Pacific Journal of Mathematics

THE NILPOTENT PART OF A SPECTRAL OPERATOR

CHARLES ALAN MCCARTHY

Vol. 9, No. 4 August 1959

THE NILPOTENT PART OF A SPECTRAL OPERATOR

CHARLES A. MCCARTHY

1. Introduction. Throughout this paper, $\mathfrak{X}$ is a Banach space, T a bounded spectral operator on $\mathfrak{X}$ with scalar part S, nilpotent part N, and resolution of the identity $E(\sigma)$ for σ a Borel set in the complex plane. M is the bound for the norms of the $E(\sigma)$; $|E(\sigma)| \leq M$ for all Borel sets σ. The resolvent function for $T, (\lambda - T)^{-1}$, is denoted by $R(\lambda, T)$. The operator $R(\lambda, T)E(\sigma)$ has an unique analytic extension from the resolvent set of T to the complement of $\bar{\sigma}$, and on the subspace $E(\sigma)\mathfrak{X}$ it is equal to the operator $R(\lambda, T_\sigma)$ where T_σ is the restriction of T to $E(\sigma)\mathfrak{X}$. For material on spectral operators, we refer to the papers on N. Dunford [1], [2]. $\chi_\sigma(\xi)$ is the characteristic function of the Borel set σ: $\chi_\sigma(\xi) = 1$ if $\xi \in \sigma, \chi_\sigma(\xi) = 0$ if $\xi \notin \sigma$. For p a non-negative real number, μ_p is Hausdorff p-dimensional measure [3, pp. 102 *ff.*]; μ_2 is Lebesgue planar measure multiplied by $\pi/4$, and μ_1 restricted to an arc is majorized by arc length.

We assume throughout that there is an integer m for which the resolvent function for T satisfies the mth order rate of growth condition

$$|R(\lambda, T)E(\sigma)| \leq K \cdot d(\lambda, \sigma)^{-m}, \lambda \notin \bar{\sigma}, |\lambda| \leq |T| + 1 ,$$

where $d(\lambda, \sigma)$ is the distance from λ to σ and K is a constant independent of σ. If $\mathfrak{X}$ is Hilbert space, it is known that this growth condition implies $N^m = 0$ [1, p. 337]. In an arbitrary Banach space, this is no longer true; the best that can be done is $N^{m+2} = 0$. If $\mathfrak{X}$ is weakly complete, $N^{m+1}=0$; or if σ is a set of μ_2 measure zero, $N^{m+1}E(\sigma) = 0$. If σ lies in an arc and either $\mathfrak{X}$ is weakly complete or σ has μ_1 measure zero, then $N^m E(\sigma) = 0$. Examples show that we cannot obtain lower indices of nilpotency in general.

2. The fundamental lemma and some easy consequences. If $f(\xi)$ is a bounded, scalar valued Borel function, the operator $\int f(\xi)E(d\xi)$ exists as a bounded operator with norm at most $4M \cdot \sup_\xi |f(\xi)|$ [1, p. 341], so that uniform convergence of a sequence of bounded Borel functions $f_n(\xi)$ implies convergence in the uniform operator topology of the operators $\int f_n(\xi)E(d\xi)$. Thus for a given bounded Borel function $f(\xi)$ and a given positive number η, there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

Received February 5, 1959. This paper is a portion of a doctoral dissertation presented to Yale University, written under the direction of Professor E. Hille, while the author was an NSF fellow. Particular thanks are due to W. G. Bade who read the manuscript and discovered an error in the author's original proof of Theorem 3.1.

$$\left|\int f(\xi)E(d\xi) - \sum_i f(\xi_i)E(\sigma_i)\right| < \eta\,.$$

Similiarly if A_n are a finite number of bounded operators and $f_n(\xi)$ are bounded Borel functions, for any positive number η, there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

$$\left|\sum_n \int A_n f_n(\xi)E(d\xi) - \sum_i \sum_n A_n f_n(\xi_i)E(\sigma_i)\right| < \eta;$$

in particular, for an integer k and a positive number η, there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

$$\left|\int (T-\xi)^k E(d\xi) - \sum_i (T-\xi_i)^k E(\sigma_i)\right| < \eta\,.$$

