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THE NILPOTENT PART OF A SPECTRAL OPERATOR

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1. Introduction. Throughout this paper, \mathfrak{X} is a Banach space, T a bounded spectral operator on \mathfrak{X} with scalar part S, nilpotent part N, and resolution of the identity $E(\sigma)$ for σ a Borel set in the complex plane. M is the bound for the norms of the $E(\sigma)$; $|E(\sigma)| \leq M$ for all Borel sets σ . The resolvent function for T, $(\lambda - T)^{-1}$, is denoted by $R(\lambda, T)$. The operator $R(\lambda, T)E(\sigma)$ has an unique analytic extension from the resolvent set of T to the complement of $\overline{\sigma}$, and on the subspace $E(\sigma)\mathfrak{X}$ it is equal to the operator $R(\lambda, T_{\sigma})$ where T_{σ} is the restriction of T to $E(\sigma)\mathfrak{X}$. For material on spectral operators, we refer to the papers on N. Dunford [1], [2]. $\chi_{\sigma}(\xi)$ is the characteristic function of the Borel set $\sigma: \chi_{\sigma}(\xi) = 1$ if $\xi \in \sigma, \chi_{\sigma}(\xi) = 0$ if $\xi \notin \sigma$. For p a nonnegative real number, μ_p is Hausdorff p-dimensional measure [3, pp. 102 ff.]; μ_2 is Lebesgue planar measure multiplied by $\pi/4$, and μ_1 restricted to an arc is majorized by arc length.

We assume throughout that there is an integer m for which the resolvent function for T satisfies the mth order rate of growth condition

$$|R(\lambda, T)E(\sigma)| \leq K \cdot d(\lambda, \sigma)^{-m}, \lambda \notin \overline{\sigma}, |\lambda| \leq |T| + 1$$

where $d(\lambda, \sigma)$ is the distance from λ to σ and K is a constant independent of σ . If \mathfrak{X} is Hilbert space, it is known that this growth condition implies $N^m = 0$ [1, p. 337]. In an arbitrary Banach space, this is no longer true; the best that can be done is $N^{m+2} = 0$. If \mathfrak{X} is weakly complete, $N^{m+1}=0$; or if σ is a set of μ_2 measure zero, $N^{m+1}E(\sigma) = 0$. If σ lies in an arc and either \mathfrak{X} is weakly complete or σ has μ_1 measure zero, then $N^m E(\sigma) = 0$. Examples show that we cannot obtain lower indices of nilpotency in general.

2. The fundamental lemma and some easy consequences. If $f(\xi)$ is a bounded, scalar valued Borel function, the operator $\int f(\xi)E(d\xi)$ exists as a bounded operator with norm at most $4M \cdot \sup_{\xi} |f(\xi)|$ [1, p. 341], so that uniform convergence of a sequence of bounded Borel functions $f_n(\xi)$ implies convergence in the uniform operator topology of the operators $\int f_n(\xi)E(d\xi)$. Thus for a given bounded Borel function $f(\xi)$ and a given positive number η , there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

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$$\left|\int f(\xi)E(d\xi) - \sum_i f(\xi_i)E(\sigma_i)\right| < \eta \; .$$

Similarly if A_n are a finite number of bounded operators and $f_n(\xi)$ are bounded Borel functions, for any positive number η , there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

$$\left|\sum_{n}\int A_{n}f_{n}(\xi)E(d\xi)-\sum_{i}\sum_{n}A_{n}f_{n}(\xi_{i})E(\sigma_{i})\right|<\eta;$$

in particular, for an integer k and a positive number η , there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

$$\left|\int (T-\xi)^{k} E(d\xi) - \sum_{i} (T-\xi_{i})^{k} E(\sigma_{i})\right| < \eta.$$

LEMMA 2.1. There exist constants M_k such that $|N^k E(\sigma)| \leq M_k \in {}^{k+1-m}$ for any choice of ε , $0 < \varepsilon \leq 1$, and Borel set σ of diameter no greater than ε .

