix} M_{12} & N_{12} \\ M_{22} & N_{22} \end{vmatrix} & F &= \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} \end{aligned}$$

These coordinates satisfy the quadratic relationship

$$(3.2) \quad FC - BD = AE,$$

and they are unique to within a constant of proportionality. Also, any constants, not all zero, which satisfy (3.2) define by (3.1) a set of conditions π of rank 2 (Hodge and Pedoe [11], p. 312).

We now define, for brevity in the sequel, four types of boundary conditions by the following sets:

$$(3.3) \quad \begin{aligned} \tau_1 &= \{\pi \mid E = B + D = 0\} \\ \tau_2 &= \{\pi \mid E \neq 0, \text{ or } E = 0 \text{ and } B + D \neq 0, \text{ or } A \neq 0 \text{ and} \\ &\quad B = C = D = E = F = 0\}, \\ \tau_3 &= \{\pi \mid F = C = 0 \text{ and one and only one of } A, B, D, E \neq 0\}, \\ \tau_4 &= \{\pi \mid A = E = 0, B = D = 1 \text{ and } FC = 1\}. \end{aligned}$$

Sets τ_1 and τ_2 have only the absorbing boundary conditions in common, i.e. $u(a) = u(b) = 0$. Sets τ_3 and τ_4 are disjoint subsets of τ_2 . The set τ_3 contains only separated endpoint boundary conditions. Representatives of these types in the form of (1.2) are easily determined by imposing the defining conditions in (3.1).

It is a simple matter to check that all boundary conditions in the set τ_2 are regular in the sense of G. D. Birkhoff. With one exception, $u(a) = u(b) = 0$, all π in the set τ_1 are non-regular.

4. $\Omega_\alpha = D^2$ in $L_1[0, 1]$ and $L_2[0, 1]$. For the special operator $\Omega_\alpha = D^2$ on $[0, 1]$ we need

LEMMA 1. Ω_α in $L_p[0, 1]$, $1 \leq p < \infty$, is a closed operator with dense domain. Except for those non-regular π given by

$$(4.1) \quad \begin{aligned} \alpha u(0) + u(1) &= 0 \\ \alpha u'(0) - u'(1) &= 0, \quad \alpha^2 = 1, \end{aligned}$$

the resolvent $R(\lambda; \Omega_\alpha)$ exists for all λ , $\Re(\lambda) > \omega_0 \geq 0$ for some ω_0 , and $R(\lambda; \Omega_\alpha)$ is expressed in all L_p , $1 \leq p < \infty$, by a Green's function as

$$(4.2) \quad R(\lambda; \Omega_\alpha)[u](\cdot) = \int_0^1 G(\cdot, t, \lambda) u(t) dt.$$

The proof of Lemma 1 is easy and is omitted.³ We, however, shall refer to the explicit expression for $G(x, t, \lambda)$ which is

$$(4.3) \quad \frac{1}{\Delta} \left\{ \begin{array}{l} G(x, t, \lambda) = \\ - F\sqrt{\lambda} sh\sqrt{\lambda}(x-t) + sh\sqrt{\lambda}t[Ash\sqrt{\lambda}(1-x) \\ + D\sqrt{\lambda}ch\sqrt{\lambda}(1-x)] + ch\sqrt{\lambda}t[B\sqrt{\lambda}sh\sqrt{\lambda}(1-x) \\ - E\lambda ch\sqrt{\lambda}(1-x)], \\ \text{for } t \leq x, \text{ and} \\ - C\sqrt{\lambda}sh\sqrt{\lambda}(t-x) + (\text{above with } x \text{ and } t \text{ interchanged}), \\ \text{for } t \geq x. \end{array} \right.$$

The function $\Delta(\lambda)$ is given in terms of (3.1) by

$$(4.4) \quad \Delta(\lambda) = (F + C)\lambda + A\sqrt{\lambda}sh\sqrt{\lambda} + (B + D)\lambda ch\sqrt{\lambda} - E\lambda^{3/2}sh\sqrt{\lambda}$$

where the principle value of $\sqrt{\lambda}$ is chosen for $\Re(\lambda) \geq 0$.

