

Pacific Journal of Mathematics

**THE SUSPENSION OF THE GENERALIZED PONTRJAGIN
COHOMOLOGY OPERATIONS**

EMERY THOMAS

THE SUSPENSION OF THE GENERALIZED PONTRJAGIN COHOMOLOGY OPERATIONS

EMERY THOMAS

1. The main theorem. In a previous paper [9] I have defined a sequence of new cohomology operations, called the *generalized Pontrjagin operations*. These operations use as coefficient groups the summands of a certain type of graded ring: namely, *a ring with divided powers* (defined by H. Cartan in [1]), which is termed a Γ -ring in [9]. Let $A = \sum_k A_k$ be a ring with divided powers such that each summand A_k is a cyclic group of infinite or prime power order; we termed such rings *p-cyclic* in [9]. Then, the Pontrjagin operations are functions

$$\mathfrak{P}_t: H^{2n}(X; A_{2k}) \longrightarrow H^{2tn}(X; A_{2tk}) \quad (k, n > 0; t = 0, 1, \dots)$$

where $H^q(Y, B; G)$ denotes the q th (singular) cohomology group of the pair (Y, B) with coefficients in the group G .

Let C be a cohomology operation relative to integers r, s and coefficient groups G, H . That is, C is a natural transformation

$$C: H^r(Y, B; G) \longrightarrow H^s(Y, B; H).$$

With each operation C we associate a second operation, $S(C)$, called the *suspension* of C . $S(C)$ is a natural transformation

$$H^{r-1}(Y, B; G) \longrightarrow H^{s-1}(Y, B; H);$$

its definition is given in § 3.

The purpose of this note is to determine $S(\mathfrak{P}_t)$, where \mathfrak{P}_t is the generalized Pontrjagin operation. In order to state our result concerning $S(\mathfrak{P}_t)$, we need an additional cohomology operation, the Postnikov square (see [3], [10]). This was defined in [9], but only for a restricted class of coefficient groups. In this paper we will define the Postnikov square as a cohomology operation

$$p: H^q(Y, B; A_{2k}) \longrightarrow H^{2q+1}(Y, B; A_{4k}), \quad (q, k > 0)$$

where A_{2k} is an even summand of a p -cyclic ring with divided powers.

We now may state the main result of the paper.

THEOREM I. *For any cohomology operation C , let $S(C)$ denote the suspension of the operation C . Then,*

Received October 18, 1957, in revised form December 19, 1958. This research has been partly supported by U. S. Air Force contract AF 49 (638)-79.

- (i) $S(\mathbb{F}_2) = \mathfrak{p}$
- (ii) $S(\mathbb{F}_t) = 0$, ($t > 2$)

where 0 denotes the zero cohomology operation.

The proof of Theorem I is given in § 5. In § 2 we define the operation \mathfrak{p} , while in § 3 we give the definition of the suspension. In § 4 we discuss relative cohomology operations, while in § 6 we give some additional properties of the operation \mathfrak{p} . In particular, we show that $S(\mathfrak{p}) = 0$. Finally, the last section gives the theorem, $\delta S(C) = C\delta$, for any operation C .

I would like to thank Professor N. E. Steenrod for the valuable suggestions made to me at the time of revising the paper. In particular the definition of the suspension in § 3 and Theorem 7.1 are due to him.

2. The definition of the Postnikov square. The definition of the Postnikov square, \mathfrak{p} , is obtained by first defining a “model operation”, p , which uses only a restricted category \mathcal{C} of coefficient groups. The category \mathcal{C} is defined as follows: let $Z_r = Z/rZ$ ($r = 0, 1, \dots$), where $Z = \text{integers} = Z_0$. Denote by \mathcal{C} the category of all groups of the form Z_θ , where θ is zero or a power of a prime. For each group Z_θ in \mathcal{C} we have defined a p -cyclic ring with divided powers,

$$G(Z_\theta) = G_0(Z_\theta) + \dots + G_t(Z_\theta) + \dots \text{ (direct sum) (see [9; 1.17]).}$$

In particular,

$$G_2(Z_\theta) = \begin{cases} Z_\theta, & \text{if } \theta \text{ is zero or odd} \\ Z_{2^\theta}, & \text{if } \theta \text{ is a power of 2.} \end{cases}$$

We define a generator for $G_2(Z_\theta)$ by

$$g_2(1_\theta) = \begin{cases} 1_\theta, & \text{if } \theta \text{ is zero or odd} \\ 1_{2^\theta}, & \text{if } \theta \text{ is a power of 2} \end{cases}$$

where $1_r = 1 \pmod r$ ($r = 0, 1, \dots$). The group $G_2(Z_\theta)$ will be the coefficient domain for the operation \mathfrak{p} . As remarked in [9; § 2], once we have defined the operation \mathfrak{p} for the category of *regular cell complexes*, the definition easily extends to the category of all topological spaces. Hence, in what follows we restrict attention to regular cell complexes, which we will simply term *complexes*.

Let K be a complex and L a subcomplex of K . Let Z_θ be a group in the category \mathcal{C} ; that is, θ is zero or a power of a prime. We define an operation

$$p: H^q(K, L; Z_\theta) \longrightarrow H^{2q+1}(K, L; G_2(Z_\theta))$$

as follows. Let $u \in H^q(K, L; Z_\theta)$; let β be the homomorphism from Z_θ to $G_2(Z_\theta)$ given by $\beta(1_\theta) = \theta g_2(1_\theta)$. Define

$$(2.1) \quad p(u) = \beta_*(u \cup \delta_* u) .$$

Here, δ_* is the Bockstein coboundary operator associated with the exact sequence

$$0 \longrightarrow Z \xrightarrow{\theta} Z \longrightarrow Z_\theta \longrightarrow 0 ,$$

and the cup-product is taken relative to the natural pairing $Z_\theta \otimes Z \approx Z_\theta$.

