ON TOEPLITZ MATRICES, ABSOLUTE CONTINUITY, AND UNITARY EQUIVALENCE

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1. Preliminaries. For $n = 0, \pm 1, \pm 2, \cdots$, let $\{c_n\}$ be real numbers satisfying

$$(1) \qquad c_0 = 0,\ c_{-n} = c_n \text{ and } \sum_1^\infty c_n^2 < \infty ,$$

and consider the associated real-valued, even function $f(\theta)$ of period 2 and of class $L^2[0, \pi]$ defined by

$$(2) \qquad f(\theta) \sim \sum_{-\infty}^\infty c_n e^{in\theta} = 2 \sum_1^\infty c_n \cos n\theta .$$

(Throughout this paper it will be assumed for the sake of convenience that $c_0 = 0$. If $c_0 \neq 0$, T (see below) is modified merely by the addition of a multiple of the unit matrix.) Let $A = (a_{ij})$, where $a_{ij} = c_{i-j}(= c_{j-i})$ or $a_{ij} = 0$ according as $i < j$ or $i \geq j(i, j = 1, 2, \cdots)$, and define the Toeplitz matrix T and the Hankel matrices H and K by

$$(3) \qquad T = (c_{i-j}) = A + A^*,\ H = (c_{i+j-1}) \text{ and } K = (c_{i+j}) .$$

The matrices T, H and K are real and Hermitian (symmetric).

Let J denote the matrix belonging to the quadratic form $2\sum_1^\infty x_n x_{n+1}$. The differential of its spectral matrix is given by $d\rho_{ij}(\theta) = 2\pi^{-1}\sin i\theta \sin j\theta d\theta$ (cf. Hilbert [5], p. 155, Hellinger [8], pp. 148 ff.). A direct calculation (cf. [11], Appendix 2) shows that

$$(4) \qquad T = F + K ,$$

where T and K are defined by (3), and F is given by

$$(5) \qquad F = \left(\int_0^\pi f(\theta) d\rho_{ij}(\theta) \right) ,$$

with $f(\theta)$ defined by (2) and (1). In particular, if $c_1 = 1$ and $c_n = 0$ for $n > 1$, then $f(\theta) = 2 \cos \theta$ and (5) is the spectral resolution of J (with the usual parameter λ being given by $\lambda = 2 \cos \theta$).

It should be noted that the L^2 assumption on the sequence $\{c_n\}$ in (1) does not imply the boundedness of the various matrices considered above, although of course, the existence, in the mean, of the integrals in (5) is assured. Moreover, all two factor products of the type A^2, AA^*, etc. surely exist and it can be verified that

$$(6) \qquad\qquad\qquad AT - TA = H^2 \, ,$$

where A, T and H are defined by (3); cf. [11], p. 517.

It is known (Toeplitz) that T is bounded if and only if $f(\theta)$ is essentially bounded, so that $|f(\theta)| \leq$ const. almost everywhere on $[0, \pi]$; [3], p. 360. Moreover, if M and m denote the essential upper and lower bounds of f, then the spectrum of T consists of the interval $[m, M]$ and, unless all $c_n = 0$, is purely continuous (so that the point spectrum is empty); [3] and [4]. Furthermore, if T is not bounded, but is still self-adjoint, then the spectrum of T is again purely continuous and is again the (unbounded) interval $[m, M]$; [4], p. 878. (Actually the results of Hartman and Wintner mentioned above are not restricted to the case of real sequences $\{c_n\}$ as in the present paper.) For necessary and sufficient conditions in order that a Hankel matrix be bounded, see [9].

The matrix A is bounded if and only if $g(\theta) \sim \sum_1^\infty c_n e^{in\theta}$ is essentially bounded (Toeplitz; cf. [4], p. 880, [11], p. 517). Clearly, if A is bounded, so also is T. In addition, if T is bounded, then $f(\theta)$ of (2) is bounded and consequently K is bounded (Toeplitz; cf. [2], p. 223, also [3], p. 365). In view of the easily verified relation

$$(7) \qquad\qquad \|Hx\|^2 = \|Kx\|^2 + \left(\sum_1^\infty c_n x_n\right)^2$$

H is bounded (or completely continuous) if and only if K is bounded (or completely continuous). It is seen that if A is bounded then all of the other matrices considered above are bounded.