LEMMA 2.1. *There exist constants M_k such that $|N^k E(\sigma)| \le M_k \varepsilon^{k+1-m}$ for any choice of ε, $0 < \varepsilon \le 1$, and Borel set σ of diameter no greater than ε.*

Proof. Pick ε, $0 < \varepsilon \le 1$, and let σ be any Borel set of diameter no greater than ε. We have [1, p, 338]

$$N^k E(\sigma) = \int_\sigma (T-\xi)^k E(d\xi)\,.$$

For any positive number η, there is a decomposition of σ into a finite number of disjoint Borel sets $\sigma_i \subset \sigma$, and points $\xi_i \in \sigma_i$ such that

$$\left|\int (T-\xi)^k E(d\xi) - \sum_i (T-\xi_i)^k E(\sigma_i)\right| < \eta\,.$$

Since σ is of diameter at most ε, there is a circle Γ of diameter 3ε which encloses σ and for which $|\gamma - \xi| \ge \varepsilon$ for all $\gamma \in \Gamma$ and $\xi \in \sigma$. Then

$$(T-\xi_i)^k E(\sigma_i) = \frac{1}{2\pi i}\int_\Gamma (\gamma - \xi\,)^k R(\gamma, T)E(\sigma_i)d\gamma\,,$$

so that

$$\sum_i (T-\xi_i)^k E(\sigma_i) = \frac{1}{2\pi i}\int_\Gamma R(\gamma, T)\sum_i (\gamma-\xi_i)^k E(\sigma_i)d\gamma\,,$$

which in norm in no greater than

$$(*) \qquad \frac{1}{2\pi}\cdot \sup_{\gamma\in\Gamma}|R(\gamma, T)E(\sigma)| \cdot \sup_{\gamma\in\Gamma}\left|\sum_i (\gamma-\xi_i)^k E(\sigma_i)\right| \cdot \text{length of } \Gamma\,.$$

The mth order rate of growth condition gives

$$\sup_{\gamma \in \Gamma} |R(\gamma, T)E(\sigma)| \leq K\varepsilon^{-m} .$$

For any $\gamma \in \Gamma$,

$$|\textstyle\sum_i (\gamma - \xi_i)^k E(\sigma_i)| \leq 4M \cdot \max_i |\gamma - \xi_i|^k \leq 4M(2\varepsilon)^k ,$$

so that (*) is no greater than

$$\frac{1}{2\pi} K\varepsilon^{-m} \cdot 4M(2\varepsilon)^k \cdot 6\pi\varepsilon = M_k \varepsilon^{k+1-m} ,$$

where $M_k = 3 \cdot 2^{k+2}KM$, and is independent of η, ε, σ, and the manner in which σ is decomposed. Thus

$$|N^k E(\sigma)| \leq M_k \varepsilon^{k+1-m} + \eta$$

for every positive η, which proves the lemma.

THEOREM 2.2. *Let σ be a Borel set whose Hausdorff p-measure is zero for a given p. Then $N^k E(\sigma) = 0$ where k is an integer and $k \geq p + m - 1$.*

Proof. Since σ has p-measure zero, for every $\varepsilon > 0$, there is a covering of σ by disjoint sets σ_i of diameter ε_i such that $\sum_i \varepsilon_i^p < \varepsilon$. By Lemma 2.1 we have

$$\begin{aligned} |N^k E(\sigma)| &\leq \textstyle\sum_i |N^k E(\sigma_i)| \leq M_k \sum_i \varepsilon_i^{k+1-m} \\ &\leq M_k \textstyle\sum_i \varepsilon_i^{(p+m-1)+1-m} \leq M_k \sum_i \varepsilon_i^p \leq M_k \varepsilon . \end{aligned}$$

Since ε may be chosen arbitrarily small, $N^k E(\sigma) = 0$.