It is easily seen that this agrees with the usual definition of the operation p (see [3] and [10]). For let $\bar{u} \in C^q(K, L; Z)$ be a cochain representing u ; that is, $\delta\bar{u} = \theta\bar{v}$, for some cochain $\bar{v} \in C^{q+1}(K, L; Z)$. Then, a cocycle representing $\beta_*(u \cup \delta_* u)$ is given by $\bar{u} \cup \delta\bar{u}$, which coincides with the definition given in [10].

In [9; 8.14] we defined a function w which goes from $H^q(K; Z_\theta)$ to $H^{2q+1}(K; Z)$. This function can be extended to the relative case, following the method given in § 4. When this is done it is easily shown that

$$(2.2) \quad p(u) = \beta_* w(u) ,$$

a result we will need later.

The Postnikov square, \mathfrak{p} , is defined using the operation p as follows: let $u \in H^q(K, L; A_{2k})$, where A_{2k} is an even summand of a p -cyclic ring with divided powers. By hypothesis, A_{2k} is a cyclic group whose order is infinite or a power of a prime. Thus, there is an integer θ such that A_{2k} is isomorphic to Z_θ , where $Z_\theta \in \mathcal{C}$. Let ν be an isomorphism from A_{2k} to Z_θ . Then, by 3.1 in [9], for each non-negative integer r we have defined a homomorphism ζ_r mapping $G_r(Z_\theta)$ to A_{2rk} , which is an extension of ν^{-1} . We define the operation \mathfrak{p} by

$$(2.3) \quad \mathfrak{p}(u) = \zeta_2^* p\nu_*(u) ;$$

that is, \mathfrak{p} is the composition of the following functions:

$$\begin{aligned} H^q(K, L; A_{2k}) &\xrightarrow{\nu_*} H^q(K, L; Z_\theta) \xrightarrow{p} \\ H^{2q+1}(K, L; G_2(Z_\theta)) &\xrightarrow{\zeta_2^*} H^{2q+1}(K, L; A_{4k}) . \end{aligned}$$

We show the independence of this definition from the particular choice of the isomorphism ν (and hence ζ_2). This is a consequence of the fact that

$$(2.4) \text{ LEMMA.} \quad p\alpha_* = G_2(\alpha)_* p ,$$

where α is a homomorphism from Z_θ to a group Z_τ in \mathcal{C} , and $G_2(\alpha)$ is the homomorphism from $G_2(Z_\theta)$ to $G_2(Z_\tau)$ induced by the functor G (see [9; 1.23]).

Using 2.2, the proof of 2.4 is entirely similar to that given for 5.22 in [9] and is omitted here. From 2.4 the proof of the independence of

the definition of \mathfrak{p} follows along exactly the same lines as 3.5 and 3.6 in [9]; we omit the details.

3. Suspension of cohomology operations. The definition of the suspension used here is due to N. E. Steenrod¹. Let I denote the unit interval, $[0, 1]$, and \dot{I} the subspace $\{0\} \cup \{1\}$. The group $H^1(I, \dot{I}; Z)$ is cyclic infinite; let v be a fixed generator. For each space X and coefficient group G define a function ϕ from $H^q(X; G)$ to $H^{q+1}(I \times X, \dot{I} \times X; G)$ by

$$(3.1) \quad \phi(u) = v \times u .$$

We use singular cohomology for X , and the natural pairing $Z \otimes G \approx G$ for the cross-product. In § 7 we prove the following lemma.

(3.2) LEMMA. *The function ϕ is an isomorphism mapping $H^q(X; G)$ onto $H^{q+1}(I \times X, \dot{I} \times X; G)$ ($q > 0$).*

Consider now any cohomology operation C , which is defined on relative cohomology groups; say, C maps $H^r(X, A; G)$ to $H^s(X, A; H)$ for each pair (X, A) . Define an absolute cohomology operation, $S(C)$, which maps $H^{r-1}(Y; G)$ to $H^{s-1}(Y; H)$, for each space Y , by

$$(3.3) \quad S(C)(u) = \phi^{-1} C\phi(u) \quad (u \in H^{r-1}(Y; G)) .$$

Using the method described in § 4 we may extend $S(C)$ to an operation defined on relative cohomology groups, an operation which we continue to denote by $S(C)$. We wish to apply this construction to the operation \mathfrak{A}_i ; as defined in [9], this is just an absolute operation. Thus, to use Definition 3.3 we must first extend the definition of \mathfrak{A}_i to the relative case.