In § 2 there will be pointed out a few consequences of the relations (4) and (5) bearing on the nature of Fourier series and the spectra of Toeplitz matrices belonging to real valued, even functions defined by (2). In §§ 3 and 4, sufficient conditions guaranteeing that a Toeplitz matrix be absolutely continuous or unitarily equivalent to a certain function of J, will be obtained. Some applications to Hilbert matrices will be given in § 5.

Condition (1) on the real sequence $\{c_n\}$ will be assumed throughout the paper.

2. Toeplitz matrices and Fourier series. First there will be proved:

(I) *If the Hankel matrix K is bounded, then necessarily T is self-adjoint.*

This is an obvious consequence of (4) if it is noted that F of (5) is always self-adjoint. Incidentally, it is seen that in this case the domain of T is identical with that of F.

By the essential range of $f(\theta)$ will be meant the (closed) set of values λ for which $|f(\theta) - \lambda| < \varepsilon$ holds on a set of positive measure

on $[0, \pi]$ for every $\varepsilon > 0$. Then one has:

(II) *If K is bounded and satisfies $\| K \| \leq C$ (= const.), then any closed interval of length $2\,C$ contained in $[m, M]$ contains at least one point of the essential range of $f(\theta)$.*

This assertion also follows from (4). For, it is clear that the spectrum of F is exactly the essential range of $f(\theta)$ (the situation being analogous to the problem of Toeplitz of determining the spectrum of a Laurent matrix; cf. [3], pp. 359-360 and the references cited there). Consequently, since relation (4) shows that F is obtained as a perturbation of T by the operator $- K$, the assertion now follows from the fact that the spectrum of T is the entire interval $[m, M]$.

A theorem similar to (II) is

(III) *If K is completely continuous then the essential range of $f(\theta)$ is $[m, M]$.*

The proof follows from the observation that, by (the generalization of) Weyl's theorem, the essential (cluster) spectra of T and F are identical. Since, by (I), T is self-adjoint, its spectrum is $[m, M]$ ([4]) and it follows that the essential spectrum of F is also $[m, M]$. But the spectrum of F is always contained in $[m, M]$ and hence must be identical with this interval. This implies (III).

A curious corollary of (III) is the following assertion:

(IV) *If $g(\theta) \sim \sum_1^\infty c_n \sin n\theta$ is continuous on $- \infty < \theta < \infty$ then the essential range of $h(\theta) \sim \sum_1^\infty c_n \cos n\theta$ is an interval (possibly unbounded).*

The assertion follows from (III) if it is noted that the continuity of $g(\theta)$ implies the complete continuity of the operator K. Cf. [3], p. 365. It is stated there that $H = (c_{i+j-1})$ is completely continuous if either $g(\theta)$ or the function $\frac{1}{2} f(\theta) \sim \sum_1^\infty c_n \cos n\theta$ is continuous on $[0, \pi]$. The proof seems to indicate however that $K = (c_{i+j})$ (or H) is completely continuous if either $g(\theta)$ or $f(\theta)$ is continuous on $[-\pi, \pi]$ (and hence on $- \infty < \theta < \infty$). See [2], p. 223. The continuity of $g(\theta)$ on $[0, \pi]$ implies its continuity on $[- \pi, \pi]$ but the corresponding assertion for $f(\theta)$ is false.

Another easy consequence of (4) is the following:

(V) *If T_1 and T_2 are two bounded Toeplitz matrices with the representations*

(4_m) $T_m = F_m + K_m$

where

$$(5_m) \qquad F_m = \left(\int_0^\pi f_m(\theta) d\rho_{ij}(\theta) \right) \text{ and } f_m(\theta) \sim 2 \sum_1^\infty c_{mn} \cos n\theta \; ,$$

and if K_1 and K_2 are completely continuous, then

$$(8) \qquad T_1 T_2 = \left(\int_0^\pi f_1(\theta) f_2(\theta) d\rho_{ij}(\theta) \right) + C \; ,$$

where C is completely continuous.