COROLLARY 2.3. $N^{m+2} = 0$.

Proof. Taking σ to be the spectrum of T and $p = 3$, $N^{m+2}E(\sigma(T)) = 0$; but $E(\sigma(T))$ is the identity mapping on $\mathfrak{X}$.

COROLLARY 2.4. *If σ has planar measure zero, then $N^{m+1}E(\sigma) = 0$.*

COROLLARY 2.5. *If σ has μ_1-measure zero, then $N^m E(\sigma) = 0$.*

3. **The case of weakly complete $\mathfrak{X}$.** Let σ be a Borel set in the plane. For any $\varepsilon > 0$, we can cover σ with disjoint Borel sets. σ_i of diameter ε_i, $\varepsilon_i \leq 1$, such that

$$\textstyle\sum_i \varepsilon_i^2 \leq \mu_2(\sigma) + \varepsilon .$$

Thus by Lemma 2.1,

$$\begin{aligned}|N^{m+1}E(\sigma)| &\leq \textstyle\sum_i |N^{m+1}E(\sigma_i)| \leq M_{m+1} \sum_i \varepsilon_i^2 \\ &\leq M_{m+1}(\mu_2(\sigma) + \varepsilon) .\end{aligned}$$

Since ε and σ are arbitrary, we have for all Borel sets σ,

$$|N^{m+1}E(\sigma)| \leq M_{m+1}\mu_2(\sigma) .$$

As a consequence, all the scalar measures $x^*N^{m+1}E(\cdot)x = [(N^*)^{m+1}E^*(\cdot)x^*]x$, $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, are absolutely continuous with respect to μ_2, and have derivative bounded by $M_{m+1}|x^*|\,|x|$.

Suppose that $f(\xi) = \sum_{p=1}^{P} \alpha_p \chi_{\sigma_p}(\xi)$ is a simple Borel function; α_p are scalar constants and σ_p are disjoint Borel sets. We have

$$\begin{aligned}\left|\int f(\xi)(N^*)^{m+1}E^*(d\xi)\right| &\leq \sum_{p=1}^{P} |\alpha_p(N^*)^{m+1}E^*(\sigma_p)| \\ &\leq \sum_{p=1}^{P} |\alpha_p| M_{m+1}\mu_2(\sigma_p) \\ &= M_{m+1}|f|_{L_1(\mu_2)} .\end{aligned}$$

Thus if $f_n(\xi)$ are simple Borel functions converging in $L_1(\mu_2)$ to $f(\xi)$, the operators $\int f_n(\xi)(N^*)^{m+1}E^*(d\xi)$ converge in the uniform operator topology to an operator which we denote by $\int f(\xi)(N^*)^{m+1}E^*(d\xi)$; this limit operator has norm bounded by $M_{m+1}|f|L_1(\mu_2)$.

THEOREM 3.1. *If $\mathfrak{X}$ is weakly complete, then $N^{m+1} = 0$.*

Proof. Assume that $N^{m+1} \neq 0$, so that also $(N^*)^{m+1} \neq 0$. We will first obtain a bicontinuous map of an infinite dimensional L_1 space into $\mathfrak{X}^*$. An analogous map into $\mathfrak{X}$ would show then that $\mathfrak{X}$ cannot be reflexive, since the image in $\mathfrak{X}$ of this L_1 space would be a closed nonreflexive subspace of $\mathfrak{X}$; however, the map into $\mathfrak{X}^*$ is needed for the slightly more general case of $\mathfrak{X}$ weakly complete.

Let the Borel set σ, $x_0 \in \mathfrak{X}$, and $x_0^* \in \mathfrak{X}^*$ be chosen so that $[(N^*)^{m+1}E^*(\sigma)x_0^*]x_0 \neq 0$, and let the derivative of the measure $[(N^*)^{m+1}E^*(\cdot)x_0^*]x_0$ be denoted by $g(\xi)$. We can then find a subset τ of σ and a constant $a > 0$ such that $\mu_2(\tau) > 0$ and $|g(\xi)| \geq a$ on τ.