4. Relative cohomology operations. Let $O(q, r; G, H)$ denote the set of absolute cohomology operations relative to dimensions q, r and coefficient groups G, H ; that is, if $C \in O(q, r; G, H)$, then $C: H^q(X; G) \rightarrow H^r(X, H)$ for each space X . As is well-known the set $O(q, r; G, H)$ is in 1-1 correspondance with the group $H^r(K; H)$, where K is an Eilenberg-MacLane space of type (G, q) . The correspondance is obtained by assigning $C(\iota)$ to ι , where ι is the fundamental class in $H^q(K; G)$. Choose now a base point $e \in K$, and let $\alpha^*: H^*(K, e; A) \approx H^*(K; A)$ be the isomorphism induced by the inclusion $K \subset (K, e)$. For any CW-complex X and subcomplex A , the homotopy classes of maps $(X, A) \rightarrow (K, e)$

¹ This definition has the advantage that it can be used in the case of cohomology with local coefficients.

are in one-to-one correspondance with $H^q(X, A; G)$. Thus we define a relative cohomology operation, C' , associated with an absolute operation, C , as follows:

$$(4.1) \quad C'(u) = f^* \alpha^{*-1} C(\ell) ,$$

where $u \in H^q(X, A; G)$ and f is a map $(X, A) \rightarrow (K, e)$ such that

$$f^* \alpha^{*-1}(\ell) = u .$$

With the operation C' defined, one is then interested in whether the properties of C extend to the operation C' . We now prove a general lemma which essentially asserts that all the properties of C' do carry over to C .

Let $O(q_1, \dots, q_n, r; G_1, \dots, G_n, H)$ denote the group of absolute cohomology operations, T , in n variables; that is, if $u_i \in H^{q_i}(X; G_i)$ ($i = 1, \dots, n$), then, $T(u_1, \dots, u_n) \in H^r(X; H)$. The operation T extends to a relative operation, T' , using the method just given for operations of a single variable. Suppose now we are given absolute cohomology operations

$$\begin{aligned} C &\in O(q_1, \dots, q_n, r; G_1, \dots, G_n, H) , \\ E &\in O(s_1, \dots, s_p, r; H_1, \dots, H_p, H) , \\ \text{and} \quad D_i &\in O(q_1, \dots, q_n, s_i; G_1, \dots, G_n, H_i) \end{aligned} \quad (i = 1, 2, \dots, p).$$

Let $C', E' D_i$, be the corresponding relative operations.

(4.2) PROPOSITION. *Suppose that for each space X and cohomology classes $u_i \in H^{q_i}(X; G_i)$ ($i = 1, \dots, n$), we have*

$$C(u_1, \dots, u_n) = E(D_1(u_1, \dots, u_n), \dots, D_p(u_1, \dots, u_n)) .$$

Then, for each pair (X, A) and classes $u'_i \in H^{q_i}(X, A; G_i)$ ($i = 1, \dots, n$), we have

$$C'(u'_1, \dots, u'_n) = E'(D'_1(u'_1, \dots, u'_n), \dots, D'_p(u'_1, \dots, u'_n)) .$$

We give the proof at the end of this section, first illustrating the theorem by giving several corollaries.

(4.3) COROLLARY 1. *Let $C \in O(q, s; R, S)$, $D_i \in O(q_i, s_i; R, S)$ ($i = 1, 2$), where R, S are rings, $q = q_1 + q_2$, and $s = s_1 + s_2$. Suppose that*

$$C(u_1 \cup u_2) = D_1(u_1) \cup D_2(u_2)$$

for all classes $u_i \in H^{q_i}(X; R)$. Then,

$$C'(u'_1 \cup u'_2) = D'_1(u'_1) \cup D'_2(u'_2) ,$$

for all classes $u'_i \in (H^{q_i}(X, A; R))$.

Proof. Let $E_R \in O(q_1, q_2, q; R, R, R)$ and $E_S \in O(s_1, s_2, s; S, S, S)$ be the respective cup-products. Let F be the composite operation $C \circ E_R$. Using Proposition 4.2 we see that $F' = C' \circ E'_R$. But since $F(u_1, u_2) = E_S(D_1(u_1), D_2(u_2))$, again using 4.2 we see that

$$F'(u'_1, u'_2) = E'_S(D'_1(u'_1), D'_2(u'_2)) ;$$

that is,

$$C'(u'_1 \cup u'_2) = D'_1(u'_1) \cup D'_2(u'_2) ,$$

as was to be shown.

Let C, D_1, D_2 be the same operations as in Corollary 1. Then,

(4.4) COROLLARY 2. $C'(u'_1 \times u'_2) = D'_1(u'_1) \times D'_2(u'_2)$, where $u_i \in H^{q_i}(X_i, A_i; R)$ ($i = 1, 2$).

Proof. Let $p_1: (X_1 \times X_2, A_1 \times X_2) \rightarrow (X_1, A_1)$, $p_2: (X_1 \times X_2, X_1 \times A_2) \rightarrow (X_2, A_2)$ be projections. Then,

$$u'_1 \times u'_2 = p_1^*(u'_1) \cup p_2^*(u'_2) .$$

Thus,

$$\begin{aligned} C'(u'_1 \times u'_2) &= C'(p_1^*u'_1 \cup p_2^*u'_2) = D'_1(p_1^*u'_1) \cup D'_2(p_2^*u'_2) \\ &= p_1^*(D'_1u'_1) \cup p_2^*(D'_2u'_2) = (D'_1u'_1) \times (D'_2u'_2) . \end{aligned}$$

Here we have used Corollary 1 and the naturality of the cohomology operations involved.

To apply this to the operations \mathfrak{A}_t , recall the way in which these operations were defined (see § 3 in [9]). We defined a set of ‘‘model operations’’, P_t , which used as coefficient groups only the groups of the category \mathcal{C} (see § 2). The operations \mathfrak{A}_t were then defined by composing the operation P_t with coefficient group homomorphisms; that is, precisely the same pattern as followed in Definition 2.3. Thus, the operations \mathfrak{A}_t are defined in the relative case by simply applying the method given in this section to the operations P_t .

Let P'_t be the relative operation obtained from P_t . We note several facts needed later.