In § 5 it will be shown that our main result, Theorem 4, follows easily from the rather difficult

THEOREM 2. *If π is regular, then $\Omega_\pi = D^2$ generates a semi-group of class (C_0) in $L_1[0, 1]$ and in $L_2[0, 1]$.*

We prove Theorem 2 by a series of lemmas. Our method of proof amounts to proving this theorem for the subsets τ_3 and τ_4 of the set τ_2 of regular π . These results are then used to define a factorization of $R(\lambda; \Omega_\pi)$ for any regular π by which we reduce estimates on $\| [R(\lambda; \Omega_\pi)]^n \|$, $n = 1, 2, \dots$, which are needed for an application of the Feller-Phillips-Miyadera Theorem, to estimates on certain functions of the complex parameter λ .

The necessity for estimating $\| [R(\lambda; \Omega_\pi)]^n \|$ for $n > 1$ results when Ω_π generates a semi-group $\{T_t\}$ for which $\|T_t\|$ is not bounded by $e^{\omega t}$ for any ω . Whether or not $\|T_t\| \leq e^{\omega t}$ for a semi-group of class (C_0) in a Banach space.⁴ In one instance, part (b) of Lemma 3, we are able to guess an equivalent norm for $L_1[0, 1]$ so that the Hille-Yosida Theorem can be applied, whereas in the L_1 norm this does not seem to be the case.

We have the easy

LEMMA 2. *For π in the set τ_3 , Ω_π generates a semi-group of class (C_0) both in $L_1[0, 1]$ and in $L_2[0, 1]$.*

³ See Coddington and Levinson [2], pp. 300-305.

⁴ See Feller [5] where it is shown that if $\{T_t\}$ is a semi-group of class (C_0) in a Banach space, then an equivalent norm can always be defined by the semi-group so that in this norm $\|T_t\| < e^{\omega t}$.

Proof. (a) For $L_2[0, 1]$ all such Ω_π are self-adjoint with negative spectrum and a set of eigenfunctions which are a basis for $L_2[0, 1]$. It follows easily that such Ω_π generate semi-groups of contracting operators in $L_2[0, 1]$. (b) In $L_1[0, 1]$ we have by Fubini's Theorem, since $G(x, t, \lambda)$ is continuous,

$$\begin{aligned} \|R(\lambda; \Omega_\pi)u\| &\leq \int_0^1 \int_0^1 |G(x, t, \lambda)| |u(t)| dt dx \\ &\leq \|u\|_1 \max_{0 \leq t \leq 1} \int_0^1 |G(x, t, \lambda)| dx . \end{aligned}$$

From (4.3) for these special π one gets easily

$$(4.6) \quad \|R(\lambda; \Omega_\pi)\| \leq \frac{1}{\lambda} .$$

By the Hille-Yosida Theorem, Ω_π generates a semi-group of contracting operators. This completes the proof.

The proof is not so easy for

LEMMA 3. For π in the set τ_4 , Ω_π generates in $L_1[0, 1]$ and in $L_2[0, 1]$ a semi-group of class (C_0) .

Proof. Any π in the set τ_4 is given by

$$(4.7) \quad \begin{aligned} au(0) + u(1) &= 0 \\ au'(0) + u'(1) &= 0 \end{aligned} \quad a \neq 0 .$$

We note that if the complex constant a in (4.7) is such that $|a| = 1$, then the conditions π are self-adjoint relative to the operator D^2 .

(a) We set $\sigma = \log |a|$ and define a linear homeomorphism H of $L_2[0, 1]$ onto $L_2[0, 1]$ by

$$(4.8) \quad H[u](x) = e^{-\sigma x} u(x) .$$

The operator $\tilde{\Omega}_\pi$ equivalent to Ω_π is

$$(4.9) \quad \tilde{\Omega}_\pi = D^2 + 2\sigma D + \sigma^2 I .$$

Now $\tilde{\Omega}_\pi$ is a perturbation by the unbounded operator

$$(4.10) \quad B = 2\sigma D + \sigma^2 I$$

of the operator $\Omega_{\tilde{\pi}}$, where $\tilde{\pi}$ is given by

$$(4.11) \quad \begin{aligned} \alpha u(0) + u(1) &= 0 \\ \alpha u'(0) + u'(1) &= 0 , \quad \alpha = \frac{a}{|a|} = e^{i\theta} . \end{aligned}$$

The domain $D(B)$ of B is the same as $D(\tilde{\Omega}_\pi) = D(\Omega_\pi)$.