Proof. Pick ε , $0 < \varepsilon \leq 1$, and let σ be any Borel set of diameter no greater than ε . We have [1, p, 338]

$$N^k E(\sigma) = \int_{\sigma} (T-\xi)^k E(d\xi) \; .$$

For any positive number η , there is a decomposition of σ into a finite number of disjoint Borel sets $\sigma_i \subset \sigma$, and points $\xi_i \in \sigma_i$ such that

$$\left|\int (T-\xi)^k E(d\xi) - \sum_i (T-\xi_i)^k E(\sigma_i)\right| < \eta \;.$$

Since σ is of diameter at most ε , there is a circle Γ of diameter 3ε which encloses σ and for which $|\gamma - \xi| \ge \varepsilon$ for all $\gamma \in \Gamma$ and $\xi \in \sigma$. Then

$$(T-\xi_i)^k E(\sigma_i) = \frac{1}{2\pi i} \int_{\Gamma} (\gamma-\xi)^k R(\gamma,T) E(\sigma_i) d\gamma ,$$

so that

$$\sum_{i} (T - \xi_i)^k E(\sigma_i) = \frac{1}{2\pi i} \int_{\Gamma} R(\gamma, T) \sum_{i} (\gamma - \xi_i)^k E(\sigma_i) d\gamma ,$$

which in norm in no greater than

(*)
$$\frac{1}{2\pi} \cdot \sup_{\gamma \in F} |R(\gamma, T)E(\sigma)| \cdot \sup_{\gamma \in F} |\sum_{i} (\gamma - \xi_i)^k E(\sigma_i)| \cdot \text{ length of } \Gamma.$$

The mth order rate of growth condition gives

$$\sup_{\gamma \in F} |R(\gamma, T)E(\sigma)| \leq K \varepsilon^{-m}$$

For any $\gamma \in \Gamma$,

$$|\sum_i (\gamma-\xi_i)^k E(\sigma_i)| \leq 4M \cdot \max_i |\gamma-\xi_i|^k \leq 4M (2arepsilon)^k$$
 ,

so that (*) is no greater than

$$rac{1}{2\pi} K arepsilon^{-m} ullet 4 M(2arepsilon)^k ullet 6 \pi arepsilon = M_k arepsilon^{k+1-m}$$
 ,

where $M_k = 3 \cdot 2^{k+2} K M$, and is independent of $\eta, \varepsilon, \sigma$, and the manner in which σ is decomposed. Thus

$$|N^k E(\sigma)| \le M_k \varepsilon^{k+1-m} + \eta$$

for every positive η , which proves the lemma.

THEOREM 2.2. Let σ be a Borel set whose Hausdorff p-measure is zero for a given p. Then $N^{k}E(\sigma) = 0$ where k is an integer and $k \ge p + m - 1$.

Proof. Since σ has *p*-measure zero, for every $\varepsilon > 0$, there is a covering of σ by disjoint sets σ_i of diameter ε_i such that $\sum_i \varepsilon_i^p < \varepsilon$. By Lemma 2.1 we have

$$egin{aligned} |N^k E(\sigma)| &\leq \sum_i |N^k E(\sigma_i)| \leq M_k \sum_i arepsilon_i^{k+1-m} \ &\leq M_k \sum_i arepsilon_i^{(p+m-1)+1-m} \leq M_k \sum_i arepsilon_i^p \leq M_k arepsilon \ . \end{aligned}$$

Since ε may be chosen arbitrarily small, $N^{k}E(\sigma) = 0$.

COROLLARY 2.3. $N^{m+2} = 0$.

Proof. Taking σ to be the spectrum of T and p = 3, $N^{m+2}E(\sigma(T)) = 0$; but $E(\sigma(T))$ is the identity mapping on \mathfrak{X} .

COROLLARY 2.4. If σ has planar measure zero, then $N^{m+1}E(\sigma) = 0$. COROLLARY 2.5. If σ has μ_1 -measure zero, then $N^m E(\sigma) = 0$.

3. The case of weakly complete \mathfrak{X} . Let σ be a Borel set in the plane. For any $\varepsilon > 0$, we can cover σ with disjoint Borel sets. σ_i of diameter $\varepsilon_i, \varepsilon_i \leq 1$, such that

$$\sum_i arepsilon_i^2 \leq \mu_2(\sigma) + arepsilon$$
 .