(4.5) LEMMA. Let $u_i \in H^{q_i}(X_i, A_i; Z_\theta)$ ($i = 1, 2$), where $Z_\theta \in \mathcal{C}$. Then

$$(1) \quad P'_t(u_1 \times u_2) = P'_t(u_1) \times P'_t(u_2) \quad (t \text{ odd}) .$$

If $t=2$ and θ is a power of 2, then,

$$(2) \quad P'_2(u_1 \times u_2) = P'_2(u_1) \times P'_2(u_2) + \nu_*[Sq_1(u_1) \times \mu_* w(u_2) + \mu_* w(u_1) \times Sq_1(u_2)].$$

Here, ν is the homomorphism of Z_2 to $G_2(Z_\theta)$ given by $\nu(1_2) = \theta g_2(1_\theta)$, and μ is the factor homomorphism $Z_n \rightarrow Z_2$. The functions Sq and w are defined respectively in 9.6 and 8.14 of [9].

Proof. The first statement is a consequence of Corollary 4.3 and the fact that the absolute operations P_i satisfy this formula². Equation 4.5(2) was remarked in [9; § 13] for the absolute operations P_i , and the case $\dim u_i$ odd. But it follows from 8.12 in [9] that 4.5(2) holds in general. In fact Theorem 8.11 in [9] can be obtained at once from equation 4.5(2). The extension of the equation to the relative operation P'_i , follows then from application of Proposition 4.2.

Combining Proposition 4.2 and 8.2 of [9] we also obtain

(4.6) LEMMA. *Let t be an integer where $t = p_k \cdots p_1$ (p_i prime). Let $u \in H^{2q}(X, A; Z)$. ($Z \in \mathcal{C}$). Then,*

$$P'_t(u) = P'_{p_k} \circ \cdots \circ P'_{p_1}(u).$$

Since it is in fact the relative operation, P'_i , we will work with, from now on we drop the prime, writing only P_i for both the relative and absolute operation.

Proof of Proposition 4.2. Let $Y = K(G_1, q_1) \times \cdots \times K(G_n, q_n)$, where each $K(G_i, q_i)$ is an Eilenberg-MacLane space of type (G_i, q_i) . Let $\pi_j: Y \rightarrow K(G_j, q_j)$ ($j = 1, \dots, n$), be the projection map and set $\bar{\tau}_j = \pi_j^*(\tau_j)$, where τ_j is the characteristic class in $H^{q_j}(K(G_j, q_j); G_j)$. Let e_j be a base point in $K(G_j, q_j)$ and set $Y' = (K(G_1, q_1), e_1) \times \cdots \times (K(G_n, q_n), e_n)$. Let $\tau'_j, \bar{\tau}'_j$ be the equivalent of τ_j and $\bar{\tau}_j$. Then, Proposition 4.2 follows at once from the following three lemmas (we keep the same notation as used in Proposition 4.2)

$$(4.7) \quad C(u_1, \dots, u_n) = E(D_1(u_1, \dots, u_n), \dots, D_p(u_1, \dots, u_n))$$

if and only if

$$C(\bar{\tau}_1, \dots, \bar{\tau}_n) = E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)).$$

$$(4.8) \quad C'(u'_1, \dots, u'_n) = E'(D'_1(u'_1, \dots, u'_n), \dots, D'_p(u'_1, \dots, u'_n))$$

if and only if

$$C'(\bar{\tau}'_1, \dots, \bar{\tau}'_n) = E'(D'_1(\bar{\tau}'_1, \dots, \bar{\tau}'_n), \dots, D'_p(\bar{\tau}'_1, \dots, \bar{\tau}'_n))$$

² The operations \mathbb{F}_i are easily defined for odd dimensional classes: see [9; § 7].

$$(4.9) \quad \text{If } C(\bar{\tau}_1, \dots, \bar{\tau}_n) = E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n))$$

then,

$$C'(\bar{\tau}'_1, \dots, \bar{\tau}'_n) = E'(D'_1(\bar{\tau}'_1, \dots, \bar{\tau}'_n), \dots, D'_p(\bar{\tau}'_1, \dots, \bar{\tau}'_n)) .$$

We give only the proof of Lemma 4.7, the others being entirely similar. Assume first we are given classes $u_i \in H^{q_i}(X; G_i)$ ($i = 1, \dots, n$). Let $f_j: X \rightarrow K(G_j; q_j)$ be mappings such that $f_j^*(\tau_j) = u_j$. Set $f = f_1 \times \dots \times f_n: X \rightarrow Y$. Then, by naturality, one has

$$(4.10) \quad \begin{aligned} (a) \quad & C(u_1, \dots, u_n) = f^*C(\bar{\tau}_1, \dots, \bar{\tau}_n) , \\ (b) \quad & D_i(u_1, \dots, u_n) = f^*D_i(\bar{\tau}_1, \dots, \bar{\tau}_n) \quad (i = 1, \dots, p). \end{aligned}$$

Suppose now that

$$C(\bar{\tau}_1, \dots, \bar{\tau}_n) = E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)) .$$

Then, by 4.10,

$$C(u_1, \dots, u_n) = f^*E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)) .$$

But E is natural with respect to mappings. Therefore,

$$\begin{aligned} & f^*E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)) \\ &= E(f^*D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, f^*D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)) \\ &= E(D_1(u_1, \dots, u_n), \dots, D_p(u_1, \dots, u_n)) , \end{aligned}$$

again by 4.10, which completes the proof of this assertion. The proof in the opposite direction is trivial.