Now Ω_π is self-adjoint in $L_2[0, 1]$ with eigenvalues $\lambda_n = -(\theta + (2n+1)\pi)^2$, $n=0, \pm 1, \dots$, and eigenfunctions $\phi_n(x) = \exp[i(\theta + (2n+1)\pi)x]$, which are a basis for $L_2[0, 1]$. Then Ω_π generates a contraction semi-group given by

$$(4.12) \quad T_t[u] = \sum_{n=-\infty}^{\infty} a_n e^{\lambda_n t} \phi_n(x), \quad a_n = (u, \phi_n).$$

We want to establish that B is in the perturbing class $\mathfrak{P}(\Omega_\pi)$ of Ω_π (Hille and Phillips [10], p. 394). Since $D(B) = D(\Omega_\pi)$ we must establish that

$$(4.13) \quad \begin{aligned} & \text{(i) } BR(\lambda; \Omega_\pi) \text{ is bounded for some } \lambda, \\ & \text{(ii) } BT_t \text{ on } D(\Omega_\pi) \text{ is bounded for all } t > 0, \text{ and therefore} \\ & \quad \text{extensible to } \overline{BT}_t \text{ on } L_2[0, 1], \text{ and} \\ & \text{(iii) } \int_0^1 \|\overline{BT}_t\| dt < \infty. \end{aligned}$$

Now (i) of (4.13) follows immediately from (4.2). For (ii) of (4.13) we compute for $u \in D(\Omega_\pi)$,

$$(4.14) \quad \begin{aligned} \frac{1}{2} \|BT_t(u)\|_2^2 & \leq 4\sigma^2(DT_t(u), DT_t(u)) + \sigma^4 \|T_t(u)\|_2^2 \\ & = 4\sigma^2 T_t(u)DT_t(u)|_0^1 - 4\sigma^2(T_t(u), D^2T_t(u)) \\ & \quad + \sigma^4 \|T_t(u)\|_2^2. \end{aligned}$$

Using the facts that $\pi(T_t(u)) = 0$, $\|T_t(u)\|_2 \leq \|u\|_2$, and $\lambda_n \leq 0$, we get

$$(4.15) \quad \frac{1}{2} \|BT_t(u)\|_2^2 \leq \sigma^4 \|u\|_2^2 + 4\sigma^2 \|u\|_2^2 \left\{ \max_{-\infty \leq n \leq \infty} -\lambda_n e^{2\lambda_n t} \right\}.$$

Therefore, since $\lambda e^{-\lambda t}$ has on $[0, \infty)$ the maximum $1/2et$,

$$(4.16) \quad \|BT_t(u)\|_2 \leq 2\sigma \left(\sigma^2 + \frac{2}{et} \right)^{1/2} \|u\|_2.$$

This proves (ii) in (4.13) as well as (iii)

Since $B \in \mathfrak{P}(\Omega_\pi)$, the operator $\tilde{\Omega}_\pi$ generates a semi-group of class (C_0) (Hille and Phillips [10], p. 400). Since $\tilde{\Omega}_\pi$ is equivalent to Ω_π , this proves our lemma for $L_2(0, 1)$.

(b) In $L_1[0, 1]$ we do not use a perturbation argument as in $L_2[0, 1]$ because of the difficulty in proving (ii) of (4.13) without using orthogonality relations.

Again let $\sigma = \log|a|$ and introduce in $L_1[0, 1]$ an equivalent norm by

$$(4.17) \quad \|f\|_0 = \int_0^1 |f(x)| e^{-\sigma x} dx.$$

The identity mapping of $L_1[0, 1]$ under these two norms is a linear homeomorphism and Ω_π is equivalent to itself.

We get by Fubini's Theorem

$$(4.18) \quad \|R(\lambda; \Omega_\pi)u\|_0 \leq \int_0^1 |u(t)| \int_0^1 |G(x, t, \lambda)| e^{-\sigma x} dx dt .$$

The Grassman coordinates for (4.7) are $A = E = 0, B = D = a, C = 1,$ and $F = a^2,$ and from (4.3) for real $\lambda, \lambda > \sigma^2 (\sigma = \log |a|),$

$$(4.19) \quad |G(x, t, \lambda)| \leq \frac{\begin{cases} |a|^2 sh \sqrt{\lambda} (x - t) + |a| sh \sqrt{\lambda} (1 + t - x), & t \leq x \\ sh \sqrt{\lambda} (t - x) + |a| sh \sqrt{\lambda} (1 + x - t), & t \geq x \end{cases}}{\lambda(-1 - |a|^2 + 2|a|ch \sqrt{\lambda})}$$