A relation similar to (8) holds of course for products with more than two factors. The proof of (V) follows easily if it is noted that the product of a bounded operator and a completely continuous operator is completely continuous. In particular, it is seen from (8) that the essential spectrum of $T_1 T_2$ is the essential range of $f_1(\theta)\, f_2(\theta)$. The situation is to be compared with that for Laurent matrices; cf. the remark following (II) above.

3. Absolute continuity. It follows from Theorem 13 of [11], p. 523, that if A is bounded, then (6) implies that T is absolutely continuous whenever 0 is not in the point spectrum of H. That is, this last condition is sufficient in order to guarantee that $\int_Z dE(\lambda) = 0$, where

$$(9) \qquad T = \int \lambda \, dE(\lambda)$$

is the spectral resolution of T and Z is any set of one-dimensional Lebesgue measure zero. However it is possible that T is absolutely continuous even if 0 does belong to the point spectrum of H. In fact each T_N, belonging to the sequence $\{c_n\}$ with $c_n = c_N \neq 0$ $(N > 0)$ if $n = \pm N$ and $c_n = 0$ otherwise, is absolutely continuous; cf. [11], pp. 519, 524. This result will be generalized in the following theorem:

(VI) *Let the real sequence $\{c_n\}$, $n = 0, \pm 1, \pm 2, \cdots$ satisfy (1) define the associated matrices A, T and F as in §1, and suppose that A is bounded. Then T is absolutely continuous whenever F is absolutely continuous.*

As remarked above, the boundedness of A implies that of all other operators considered. It follows from the argument of [10] (cf. p. 1027, formula line (4)) when applied to (6) that

$$(10) \qquad H \int_Z dE(\lambda) = 0 \; ,$$

where Z denotes any set of one-dimensional Lebesgue measure zero and H is defined by (3). (The square root $H^{1/2}$ appearing in [10] *loc. cit.* can clearly be taken to be any self-adjoint square root of the non-negative

self-adjoint operator H. The H appearing there corresponds to a positive multiple of the operator H^2 of the present paper.) Next let y be any element in the range of $\int_z dE(\lambda)$ so that

$$(11) \qquad\qquad y = \int_z dE(\lambda)x , \qquad\qquad \|x\| < \infty .$$

Since, by (10), $Hy = 0$, it follows from (7) that $Ky = 0$. Consequently, by (4), $Ty = Fy$. For $n = 0, 1, 2, \cdots$, $T^n y = \int_z dE(\lambda) T^n x$ is also in the range of $\int_z dE(\lambda)$, and it follows that $T^{n+1}y = FT^n y$. Hence

$$(12) \qquad\qquad T^n y = F^n y \quad (n = 0, 1, 2, \cdots; T^0 = F^0 = I) ,$$

where y is defined by (11). But (12) implies $E(\lambda)y = F(\lambda)y$, where

$$(13) \qquad\qquad F = \int \lambda dF(\lambda)$$

is the spectral resolution of F, and hence $\int_z dE(\lambda)y = \int_z dF(\lambda)y$. But, whenever F is absolutely continuous, $\int_z dF(\lambda) = 0$ and so, by (11), $\int_z dE(\lambda)x = 0$ for all x. That is, T is absolutely continuous and the proof of (VI) is now complete.

4. **Unitary equivalence.** It was shown in [11] that each T_N (see the beginning of § 3 above) is absolutely continuous and that moreover T_N is unitarily equivalent to the corresponding $F = F_N$. This result will be considerably refined in the following theorem:

(VII) *Let the real sequence $\{c_n\}$ satisfy* (1) *and the condition*

$$(14) \qquad\qquad |c_n| \leq \text{const. } \alpha^n \qquad\qquad (n = 1, 2, \cdots)$$

for some constant α, $0 < \alpha < 1$. Then the associated matrices T and F are unitarily equivalent; thus, there exists a unitary matrix U such that

$$(15) \qquad\qquad T = UFU^* .$$

The condition (14) easily assures $\sum |c_n| < \infty$ and hence the boundedness of A and therefore (cf. § 1 above) that of all other operators considered. If all $c_n = 0$, then T and F are both the zero operator (matrix) and (15) is trivial. Suppose then that not all c_n are 0. It will first be shown that F is absolutely continuous.