5. The proof of Theorem I. Recall that the operation \mathfrak{F}_t is defined by means of the model operations P_t and coefficient group homomorphisms. But it is clear that the isomorphism ϕ , defined in 3.1, commutes with coefficient group homomorphisms. Thus, it suffices to prove Theorem I with \mathfrak{F}_t replaced by P_t , the operation \mathfrak{p} replaced by p , and the group A_{2k} taken to be a group in the category \mathcal{C} , say $A_{2k} = Z_0$.

Assume first that t is an odd prime p . Since ϕ is an isomorphism, the proof of Theorem I (ii) consists simply in showing

$$P_p\phi(u) = 0 , \quad u \in H^r(X; Z_0).$$

But this is immediate; for

$$P_p\phi(u) = P_p(v \times u) = P_p(\bar{v} \times u) = P_p(\bar{v}) \times P_p(u) ,$$

by Lemma 4.5(1). Here, \bar{v} is a generator of $H^r(I, \dot{I}; Z_0)$. However, $P_p(\bar{v}) = 0$, by dimensionality considerations. Thus, $P_p\phi(u) = 0$; and hence, $S(P_p) = 0$.

Now, suppose that t is any integer > 1 which is not a power of 2; say, $t = mp$, where p is an odd prime. Then, by Lemma 4.6

$$P_t\phi(u) = P_m \circ P_p\phi(u) = P_m(0) = 0 .$$

Consequently,

$$S(P_t) = 0 .$$

Thus, we have proved Theorem I(ii) for the case t is not a power of 2. Before concluding the proof of part (ii), we must prove part (i). Let the classes u and v be as above, where u has coefficients in the group Z_θ . If θ is zero or odd, then by Proposition 7.4 in [9], we have

$$P_2(v \times u) = P_2(\bar{v} \times u) = (\bar{v} \times u)^2 = \pm \bar{v}^2 \times u^2 = 0 ,$$

since $\bar{v}^2 = 0$. Thus, in this case $S(P_2) = 0$. Suppose now that θ is a power of 2.

Let η be the factor map $Z \rightarrow Z_\theta$. Then, $v \times u = (\eta_*v) \times u$, where the right hand side uses the pairing $Z_\theta \otimes Z_\theta \approx Z_\theta$. Thus, using Lemma 4.5(2), we have

$$P_2(v \times u) = P_2(\eta_*v \times u) = P_2(\eta_*v) \times P_2(u) + \nu_*[Sq_1(\eta_*v) \times \mu_*w(u) + \mu_*w(\eta_*v) \times Sq_1(u)] .$$

Now, $P_2(\eta_*v) = 0$, $w(\eta_*v) = 0$ by dimensionality considerations. Also, since η_*v is a 1-dimensional class, $Sq_1(\eta_*v) = \xi_*v$, where ξ is the natural map $Z \rightarrow Z_2$ (see Steenrod [4; 12.6]). Thus,

$$(5.1) \quad P_2(v \times u) = \nu_*[\xi_*v \times \mu_*w(u)] .$$

Consider the following commutative diagram:

$$\begin{array}{ccc} Z \otimes Z_\theta & \xrightarrow{1 \otimes \beta} & Z \otimes G_2(Z_\theta) \\ \xi \otimes \mu \downarrow & & \approx \downarrow \omega \\ Z_2 \otimes Z_2 \approx Z_2 & \xrightarrow{\omega'} & G_2(Z_\theta) , \end{array}$$

where β is the homomorphism of Z_θ to $G_2(Z_\theta)$ given by $\beta(1_\theta) = \theta g_2(1_\theta)$ (see 2.1). Then, from 5.1,

$$\begin{aligned} P_2(v \times u) &= \nu_*\omega'_*(\xi \otimes \mu)_*[v \otimes w(u)] \\ &= \omega_*(1 \otimes \beta)_*[v \otimes w(u)] \\ &= v \times \beta_*w(u) \\ &= v \times p(u) , \text{ by 2.2 .} \end{aligned}$$

Therefore,

$$P_2\phi(u) = P_2(v \times u) = v \times p(u) = \phi p(u) .$$

That is,

$$S(P_2) = p .$$

This proves part (i) of Theorem I. To complete the proof of the theorem we must show that

$$P_{2^r}\phi(u) = 0 , \tag{r > 1}.$$

But by part (i) of Theorem I and Lemma 4.6, we have

$$\begin{aligned} P_{2^r}\phi(u) &= P_{2^{r-1}} P_2\phi(u) = P_{2^{r-1}} \phi p(u) \\ &= P_{2^{r-2}} P_2\phi p(u) = P_{2^{r-2}} \phi p(p(u)) = 0 . \end{aligned}$$

Here, we use property 6.6 of the function p , which is proved independently in the next section. This completes the proof of Theorem I.

6. The properties of the operation \mathfrak{p} . We give here the main properties of the Postnikov square, \mathfrak{p} .

(6.1) **THEOREM.** *Let X be a space, and let $A = \sum_k A_k$ be a p -cyclic ring with divided powers. Suppose that $u \in H^q(X; A_{2k})$ ($q, k > 0$). Then,³*

$$(6.2) \quad \mathfrak{p}(u) = 0, \text{ if order } A_{2k} \text{ is odd or infinite,}$$

$$(6.3) \quad 2\mathfrak{p}(u) = 0 ,$$

$$(6.4) \quad \mathfrak{p} \text{ is a homomorphism,}$$

$$(6.5) \quad \text{if order } A_{2k} = 2^i \text{ (} i > 1 \text{) and } 2u = 0, \text{ then } \mathfrak{p}(u) = 0,$$

$$(6.6) \quad \mathfrak{p}(\mathfrak{p}(u)) = 0 ,$$

$$(6.7) \quad f^*\mathfrak{p}(u) = \mathfrak{p}f^*(u) ,$$

$$(6.8) \quad \alpha_*\mathfrak{p}(u) = \mathfrak{p}\alpha_*(u) ,$$

where f^* is induced by a map f from a space Y to X , and α_* is induced by a homomorphism α from A to a p -cyclic ring with divided powers A' .