We recognize the right-hand side of (4.19) as the Green's function, $G_1(x, t, \lambda),$ for d^2/dx^2 and the real boundary conditions π_1 given by

$$(4.20) \quad \begin{aligned} -|a|u(0) + u(1) &= 0 \\ -|a|u'(0) + u'(1) &= 0 \end{aligned}$$

for which $A = E = 0, B = D = |a|, C = -1,$ and $F = -|a|^2.$

Now the function $e^{-\sigma x}$ is an eigenfunction of the operator $\Omega_{\pi_1^+}$ for the eigenvalue $\sigma^2,$ where π_1^+ is the adjoint of $\pi_1,$ which is represented by (4.20) if $|a|$ is replaced by $|a|^{-1}.$ Since these are real boundary conditions, $G_1(x, t, \lambda),$ for real $\lambda,$ defines the Green's function for $\Omega_{\pi_1^+}$ if integration is done with respect to the variable $x.$ Therefore for (4.18) we have with λ real

$$(4.21) \quad \begin{aligned} \|R(\lambda; \Omega_\pi)u\|_0 &\leq \int_0^1 \frac{|u(t)| e^{-\sigma t}}{\lambda - \sigma^2} dt \\ &\leq \frac{\|u\|_0}{\lambda - \sigma^2}, \lambda > \sigma^2 . \end{aligned}$$

This proves that Ω_π generates a semi-group of class (C_0) in L_1 normed by $\|u\|_0,$ and therefore in L_1 with the usual norm. This completes the proof of our lemma.

The extension to all π in the set τ_2 is based on

LEMMA 4. *Let π be in the set $\tau_2.$ Then*

$$(4.22) \quad R(\lambda; \Omega_\pi) = \sum_{i=1}^6 f_i(\lambda) R(\lambda; \Omega_{\pi_i}) ,$$

where π_1 and π_2 are in the set τ_4 and π_3, \dots, π_6 are in the set $\tau_3.$ The functions $f_i(\lambda)$ are given by

$$(42.3) \quad f_i(\lambda) = \alpha_i \frac{\Delta_i(\lambda)}{\Delta(\lambda)}, \quad i = 1, 2, \dots, 6,$$

where the α_i are constants and $\Delta(\lambda)$ for π and $\Delta_i(\lambda)$ for π_i are defined by (4.4).

Proof. We use the Grassmann coordinates to define the π_i as follows. By adding and subtracting constants we write π as $\sum_{i=1}^6 \alpha_i \pi_i$ where

$$(4.24) \quad \begin{aligned} \pi: & (A, B, C, D, E, F), \\ \pi_1: & (0, 1, C - X, 1, 0, F - \bar{X}), \\ \pi_2: & \left(0, 1, \frac{X}{|X|}, 1, 0, \frac{\bar{X}}{|X|}\right), \\ \pi_3: & (1, 0, 0, 0, 0, 0), \\ \pi_4: & (0, 1, 0, 0, 0, 0), \\ \pi_5: & (0, 0, 0, 1, 0, 0), \\ \pi_6: & (0, 0, 0, 0, 1, 0), \end{aligned}$$

$\alpha_1 = 1$, $\alpha_2 = |X|$, $\alpha_3 = A$, $\alpha_4 = B - 1 - |X|$, $\alpha_5 = D - 1 - |X|$ and $\alpha_6 = E$. Now X has to be chosen so that the coordinates of π_i satisfy (3.2). X is given by

$$(4.25) \quad \begin{aligned} X &= C - \rho e^{i\theta}, \quad \theta = \arg(C - \bar{F}) \text{ and} \\ \rho &= \frac{|C - \bar{F}| + \sqrt{|C - \bar{F}|^2 + 2}}{2} \end{aligned}$$

Using the linearity of the numerator of the Green's function (4.3) in the constants A, B, C, D, E , and F , we get the expression (4.23).