To this end, consider $f(z) = 2 \sum_1^\infty c_n \cos nz$ for the complex variable

$z = x + iy$. It is clear that $|\cos nz| = \frac{1}{2}|e^{inz} + e^{-inz}| \leq \frac{1}{2}(e^{ny} + e^{-ny}) \leq$
$e^{n|y|}$ and hence, by (14), $|c_n \cos nz| \leq \mathrm{const.}\,(\alpha e^{|y|})^n$. Since $0 < \alpha < 1$, it
follows that $\alpha e^{|y|} < 1$ for y sufficiently small and so $f(z) \not\equiv 0$ and is
analytic in a strip containing the real axis. Consequently $df(\theta)/d\theta$ can
be zero at most a finite number of times on $0 \leq \theta \leq \pi$ and it follows
that the (possibly many-valued) inverse function of $f(\theta)$ on $[0, \pi]$ is
absolutely continuous (more precisely, that each of the finite number of
branches of the inverse of $f(\theta)$ on $0 \leq \theta \leq \pi$ is absolutely continuous).
Moreover, if $\lambda = 2 \cos \theta$, the operator F can be represented (cf. (5)) as
$F = \displaystyle\int_{-2}^{2} h(\lambda)dE_1(\lambda)$ where $E_1(\lambda)$ is the resolution of the identity belong-
ing to the matrix J. Since $h(\lambda) = f(\theta)$ via the substitution $\lambda = 2 \cos \theta$
it is clear that $h(\lambda)$ has a (possibly many-valued) absolutely continuous
inverse and it follows (cf. [11], pp. 521–522) that F is absolutely con-
tinuous, as was to be shown. In fact, if one considers the spectral
resolution of F as given by (13), it is seen from a comparison with (5)
that zero sets on the λ-interval $-2 \leq \lambda \leq 2$ correspond to zero on the
θ-interval $0 \leq \theta \leq \pi$ via the mapping $\lambda = 2 \cos \theta$ and that F is absolutely
continuous if and only if the relation

(16) $\{\theta; f(\theta)$ in $Z\}$ is a zero set

holds whenever Z is a zero set.

By (VI) it now follows that T also is absolutely continuous. More-
over, since by (14),

$$\sum_i \sum_j c_{i+j}^2 = \sum_n nc_{n+1}^2 < \infty ,$$

K is completely continuous. In order to complete the proof it will be shown
that

(17) $tr\,|K| < \infty ,$

where $|K|$ denotes the non-negative square root of K^2. An application
of a theorem of Rosenblum ([12], p. 998, will then yield the desired
unitary equivalence relation (15). See also Kato [6].

There remains then to prove (17). Let $\{\phi_n\}$, $n = 1, 2, \cdots$, denote
the complete orthonormal sequence of vectors for which the n-th com-
ponent of ϕ_n is 1 and all others are 0. Then

$$tr\,|K| = \sum (|K|\phi_n, \phi_n) \leq \sum \||K|\phi_n = \sum \|K\phi_n\|$$
$$= \sum_n (\sum_m c_{n+m}^2)^{1/2} \leq \sum_n \sum_m |c_{n+m}| \leq \sum n|c_{n+1}| < \infty ,$$

the last inequality by (14). Thus (17) is proved and, as remarked earlier,
the proof of (VII) is complete.

The proof of (VII) makes clear the following assertion:

(VIII) *Let the real sequence* $\{c_n\}$ *satisfy* (1) *and suppose that* (16) *holds for every zero set Z. In addition, suppose that*

$$(18) \qquad \sum_n n\,|c_{n+1}| < \infty \,, \ \text{ or even } \sum_n (\sum_m c_{n+m}^2)^{1/2} < \infty \,.$$

Then (15) *holds.*

It is clear that (18) implies $\sum |c_n| < \infty$ and hence that A is bounded (cf. § 1 above). In addition (18) implies $\sum n c_{n+1}^2 < \infty$ and hence the complete continuity of K; as shown before, (18) implies (17). Moreover, unless T and F are both 0, it follows from (16) that F (hence, by (VI), also T) is absolutely continuous. Relation (15) now follows from Rosenblum's theorem as before.