The proof of Theorem 6.1 falls into 2 parts. Suppose first that we have proved 6.2 through 6.7 with the operation \mathfrak{p} replaced by the operation p , and the coefficient group A_{2k} restricted to be a group in the category \mathcal{C} . Then, the proof of 6.2-6.7 for the general case of the

³ With the exception of 6.5 and 6.6, these properties are noted by J. H. C. Whitehead in [10].

function ν follows at once, using definition 2.3; that is, $\nu = \zeta_2^* p \nu_*$. In particular, 6.2–6.5 are simple consequences of the fact that ζ_2^* and ν_* are homomorphisms; 6.6 follows from 6.3 and 6.5, and 6.7 follows from the fact that f^* commutes with all coefficient group homomorphisms. Finally, to prove 6.8 for the operation ν , one uses 2.4 and exactly the same argument as that used to prove I(9) in § 4 of [9]. Thus, we are left with proving 6.2 through 6.7 for the operation p . Let $u \in H^q(K; Z)$, where $Z_\theta \in \mathcal{C}$. Then,

(i)
$$p(u) = 0, \text{ if } \theta \text{ is zero or odd.}$$

This follows at once from 2.1. For if θ is zero or odd, the homomorphism β is zero.

(ii)
$$2p(u) = 0$$

This again is immediate from 2.1; for it is always the case that $2\beta = 0$.

(iii)
$$p \text{ is a homomorphism}$$

In § 5 we showed that the operation p is the suspension of the operation P_2 . But by 7.4 in [6], all operations which are suspensions are homomorphisms.

(iv)
$$\text{If } \theta = 2^i \text{ (} i > 1\text{), and } 2u = 0, \text{ then } p(u) = 0.$$

Since $2u = 0$, we may use Lemma 13.3 of [9]: namely, there are classes $x \in H^{q-1}(K; Z_2)$ and $y \in H^q(K; Z_2)$ such that

$$u = \lambda_* \delta_*(x) + \nu_*(y),$$

where δ_* is the coboundary associated with the exact sequence

$$0 \longrightarrow Z \xrightarrow{2} Z \longrightarrow Z_2 \longrightarrow 0,$$

λ is the natural factor map $Z \rightarrow Z_2$, and ν maps Z_2 to Z_θ by $\nu(1_2) = (\theta/2)1_\theta$ (recall that $\theta = 2^i$, $i > 1$). Hence, by (iii) above,

$$\begin{aligned} p(u) &= p\lambda_* \delta_*(x) + p\nu_*(y) \\ &= G_2(\lambda)_* p\delta_*(x) + G_2(\nu)_* p(y) \\ &= G_2(\nu)_* p(y), \end{aligned}$$

by 2.4 and (i) above, since $\delta_*(u)$ has integer coefficients. Now,

$$G_2(\nu)_* p(y) = G_2(\nu)_* \beta_* w(u),$$

by 2.2. We show that $p(u) = 0$ by showing that

$$G_2(\nu)\beta = 0.$$

From Definition 2.1 we recall that β maps Z_2 to $G_2(Z_2)$ by $\beta(1_2) = 2g_2(1_2)$. Hence, using 1.21 and 1.24 in [9],

$$\begin{aligned} G_2(\nu)\beta(1_2) &= 2G_2(\nu)g_2(1_2) = 2g_2(\nu 1_2) \\ &= 2g_2((\theta/2)1_\theta) = 2(\theta^2/4)g_2(1) = (\theta^2/2)1_{2\theta} = 0 . \end{aligned}$$

For, $\theta^2/2 = 2^{2i}/2 = 2^{2i-1}$; and, $2\theta = 2^{i+1}$. But by hypothesis, $i \geq 2$; thus $2i - 1 \geq i + 1$.

(v)
$$p(p(u)) = 0$$

This follows at once from (ii) and (iv) above.

(vi)
$$f^*p(u) = pf^*(u) .$$

This is simply a special case of Theorem 3.6 of [7]. This, then completes the proof of Theorem 6.1.

We consider one more property of the operation \mathfrak{p} : namely, its behaviour with respect to suspension. We continue to denote by $S(C)$ the suspension of a cohomology operation C .

(6.9) PROPOSITION. $S(\mathfrak{p}) = 0$, where 0 denotes the trivial cohomology operation.

Proof. By the same reasoning as given in § 5, it suffices to prove Proposition 6.9 with \mathfrak{p} replaced by the operation p , and the coefficient group A_{2k} taken to be a group in the category \mathcal{C} , say $A_{2k} = Z$. Thus, we need simply show that $p\phi(u) = 0$, where $u \in H^q(L; Z_i)$. Now by Nakaoka [2] we have⁴:

$$p(v_1 \times v_2) = P_2(v_1) \times p(v_2) + p(v_1) \times P_2(v_2) ,$$

for classes $v_i \in H^{q_i}(X_i, A_i; Z)$ ($i = 1, 2$).