We shall apply to the functions $f_i(\lambda)$ of Lemma 6 the following:

THEOREM 3. *Let $f(\lambda)$ be analytic in a half plane $\Re(\lambda) > \alpha$. Let $f(\lambda)$ satisfy either of the following conditions:*

- (i) $f(\lambda)$ is real for real λ and $(-1)^k f^{(k)}(\lambda) \geq 0$ (or ≤ 0) for all real λ , $\lambda > \alpha$, $k = 0, 1, \dots$, i.e., f is completely monotonic in $(\alpha, +\infty)$.
- (ii) (a) $\int_{-\infty}^{\infty} |f(\sigma + i\tau)| d\tau < M < +\infty$, $\sigma > \alpha$, M independent of σ .
- (b) $\lim_{|\tau| \rightarrow \infty} f(\sigma + i\tau) = 0$ uniformly in every closed subinterval of $\alpha < \sigma < +\infty$.

Then there exist real numbers $K > 0$ and ω such that

$$(4.26) \quad \sum_{k=0}^n \frac{|f^{(k)}(\lambda)|}{k!} (\lambda - \omega)^k < K, \text{ for } n = 0, 1, \dots,$$

and λ real, $\lambda > \omega$.

Proof. Suppose that (i) holds and that $f(\lambda) > 0$ for real λ (otherwise replace f by $-f$). Then $|f_i^{(k)}(\lambda)| = (-1)^k f_i^k(\lambda)$ and with $\omega = \alpha + 1$

$$(4.27) \quad \sum_{k=0}^{\infty} \frac{|f_i^{(k)}(\lambda)|}{k!} (\lambda - \omega)^k = \sum_{k=0}^{\infty} \frac{f_i^k(\lambda)}{k!} (\omega - \lambda)^k = f(\omega), \lambda \geq \omega,$$

since f is analytic in the region $\Re(\lambda) > \alpha$. Then (4.26) follows with $K = |f(\alpha + 1)|$ and $\omega = \alpha + 1$.

Suppose that condition (ii) holds. Then f is the Laplace transform (Widder [13], p. 265) of a function $\phi(t)$ for which $\phi(t) = 0, t < 0$ and $|\phi(t)| \leq M e^{\sigma-t}, \sigma > \alpha$. We have (Widder [13], p. 57)

$$(4.28) \quad f^{(k)}(\lambda) = \int_0^{\infty} (-t)^k e^{-\lambda t} \phi(t) dt \quad \Re(\lambda) > \alpha.$$

So with $\omega = \alpha + 2$ and real $\lambda, \lambda > \omega$,

$$(4.39) \quad \sum_{k=0}^n \frac{|f^{(k)}(\lambda)|}{k!} (\lambda - \omega)^k \leq \int_0^{\infty} e^{-\omega t} |\phi(t)| dt \leq M.$$

Therefore (4.26) follows with $K = M$ and $\omega = \alpha + 2$.

We finally come to

Proof of Theorem 2. We shall establish the existence of real constants M and $\omega > 0$ such that in both L_1 and L_2 for real λ

$$(4.30) \quad \|[R(\lambda; \Omega_{\pi})]^{n+1}\| \leq \frac{M}{(\lambda - \omega)^n}, \quad \lambda > \omega, n = 1, 2, \dots$$

By the Feller-Phillips-Miyadera Theorem this will prove our theorem.

In the representation (4.22) for $R(\lambda; \Omega_{\pi})$, each Ω_{π_i} generates a semi-group of class (C_0) in L_1 and in L_2 , either by Lemma 2 or by Lemma 3. Then for each $R(\lambda; \Omega_{\pi_i}), i = 1, 2, \dots, 6$ (4.30) holds in $L_p, p = 1, 2$, and M and $\omega > 0$ can be chosen independently of i and p .

Iterates of a resolvent can be computed by

$$(4.31) \quad [R(\lambda; \Omega_{\pi})]^{n+1} = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \lambda^n} R(\lambda; \Omega_{\pi}),$$

(Hille and Phillips [10], p. 184). Making use of (4.22), (4.31) and (4.30) for each $R(\lambda; \Omega_{\pi_i})$, we get

$$(4.32) \quad \|[R(\lambda; \Omega_{\pi})]^{n+1}\| \leq \frac{M}{(\lambda - \omega)^{n+1}} \sum_{i=1}^6 \sum_{k=0}^n \frac{|f_i^{(k)}(\lambda)|}{k!} (\lambda - \omega)^k,$$

real $\lambda, \lambda > \omega$, and $n = 0, 1, \dots$.

We suppose now that π is such that either $E \neq 0$ or $B + D \neq 0$.