It was shown in [4], p. 878, that whenever T is self-adjoint (not even necessarily bounded) it has no point spectrum. On the other hand, F has a point spectrum whenever $f(\theta)$ has an interval of constancy, or more generally, whenever $f(\theta) \equiv$ const. holds on a set of positive measure. This situation can of course easily obtain for non-trivial $f(\theta)$ ($f(\theta) \not\equiv$ const., i.e., since $c_0 = 0$, $f(\theta) \not\equiv 0$) possessing derivatives of arbitrarily high order (but, of course, for which $f(z)$ is not analytic). But if $f(\theta)$ is of class C^p, its Fourier coefficients are $O(n^{-p-2})$ and so it is clear that the hypothesis (14) of (VII) guaranteeing unitary equivalence cannot be weakened to, say, $|c_n| \le$ const. n^{-m} ($n = 1, 2, \cdots$) for any positive constant m. Of course, as (VIII) implies, relation (14) is not necessary for (15).

5. Hilbert matrices. A case of special interest is afforded by the sequence $\{c_n\}$ defined by $c_0 = 0$, $c_n = n^{-1}$ if $n > 0$ and $c_{-n} = c_n$. This sequence is of the type considered at the beginning of this paper and moreover $T = (|i - j|^{-1})$, $H = ((i + j - 1)^{-1})$ and $S = A - A^* = ((i - j)^{-1})$, with the understanding of course that the (i, i) elements of T and S are 0. The matrices S and H are known to be bounded (Hilbert; cf., e.g., [2], pp. 212-213, 223). Moreover, the spectrum of H is exactly the interval $[0, \pi]$ and, in fact, is purely continuous ([6]). The matrix T is known to be unbounded ([2], p. 214). Concerning T, there will be proved the following theorem:

(IX) *The matrix* $T = (|i - j|^{-1})$ *is a self-adjoint operator and is absolutely continuous; thus if* $T = \int \lambda\, dE(\lambda)$ *is the spectral resolution of* T, *then* $\int_Z dE(\lambda) = 0$ *for every set Z of one-dimensional Lebesgue measure zero.*

That Z is self-adjoint follows from an application of a theorem of

Hartman and Wintner [4], p. 878, if it is noted that $\frac{1}{2} f(\theta) = -\log(2|\sin\frac{1}{2}\theta|)$ $\sim \sum_1^\infty n^{-1} \cos n\theta$ on $(-\pi, \pi)$ is half-bounded. Another proof of the assertion follows from (I) if it is noted that H, hence also K (cf. (7)), is bounded, since the odd function $g(\theta)$ defined by $g(\theta) = \frac{1}{2}(\pi - \theta)$ $\sim \sum_1^\infty n^{-1} \sin n\theta$ on $(0, \pi)$ is bounded.

It is easy to verify that

$$(19) \qquad\qquad\qquad ST - TS = 2H^2 \,,$$

a relation similar to (6). Moreover, since 0 is not in the point spectrum of H (cf., e.g. [7], p. 699 and the reference there to [1]), Theorem 13 of [11] implies, at least formally, the absolute continuity of T. The trouble stems from the fact that boundedness restrictions were imposed in [10] and [11] and that, although S and H in (19) are bounded, T is not. As a consequence, equation (19), although a valid matrix equation, conceivably cannot be regarded as an operator equation in Hilbert space. More precisely, it is not clear that whenever x is in the domain of T, D_T, then (19) holds, so that

$$(20) \qquad\qquad\qquad STx - TSx = 2H^2x \,,$$

with the understanding that STx and TSx of (20) should mean $S(Tx)$ and $T(Sx)$ respectively. (For operator equations the associative law is of course essentially a matter of definition.) It will be shown below that in fact (20) does hold as an operator equation valid at least for all x in D_T. Once this has been established, it is easy to carry out the same reasoning as in [10], cf. pp. 1027–1028 (where the boundedness of all operators was supposed) and to obtain the equation (10) above, corresponding to formula line (4) in [10]. The absolute continuity of T then follows (cf. Theorem 13 of [11]) from the fact that 0 is not in the point spectrum of H.