Define the map Φ of $L_1(\tau, \mu_2)$ into $\mathfrak{X}^*$ by

$$\Phi(f) = \int_\tau f(\xi)(N^*)^{m+1}E^*(d\xi)x_0^* .$$

Φ is a linear map with bound $M_{m+1}|x_0^*|$. Now take

$$x = \int_\tau [g(\xi)]^{-1} \operatorname{sgn} \overline{f(\xi)} E(d\xi)x_0 ;$$

The norm of x is no greater than $4M \cdot a^{-1} \cdot |x_0|$. But we have

$$\begin{aligned}[\Phi(f)](x) &= \int_\tau f(\xi)[g(\xi)]^{-1} \operatorname{sgn} \overline{f(\xi)}[(N^*)^{m+1}E^*(d\xi)x_0^*]x_0 \\ &= \int_\tau |f(\xi)|[g(\xi)]^{-1}g(\xi)\mu_2(d\xi) \\ &= |f|_{L_1} ,\end{aligned}$$

which shows that

$$|\Phi(f)| \geq |f|_{L_1} \cdot a \cdot (4M|x_0|)^{-1} ,$$

so that Φ is one-to-one and has a continuous inverse.

Now let Ψ be the map of $L_\infty(\tau, \mu_2)$ into $\mathfrak{X}$:

$$\Psi(h) = \int_\tau [g(\xi)]^{-1}h(\xi)E(d\xi)x_0 ,$$

Ψ is a continuous map with bound no greater than $4M \cdot a^{-1}|x_0|$; we will show that Ψ is one-to-one and bicontinuous. We have

$$\begin{aligned}\Phi(f)\Psi(h) &= \int_\tau f(\xi)[g(\xi)]^{-1}h(\xi)[(N^*)^{m+1}E^*(d\xi)x_0^*]x_0 \\ &= \int_\tau f(\xi)h(\xi)\mu_2(d\xi) ,\end{aligned}$$

so that

$$\begin{aligned}\sup_{|f|_{L_1} \leq 1} |\Phi(f)\Psi(h)| &= \sup_{|f|_{L_1} \leq 1} \left|\int_\tau f(\xi)h(\xi)\mu_2(d\xi)\right| \\ &= |h|_{L_\infty} .\end{aligned}$$

But since Φ is bounded,

$$\begin{aligned}\sup_{|f|_{L_1} \leq 1} |\Phi(f)\Psi(h)| &\leq \sup_{\substack{x^* \in X^* \\ |x^*| \leq |\Phi|}} |x^*\Psi(h)| \\ &= |\Phi|\,|\Psi(h)| ,\end{aligned}$$

so that

$$|h|_{L_\infty} \leq |\Phi|\,|\Psi(h)| ;$$

thus Ψ is one-to-one and bicontinuous, The range $\mathfrak{Y}$ of Ψ in $\mathfrak{X}$ is then a closed non weakly complete subspace of $\mathfrak{X}$. But this is impossible, because every closed subspace of a weakly complete Banach space is again weakly complete; the proof of this last remark is as follows.

Let $\mathfrak{X}$ be a weakly complete Banach space, $\mathfrak{Y}$ a closed subspace. Let y_n be a weakly Cauchy sequence in $\mathfrak{Y}$, so that y^*y_n is a Cauchy sequence of numbers for every y^* in Y^*. Since any x^* in X^*, when restricted to $\mathfrak{Y}$, is an element of $\mathfrak{Y}^*$, x^*y_n is a Cauchy sequence of numbers for every x^* in $\mathfrak{X}^*$. Since $\mathfrak{X}$ is weakly complete, there is an x_0 in $\mathfrak{X}$ such that $\lim_{n\to\infty} x^*y_n = x^*x_0$ for every x^* in $\mathfrak{X}^*$; and since $\mathfrak{Y}$ is strongly closed in $\mathfrak{X}$, it is weakly closed, so that x_0 must lie in $\mathfrak{Y}$. Finally since every y^* in $\mathfrak{Y}^*$ is, by the Hahn-Banach theorem, the restriction of an x^* in $\mathfrak{X}^*$, $\lim y^*y_n = y^*x_0$ for every y^* in $\mathfrak{Y}^*$, so that $\mathfrak{Y}$ is weakly complete.