The only other regular π is in the set τ_3 , and has been dealt with in Lemma 2. With this assumption, each of the functions $f_i(\lambda)$ of Lemma 4 can be written as

$$(4.33) \quad f_i(\lambda) = J_i + \frac{K_i}{\sqrt{\lambda}} + \frac{L_i}{\lambda} + F_i(\lambda), \quad i = 1, 2, \dots, 6$$

for uniquely determined constants and a unique analytic function $F_i(\lambda)$.

For $\Re(\lambda) > 0$ we have chosen a branch of $\lambda^{1/2}$, so that the first three functions in (4.33) are analytic and satisfy condition (i) of Theorem 3. The functions $F_i(\lambda)$ are analytic and can be shown to satisfy conditions (ii) of Theorem 3. Then (4.32) and (4.33) together with Theorem 3 give our desired result (4.30). This proves our theorem.

5. A_π in $L_p[a, b]$, $1 \leq p < \infty$. With the tedious work done in § 4, we now come to our main result

THEOREM 4. *If π is regular, the operator A_π is the infinitesimal generator of a semi-group of class (C_0) in $L_p[a, b]$, $1 \leq p < \infty$.*

Proof. The assumptions on the coefficients of A in (1.3) are such that standard changes of independent and dependent variables⁵ can be made to show that A_π in $L_p[a, b]$ is equivalent in the sense of Definition 2 to \tilde{A}_π in $L_p[0, 1]$, where

$$(5.1) \quad \tilde{A}_\pi = \Omega_\pi + r_1 I.$$

The conditions $\tilde{\pi}$ are as in (1.2) and can readily be shown to be regular if and only if conditions π are regular.

The function r_1 in (5.1) is in $L_\infty[0, 1]$, and therefore $r_1 I$ is a bounded operator in any L_p . So \tilde{A}_π is obtained by perturbing Ω_π by a bounded operator. Perturbation theory shows that \tilde{A}_π generates a semi-group of class (C_0) if and only if Ω_π does (see Hille and Phillips [10], Theorem 13.2.1).

This reduces our proof to that of showing that for regular π the operators $\Omega_\pi = D^2$ generate semi-groups of class (C_0) in any $L_p[0, 1]$, $1 \leq p < \infty$. This extension of Theorem 2 we shall now give.

Let π^+ denote the boundary conditions adjoint to π relative to the operator D^2 (Coddington and Levinson [2], pp. 288-293). It is readily checked that the Grassmann coordinates (A', B', C', D', E', F') of π^+ are obtained from those of π by interchanging F and C and taking complex conjugates. From (3.3) it follows that π^+ is in the set τ_2 if and only if π is.

⁵ See Courant and Hilbert [3], p. 250.

Let π , and therefore π^+ , be regular boundary conditions. Then by Lemma 1 the resolvent $R(\lambda; \Omega_\pi)$ exists for $\Re(\lambda)$ greater than some ω_0 , and it is expressed by (4.2).

We denote the norm of a bounded linear operator T in L_p by $N_p\{T\}$. Then by Theorem 2 and the Feller-Phillips-Miyadera Theorem (Hille and Phillips [10], p. 360), we have

$$(5.2) \quad N_p\{[R(\lambda; \Omega_\pi)]^n\} \leq M_p(\lambda - \omega_0)^{-n}, \Re(\lambda) > \omega_0,$$

$p = 1, 2$ and $n = 1, 2, \dots$

Now $R(\lambda; \Omega_\pi)$ is defined by (4.2) on the space of continuous functions, which is dense in $L_p[0, 1]$, $1 \leq p < \infty$. If we let $M = \max(M_1, M_2)$ and apply the Riesz Convexity Theorem (Zygmund [14], p. 198), we obtain (5.2) for $1 \leq p \leq 2$. By the Feller-Phillips-Miyadera Theorem, this is sufficient for Ω_π to generate a semi-group of class (C_0) in L_p , $1 \leq p \leq 2$.

Also by Theorem 2 and the above argument, Ω_{π^+} generates a semi-group of class (C_0) in any $L_p[0, 1]$, $1 \leq p \leq 2$. It is readily shown that Ω_{π^+} in L_q and Ω_π in L_p , $1/p + 1/q = 1$, $1 < p \leq 2$, are adjoints of each other. The theory of adjoint semi-groups (Hille and Phillips [10], Chapter IV) shows that Ω_π in L_q generates a semi-group of class (C_0) , since Ω_{π^+} does in L_p . This completes the proof of our theorem.