In order to complete the proof there remains to be shown that if x is in D_T then (20) holds. To this end, it will be shown that if x is in Hilbert space, that is if $\|x\| < \infty$, then each of the series

$$(21) \qquad\qquad \sum_m \sum_n s_{im} t_{mn} x_n \text{ and } \sum_m \sum_n t_{im} s_{mn} x_n$$

is absolutely convergent for $i = 1, 2, \cdots$, where, for convenience, $T = (t_{ij})$ and $S = (s_{ij})$. Grant, for the moment, that this has been shown. Then, from the absolute convergence of the first series of (21), it follows that in the iterated series the orders of summation may be interchanged, and hence that, for x in Hilbert space, the corresponding components of the vectors $(ST)x$ and $S(Tx)$ are identical. Now, if it is assumed in addition that x is in D_T, then the vector $S(Tx)$ is in Hilbert space, since S is bounded. Consequently $(ST)x$ is in Hilbert space and, since H^2 is

bounded, it follows from the (matrix) equation (19) that $(TS)x$ is in Hilbert space. The absolute convergence of the second series of (21) then implies that $(TS)x = T(Sx)$, so that $T(Sx)$ is in Hilbert space (that is, essentially, that Sx is in D_T). Moreover, it is now seen that (19) implies the validity of (20) as an operator equation valid at least for all vectors in D_T.

Thus, in order to complete the proof of (IV) there now remains to be shown that the series of (21) are absolutely convergent whenever $\|x\| < \infty$. Consider the series $S_i = \sum_m \sum_n |s_{im} t_{mn} x_n|$. Since $t_{ij} \geq 0$ for all i, j and $s_{ij} \geq 0$ or $s_{ij} < 0$ according as $i \geq j$ or $i < j$, it is clear that

$$(22) \qquad S_i = -\sum_{n=1}^{\infty} \left(\sum_{m=1}^{i} s_{im} t_{mn} |x_n| \right) + \sum_{n=1}^{\infty} \left(\sum_{m=i+1}^{\infty} s_{im} t_{mn} |x_n| \right) .$$

But the inside series of the first double series on the right of equation (22) is finite and, consequently, the orders of summation may be interchanged to obtain $- \sum_{m=1}^{i} s_{im} (\sum_{n=1}^{\infty} t_{mn} |x_n|)$. Since x and the rows of T are in Hilbert space, the inside summation of this last series is always convergent by the Schwarz inequality. Hence the first series of (21) is absolutely convergent if and only if the series

$$(23) \qquad \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} s_{im} t_{mn} \right) |x_n| ,$$

obtained through modification of S_i in (22) by changing the sign of the first series, is convergent. Now the inside summation of (23) is the (i, n) element of $ST = D = (d_{in})$. Since S is bounded and the columns of T are in Hilbert space, the columns of D are in Hilbert space, that is $\sum_i d_{in}^2 < \infty$. But the matrix equation (19) can be written as $D + D^* = 2H^2$; hence, since H^2 is bounded, the columns of D^* and therefore the rows of D, are also in Hilbert space. Hence $\sum_n d_{in}^2 < \infty$ and so (23) is convergent by the Schwarz inequality. It has now been proved that the first series of (21) is absolutely convergent (for $i = 1, 2, \cdots$). Using the fact that $|s_{ij}| = t_{ij}$ it is seen that the absolute convergence of the first series of (21), that is, the convergence of $\sum_m \sum_n t_{im} t_{mn} |x_n|$, whenever $\|x\| < \infty$, implies the absolute convergence of the second series of (21). Thus both series are convergent for all x in Hilbert space and (IX) follows as indicated above.

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