THEOREM 3.2. *If $\mathfrak{X}$ is weakly complete, then $N^mE(\sigma) = 0$ for every set σ of finite μ_1-measure.*

Proof. Follow exactly the same discussion above, replacing the number $m + 1$ by m and the measure μ_2 by μ_1.

Note that Theorems 3.1 and 3.2 also hold if $\mathfrak{X}$ is assume to be separable instead of weakly complete, for the image of the L_∞ space in $\mathfrak{X}$ would be a nonseparable closed subspace of $\mathfrak{X}$; but every closed subspace of a separable space is again separable.

4. Examples.

In the following examples we will need two computational lemmas.

LEMMA 4.1. *For each real number $p \geq 1$ and Borel set σ,*

$$\int_\tau |\lambda - \xi|^{-(p+2)}\mu_2(d\xi) \leq 8d(\lambda, \sigma)^{-p}, \text{ for all } \lambda \notin \bar{\sigma} .$$

Proof.

$$\begin{aligned}
&\int_\sigma |\lambda - \xi|^{-(p+2)}\mu_2(d\xi) \\
&\quad \leq \int_{|\lambda-\xi|\geq d(\lambda,\sigma)} |\lambda - \xi|^{-(p+2)}\mu_2(d\xi) \\
&\quad = \frac{4}{\pi}\int_0^{2\pi} d\theta \int_{d(\lambda,\sigma)}^{\infty} r^{-(p+2)} r\, dr \qquad\qquad (\lambda - \xi = re^{i\vartheta}) \\
&\quad \leq 8d(\lambda, \sigma)^{-p} .
\end{aligned}$$

LEMMA 4.2. *For each real number $p \geq 1$ and Borel subset σ of the real line,*

$$\int_\sigma |\lambda - \xi|^{-(p+1)}\mu_1(d\xi) \leq 2^{p+1}\pi d(\lambda, \sigma)^{-p} ,$$

where μ_1 is Lebesgue measure along the line, and λ is any complex number, $\lambda \notin \bar{\sigma}$.

Proof. Let $\lambda = \alpha + i\beta$, α, β real. Then either, (i), $d(\alpha, \sigma) \geq d(\lambda, \sigma)/2$ or, (ii) $|\beta| \geq d(\lambda, \sigma)/2$. In case (i) we have

$$\int_\sigma |\lambda - \xi|^{-(p+1)}\mu_1(d\xi) \leq \int_{d(\lambda,\sigma)/2}^{\infty} \eta^{-(p+1)}d\eta \qquad (\lambda - \xi = \eta)$$
$$\leq 2^{p+1}p^{-1}d(\lambda, \sigma)^{-p} .$$

In case (ii) we have

$$\int_\sigma |\lambda - \xi|^{-(p+1)}\mu_1(d\xi) \leq \int_{-\infty}^{\infty} |\xi - i\beta|^{-(p+1)}d\xi$$
$$\leq \int_{-\infty}^{\infty} (\xi^2 + \beta^2)^{-\frac{1}{2}(p+1)}d\xi$$
$$\leq 2^{p+1}\pi d(\lambda, \sigma)^{-p} .$$

EXAMPLE 4.3. Let Σ be a disc in the plane with μ_2-measure 1. Let

$$x = L_\infty(\Sigma) \oplus L_2(\Sigma) \oplus \cdots \oplus L_2(\Sigma) \oplus L_1(\Sigma) ,$$

where m copies of $L_2(\Sigma)$ are taken. Let T be the operator $S + N$ where S and N are defined as

$$S[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)]$$
$$= [\xi f(\xi) \oplus \xi g_1(\xi) \oplus \cdots \oplus \xi g_m(\xi) \oplus \xi h(\xi)] ,$$

$$N[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)]$$
$$= [0 \oplus f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi)] .$$