Thus,

$$p\phi(u) = p(\bar{v} \times u) = P_2(\bar{v}) \times p(u) + p(\bar{v}) \times P_2(u) = 0 ,$$

since $P_2(\bar{v}) = p(\bar{v}) = 0$ by dimensionality considerations. Here, \bar{v} is the image of v in $H^1(I, \dot{I}; Z_\theta)$. Hence, $S(p) = 0$, as was to be proved.

7. The relation $\delta S(C) = C\delta$. We give here a theorem, whose proof is due to N. E. Steenrod.

(7.1) THEOREM. *Let C be a cohomology operation, and let δ be the relative cohomology coboundary operator. Then,*

⁴ Nakaoka only proves this for the case $\dim v_1, v_2$ even; but the result is true in general, as is easily shown using Definition 2.1.

$$\delta S(C) = C\delta ,$$

where $S(C)$ is the suspension of C .

We sketch the proof; let X be a space and $A \subset X$ a subspace. Let X' denote the mapping cylinder of the inclusion map $A \subset X$. That is, unite $I \times A$ and X by identifying $1 \times A$ with A in X . Let $A' = 0 \times A$. The inclusions

$$(X', A') \longrightarrow (X', I \times A) \longleftarrow (X, A)$$

induce isomorphisms of the cohomology sequence of (X, A) and (X', A') with local coefficients. Thus, we may discuss the behaviour of the coboundary δ in the cohomology sequence of the pair (X', A') .

Consider the following hexagonal diagram (see [8], page 42):

$$(7.2) \quad \begin{array}{ccccc} & & H^q(I \times X) & & \\ & n_1^* \swarrow & \downarrow j^* & \searrow n_0^* & \\ H^q(0 \times X) & & & & H^q(1 \times X) \\ & \swarrow d_1^* & & \searrow d_0^* & \\ & & H^q(\dot{I} \times X) & & \\ & k_1^* \uparrow & & \downarrow k_0^* & \\ H^q(\dot{I} \times X, 1 \times X) & & & & H^q(\dot{I} \times X, 0 \times X) \\ & \swarrow \delta_1 & \downarrow \delta & \searrow \delta_0 & \\ & & H^{p+1}(\dot{I} \times X, \dot{I} \times X) & & \end{array}$$

Here all homomorphisms other than δ , δ_1 , and δ_2 are induced by inclusions. Standard arguments, using exactness and homotopy equivalence, show that the arrows around the peripheries are isomorphisms. We agree to identify $H^q(X)$ with $H^q(0 \times X)$ by sending $u \rightarrow e \times u$, where e is the unit of $H^0(0; Z)$. At the end of this section we will use diagram 7.2 to prove the following lemma:

(7.3) LEMMA. *Let ϕ be the function defined in 3.1. Then,*

$$\phi = \delta_1 k_1^{*-1} ,$$

where k_1^* , δ_1 are the functions defined in diagram 7.2

Notice that this proves Lemma 3.2; for the functions δ_1 , k_1^* are isomorphisms. Now let $g^*: H^{q+1}(X', A' \cup X) \rightarrow H^{q+1}(I \times A, \dot{I} \times A)$ be induced by the inclusion. Using the fact that \dot{I} is a strong deformation retract of a neighborhood of \dot{I} in I (see [8]; Chapter 1, 11.6), together with excision, one shows that g^* is an isomorphism onto.

(7.4) LEMMA. *The following diagram is commutative, where f^* is induced by the inclusion*

$$\begin{array}{ccc}
 H^{q+1}(I \times A, \dot{I} \times A) & \xrightarrow{g^{*-1}} & H^{q+1}(X', A' \cup X) \\
 \uparrow \phi & & \downarrow f^* \\
 H^q(A') & \xrightarrow{\delta} & H^{q+1}(X', A') .
 \end{array}$$

Thus $\delta = f^*g^{*-1}\phi$.

This is a consequence of Lemma 7.3 and commutativity relations in a slightly enlarged diagram. We omit the details.

The proof of Theorem 7.1 is an immediate consequence of Lemma 7.4. For let $u \in H^q(A')$. Then, by this lemma,

$$C\delta(u) = Cf^*g^{*-1}\phi(u) .$$

Using the naturality of the operation C , we have

$$Cf^*g^{*-1}\phi(u) = f^*g^{*-1}C\phi(u) .$$

But by Definition 3.1, $C\phi = \phi S(C)$.

Thus,

$$C\delta(u) = f^*g^{*-1}\phi S(C)(u) = \delta S(C)(u) ,$$

again using Lemma 7.4. This completes the proof of Theorem 7.1.

Proof of Lemma 7.3. We apply diagram 7.2 to the case $X = \emptyset$, $q = 0$, and coefficient group = integers. Then, the unit class of $H^0(\dot{I}; Z)$ can be represented as a sum $v_0 + v_1$, where

$$v_0 = i_1^*k_1^{*-1}d_1^*(v_0 + v_1), \quad v_1 = i_0^*k_0^{*-1}d_0^*(v_0 + v_1).$$

Thus,

$\delta(v_0) = -\delta(v_1) = v =$ a generator of $H^1(I, \dot{I}; Z)$. Therefore, by Definition 3.1,

$$\phi(u) = v \times u = (\delta v_0) \times u .$$

But by the axioms for the cross-product, we may write

$$(\delta v_0) \times u = \delta(v_0 \times u) .$$

Furthermore, we have

$$v_0 = i_1^*k_1^{*-1}(e) ,$$

where $e = d_1^*(v_0 + v_1) =$ unit of $H^0(0; Z)$. Thus,

$$\begin{aligned}
 \delta(v_0 \times u) &= \delta(i_1^*k_1^{*-1}(e) \times u) \\
 &= \delta i_1^*k_1^{*-1}(e \times u) = \delta_1 k_1^{*-1}(e \times u) .
 \end{aligned}$$

Here we have used the naturality of the cross-product and the commutativity of diagram 7.2. If we now identify $H^q(X)$ with $H^q(0 \times X)$ by sending $u \rightarrow e \times u$, we then have

$$\phi(u) = \delta(v_0 \times u) = \delta_1 k_1^{*-1}(u),$$

as was asserted.