6. Non-regular π . One result relating to the necessity of regularity of π for A_π to generate a semi-group of class (C_0) in $L_p[a, b]$ is given in

LEMMA 5. *If A_π generates a semi-group of class (C_0) in $L_2[a, b]$, then π is regular.*

Proof. As we saw in the proof of Theorem 4, it is sufficient to prove this result for $\Omega_\pi = D^2$ in $L_2[0, 1]$.

Let π be a set of non-regular boundary conditions. It is simply a matter of computation to show that for the function $u(x) = 1$, $0 \leq x \leq \frac{1}{2}$, and $u(x) = 0$, $\frac{1}{2} < x \leq 1$ we get in (4.2)

$$(6.1) \quad \|R(\lambda; \Omega_\pi)u\|_2 > C\lambda^{-3/4}$$

for all real λ sufficiently large and $C > 0$. Thus, by the Feller-Phillips-Miyadera Theorem, Ω_π does not generate a semi-group of class (C_0) in $L_2[0, 1]$.⁶ This proves our result.

We now have⁷

⁶ Indeed, this proves that Ω_π does not generate a semi-group of the more general class (A) in $L_2[0, 1]$ since it is not true that $\lambda R(\lambda; \Omega_\pi)u \rightarrow u$ as $\lambda \rightarrow +\infty$ (Hille and Phillips [10], p. 322).

⁷ By a more careful analysis, the complete result can probably be proven that regularity of π is necessary for A_π to generate a semi-group of class (C_0) in $L_p[a, b]$.

THEOREM 5. *Let π and π^+ be adjoint boundary conditions relative to the operator D^2 . If both Ω_π and Ω_{π^+} generate semi-groups of class (C_0) in any $L_p[0, 1]$, $1 < p < \infty$, then π and π^+ are regular.*

Proof. Suppose that Ω_π and Ω_{π^+} generate semi-groups of class (C_0) in some $L_p[0, 1]$. Then Ω_π generates a semi-group of class (C_0) in $L_q[0, 1]$, $1/p + 1/q = 1$. An application of the Riesz Convexity Theorem, as in Theorem 4, shows that Ω_π generates a semi-group of class (C_0) in $L_2[0, 1]$. By Lemma 5, π is regular, and therefore also π^+ . This completes the proof.

For certain of the non-regular π , other Lebesgue spaces can be chosen in which operators Ω_π are defined and generate semigroups of class (C_0) . The construction of these spaces is suggested by the method of proof used in part (b) of Lemma 3.

Suppose that conditions π are given by

$$(6.2) \quad \begin{aligned} u(0) &= au'(1) \\ u(1) &= 0 \end{aligned} \quad |a| \geq 1.$$

Then, if $G(x, \tau, \lambda)$ is the Green's function of Ω_π , it can be shown that $G_1(x, \tau, \lambda) \equiv |G(\tau, x, \lambda)|$ is the Green's function for Ω_{π_1} , where conditions π_1 are given by

$$(6.3) \quad \begin{aligned} u(0) &= 0 \\ u(1) &= |a|u'(0). \end{aligned}$$

Also Ω_{π_1} has the real, non-negative eigenfunction $\phi(x) = \sigma^{-1}sh\sigma x$ where σ is the largest real root of $sh\sigma = |a|\sigma$. In a manner similar to that in part (b) of Lemma 3, one can show that Ω_π can be defined in the Lebesgue space $L_1([0, 1], \phi(x)dx)$ as the generator of a semi-group of class (C_0) . This space is also norm equivalent to the space $L_1([0, 1], dx^2)$.

The linear homeomorphism of $L_1([0, 1], dx^2)$ onto $L_1([0, 1], d(1-x)^2)$ defined by $u(x) \rightarrow u(1-x)$, shows that Ω_π generates a semi-group of class (C_0) in $L_1([0, 1], d(1-x)^2)$ where the conditions $\tilde{\pi}$ are given by

$$(6.4) \quad \begin{aligned} u(0) &= 0 \\ u(1) &= -au'(0). \end{aligned}$$

In each of these spaces, $L_1[0, 1]$ can be shown to be a dense subspace. The operators Ω_π and $\Omega_{\tilde{\pi}}$ can be shown to be equivalent to singular operators in $L_1[0, 1]$.

We do not know whether similar results hold for other non-regular π .

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