Since Σ has measure 1, any function in L_r is in L_s for all $s \leq r$, and the L_s norm is no greater than the L_r norm; thus N is a bounded operator with norm 1. Also N is a nilpotent for which $N^{m+1} \neq 0$. The operator T is a spectral operator with resolution of the identity

$$E(\sigma)[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)]$$
$$= [f(\xi)\chi_\sigma(\xi) \oplus g_1(\xi)\chi_\sigma(\xi) \oplus \cdots \oplus g_m(\xi)\chi_\sigma(\xi) \oplus h(\xi)\chi_\sigma(\xi)] .$$

The resolvent function is

$$R(\lambda, T)E(\sigma)[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)]$$
$$= \left[\frac{f(\xi)\chi_\sigma(\xi)}{\lambda - \xi} \oplus \left(\frac{g_1(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \frac{f(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^2}\right) \oplus \cdots \oplus\right.$$
$$\left(\frac{g_m(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \cdots + \frac{g_1(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^m} + \frac{f(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^{m+1}}\right)$$
$$\left.\oplus \left(\frac{h(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \frac{g_m(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)} + \cdots + \frac{g_1(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^{m+1}} + \frac{f(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^{m+2}}\right)\right] .$$

All the terms are clearly of mth order rate of growth except possibly for

$$\text{(a)}\left|\frac{f(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^{m+1}}\right|_{L_2}, \quad \text{(b)}\left|\frac{f(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^{m+2}}\right|_{L_1}, \text{ and } \text{(c)}\left|\frac{g_1(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^{m+1}}\right|_{L_1}.$$

For (a) we have

$$\begin{aligned}\left\{\int_\sigma |f(\xi)(\lambda-\xi)^{-(m+1)}|^2\mu_2(d\xi)\right\}^{1/2} &\le |f|_{L_\infty}\left\{\int_\sigma \lambda-\xi|^{-2m-2}\mu_2(d\xi)\right\}^{1/2} \\ &\le |f|_{L_\infty}\sqrt{8}\,d(\lambda,\sigma)^{-m},\end{aligned}$$

for (b) we have

$$\begin{aligned}\int_\sigma |f(\xi)(\lambda-\xi)^{-(m+2)}|\mu_2(d\xi) &\le |f|_{L_\infty}\int_\sigma |\lambda-\xi|^{-(m+2)}\mu_2(d\xi) \\ &\le |f|_{L_\infty}\cdot 8d(\lambda,\sigma)^{-m},\end{aligned}$$

and for (c) we have

$$\begin{aligned}\int_\sigma |g_1(\xi)(\lambda-\xi)^{-(m+1)}|\mu_2(d\xi) &\le \left\{\int_\sigma |g_1(\xi)|^2\mu_2(d\xi)\right\}^{1/2}\left\{\int_\sigma |\lambda-\xi|^{-2m-2}\mu_2(d\xi)\right\}^{1/2} \\ &\le |g_1|_{L_2}\cdot\sqrt{8}\cdot d(\lambda,\sigma)^{-m}.\end{aligned}$$

Thus each term of the resolvent, and hence the resolvent itself satisfies the mth order rate of growth condition; this shows that Corollary 2.3 cannot be improved.

EXAMPLE 4.4. Let Σ be as in the previous example and let

$$\mathfrak{X} = L_r(\Sigma)\oplus\cdots\oplus L_r(\Sigma)\oplus L_s(\Sigma)$$

where m copies of L_r are taken. r and s are to satisfy $1 < s < r < \infty$ and $rs \le 2(r-s)$. Let $T = S + N$, where S and N are defined in essentially the same way as in the previous example. The resolvent function is given by

$$\begin{aligned}&R(\lambda, T)E(\sigma)[f_1(\xi)\oplus\cdots\oplus f_m(\xi)\oplus g(\xi)] \\ &\quad = \left[\frac{f_1(\xi)\chi_\sigma(\xi)}{\lambda-\xi}\oplus\cdots\oplus\left(\frac{f_m(\xi)\chi_\sigma(\xi)}{\lambda-\xi}+\cdots+\frac{f_1(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^m}\right)\right. \\ &\qquad \left.\oplus\left(\frac{g(\xi)\chi_\sigma(\xi)}{\lambda-\xi}+\frac{f_m(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^2}+\cdots+\frac{f_1(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^{m+1}}\right)\right].\end{aligned}$$