REFERENCES

1. H. Cartan, *Seminaire H. Cartan 1954/55*, Paris (mimeographed).
2. M. Nakaoka, *Note on cohomological operations*, J. Inst. Polytech., Osaka City Univ., **4** (1953), 51-58.
3. M. Postnikov, *The classification of continuous mappings of a 3-dimensional polyhedron into a simply connected polyhedron of arbitrary dimension*, C. R. (Doklady) Acad. Sci. U. R. S. S., **64**(1949), 461-462.
4. N. Steenrod, *Products of cocycles and extensions of mappings*, Ann. Math., **48** (1947), 290-320.
5. ———, *Homology groups of symmetric groups and reduced operations*, Proc. Nat. Acad. Sci. U. S. A., **39** (1953), 213-223.
6. ———, *Cohomology operations*, Symposium Int. Topologie algebraica, Univ. de Mexico (1957), (to appear).
7. ———, *Cohomology operations derived from the symmetric group*, Comment. Math. Helv., **31** (1957), 195-218.
8. N. Steenrod and E. Eilenberg, *Foundations of Algebraic Topology*, Princeton University Press, 1952.
9. E. Thomas, *The generalized Pontrjagin cohomology operations and rings with divided powers*, Memoir Number 27 A. M. S. (1957).
10. J. H. C. Whitehead, *On the theory of obstructions*, Ann. Math., **54** (1951), 68-84.

UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG
Stanford University
Stanford, California

R. A. BEAUMONT
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

L. J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
C. E. BURGESS
E. HEWITT
A. HORN

V. GANAPATHY IYER
R. D. JAMES
M. S. KNEBELMAN
L. NACHBIN

I. NIVEN
T. G. OSTROM
H. L. ROYDEN
M. M. SCHIFFER

E. G. STRAUS
G. SZEKERES
F. WOLF
K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chivoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 9, No. 3

July, 1959

| | |
|--|-----|
| Errett Albert Bishop, <i>A minimal boundary for function algebras</i> | 629 |
| John W. Brace, <i>The topology of almost uniform convergence</i> | 643 |
| Cecil Edmund Burgess, <i>Chainable continua and indecomposability</i> | 653 |
| L. Carlitz, <i>Multiplication formulas for products of Bernoulli and Euler polynomials</i> | 661 |
| Eckford Cohen, <i>A class of residue systems (mod r) and related arithmetical functions. II. Higher dimensional analogues</i> | 667 |
| Shaul Foguel, <i>Boolean algebras of projections of finite multiplicity</i> | 681 |
| Richard Robinson Goldberg, <i>Averages of Fourier coefficients</i> | 695 |
| Seymour Goldberg, <i>Ranges and inverses of perturbed linear operators</i> | 701 |
| Philip Hartman, <i>On functions representable as a difference of convex functions</i> | 707 |
| Milton Vernon Johns, Jr. and Ronald Pyke, <i>On conditional expectation and quasi-rings</i> | 715 |
| Robert Jacob Koch, <i>Arcs in partially ordered spaces</i> | 723 |
| Gregers Louis Krabbe, <i>A space of multipliers of type $L^p(-\infty, \infty)$</i> | 729 |
| John W. Lamperti and Patrick Colonel Suppes, <i>Chains of infinite order and their application to learning theory</i> | 739 |
| Edith Hirsch Luchins, <i>On radicals and continuity of homomorphisms into Banach algebras</i> | 755 |
| T. M. MacRobert, <i>Multiplication formulae for the E-functions regarded as functions of their parameters</i> | 759 |
| Michael Bahir Maschler, <i>Classes of minimal and representative domains and their kernel functions</i> | 763 |
| William Schumacher Massey, <i>On the imbeddability of the real projective spaces in Euclidean space</i> | 783 |
| Thomas Wilson Mullikin, <i>Semi-groups of class (C_0) in L_p determined by parabolic differential equations</i> | 791 |
| Steven Orey, <i>Recurrent Markov chains</i> | 805 |
| Ernest Tilden Parker, <i>On quadruply transitive groups</i> | 829 |
| Calvin R. Putnam, <i>On Toeplitz matrices, absolute continuity, and unitary equivalence</i> | 837 |
| Helmut Heinrich Schaefer, <i>On nonlinear positive operators</i> | 847 |
| Robert Seall and Marion Wetzel, <i>Some connections between continued fractions and convex sets</i> | 861 |
| Robert Steinberg, <i>Variations on a theme of Chevalley</i> | 875 |
| Olga Taussky and Hans Zassenhaus, <i>On the similarity transformation between a matrix and its transpose</i> | 893 |
| Emery Thomas, <i>The suspension of the generalized Pontrjagin cohomology operations</i> | 897 |
| Joseph L. Ullman, <i>On Tchebycheff polynomials</i> | 913 |
| Richard Steven Varga, <i>Orderings of the successive overrelaxation scheme</i> | 925 |
| Orlando Eugenio Villamayor, Sr., <i>On weak dimension of algebras</i> | 941 |