Each of the terms is clearly of mth order rate of growth except possibly for the L_s norm of $f_1(\xi)(\lambda-\xi)^{-(m+1)}\chi_\sigma(\xi)$, and for this we have

$$\begin{aligned}&\left\{\int_\sigma |f_1(\xi)(\lambda-\xi)^{m+1}|^s\mu_2(d\xi)\right\}^{1/s} \\ &\quad\le \left\{\int_\sigma |f_1(\xi)|^r\mu_2(d\xi)\right\}^{1/r}\left\{\int |\lambda-\xi|^{-\frac{(m+1)rs}{r-s}}\mu_2(d\xi)\right\}^{\frac{r-s}{rs}} \\ &\quad\le |f_1|_{L_r}\cdot 8^{\frac{s-r}{rs}}\cdot d(\lambda,\sigma)^{-m-\left(1-\frac{2(r-s)}{rs}\right)} \\ &\quad\le |f_1|_{L_r}\cdot 8^{\frac{r-s}{rs}}d(\lambda,\sigma)^{-m}\end{aligned}$$

Thus the resolvent satisfies the mth order rate of growth condition, and $N^m = 0$. Since $\mathfrak{X}$ is reflexive, this shows that Theorem 3.1 cannot be improved. Note that $\mathfrak{X}$ is also separable.

EXAMPLE 4.5. Let Σ be the interval $[0, 1]$ endowed with μ_1-measure, and let

$$\mathfrak{X} = L_\infty(\Sigma) \oplus \cdots \oplus L_\infty(\Sigma) \oplus L_1(\Sigma)$$

where m copies of L_∞ are taken. Let $T = S + N$ where S and N are defined in essentially the same way as in the previous examples. The resolvent function is given by

$$\begin{aligned} &R(\lambda, T)E(\sigma)[f_1(\xi) \oplus \cdots \oplus f_m(\xi) \oplus g(\xi)] \\ &\quad = \left[\frac{f_1(\xi)\chi_\sigma(\xi)}{\lambda - \xi} \oplus \cdots \oplus \left(\frac{f_m(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \cdots + \frac{f_1(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^m}\right)\right. \\ &\qquad \left. \oplus \left(\frac{g(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \frac{f_m(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^2} + \cdots + \frac{f_1(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^{m+1}}\right)\right]. \end{aligned}$$

Each of the terms is clearly of mth order rate of growth except for the L_1 norm of $f_1(\xi)(\lambda - \xi)^{-(m+1)}\chi_\sigma(\xi)$, and for this we have

$$\begin{aligned} \int_\sigma |f(\xi)(\lambda - \xi)^{-(m+1)}|\mu_1(d\xi) &\leq |f|_{L_\infty} \int_\sigma |\lambda - \xi|^{-(m+1)}\mu_1(d\xi) \\ &\leq |f|_{L_\infty} 2^{m+1}\pi d(\lambda, \sigma)^{-m} . \end{aligned}$$

Thus we have an example of an operator with spectrum in a rectifiable arc which satisfies the mth order rate of growth condition, but for which $N^m \neq 0$.

BIBLIOGRAPHY

1. N. Dunford, *Spectral operators*, Pacific J. Math. **4** (1954), 321-354.
2. ———, *A survey of the theory of spectral operators*, Bul. AMS **64** (1958), 217-274.
3. Hurewicz and Wallman, *Dimension theory*, Princeton University press, 1948.

YALE UNIVERSITY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 9, No. 4 August, 1959