Thus by Lemma 2.1,

$$egin{aligned} |N^{m+1}\!E(\sigma)| &\leq \sum_i |N^{m+1}\!E(\sigma_i)| \leq M_{m+1}\sum_i arepsilon_i^2 \ &\leq M_{m+1}(\mu_2(\sigma)+arepsilon) \;. \end{aligned}$$

Since ε and σ are arbitrary, we have for all Borel sets σ ,

$$|N^{m+1}E(\sigma)| \leq M_{m+1}\mu_2(\sigma)\;.$$

As a consequence, all the scalar measures $x^*N^{m+1}E(\cdot)x = [(N^*)^{m+1}E^*(\cdot)x^*]x$, $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, are absolutely continuous with respect to μ_2 , and have derivative bounded by $M_{m+1}|x^*||x|$.

Suppose that $f(\xi) = \sum_{p=1}^{P} \alpha_p \chi_{\sigma_p}(\xi)$ is a simple Borel function; α_p are scalar constants and σ_p are disjoint Borel sets. We have

$$\begin{split} \left| \int f(\xi)(N^*)^{m+1} E^*(d\xi) \right| &\leq \sum_{p=1}^{P} |\alpha_p(N^*)^{m+1} E^*(\sigma_p)| \\ &\leq \sum_{p=1}^{P} |\alpha_p| M_{m+1} \mu_2(\sigma_p) \\ &= M_{m+1} |f|_{L_1(\mu_2)} \,. \end{split}$$

Thus if $f_n(\xi)$ are simple Borel functions converging in $L_1(\mu_2)$ to $f(\xi)$, the operators $\int f_n(\xi)(N^*)^{m+1}E^*(d\xi)$ converge in the uniform operator topology to an operator which we denote by $\int f(\xi)(N^*)^{m+1}E^*(d\xi)$; this limit operator has norm bounded by $M_{m+1} | f | L_1(\mu_2)$.

THEOREM 3.1. If \mathfrak{X} is weakly complete, then $N^{m+1} = 0$.

Proof. Assume that $N^{m+1} \neq 0$, so that also $(N^*)^{m+1} \neq 0$. We will first obtain a bicontinuous map of an infinite dimensional L_1 space into \mathfrak{X}^* . An analogous map into \mathfrak{X} would show then that \mathfrak{X} cannot be reflexive, since the image in \mathfrak{X} of this L_1 space would be a closed non-reflexive subspace of \mathfrak{X} ; however, the map into \mathfrak{X}^* is needed for the slightly more general case of \mathfrak{X} weakly complete.

Let the Borel set $\sigma, x_0 \in \mathfrak{X}$, and $x_0^* \in \mathfrak{X}^*$ be chosen so that $[(N^*)^{m+1}E^*(\sigma)x_0^*]x_0 \neq 0$, and let the derivative of the measure $[(N^*)^{m+1}E^*(\cdot)x_0^*]x_0$ be denoted by $g(\xi)$. We can then find a subset τ of σ and a constant a > 0 such that $\mu_2(\tau) > 0$ and $|g(\xi)| \geq a$ on τ .

Define the map Φ of $L_1(\tau, \mu_2)$ into \mathfrak{X}^* by

$$arPhi(f) = \int_{\tau} f(\xi) (N^*)^{m+1} E^*(d\xi) x_0^* \; .$$

 Φ is a linear map with bound $M_{m+1}|x_0^*|$. Now take

$$x = \int_{\tau} [g(\xi)]^{-1} \operatorname{sgn} \overline{f(\xi)} E(d\xi) x_0;$$

The norm of x is no greater than $4M \cdot a^{-1} \cdot |x_0|$. But we have

$$\begin{split} [\varPhi(f)](x) &= \int_{\pi} f(\xi) [g(\xi)]^{-1} \operatorname{sgn} \overline{f(\xi)} [(N^*)^{m+1} E^* (d\xi) x_0^*] x_0 \\ &= \int_{\pi} |f(\xi)| [g(\xi)]^{-1} g(\xi) \mu_2 (d\xi) \\ &= |f|_{L_1}, \end{split}$$

which shows that

$$|arPsi_{(f)}| \geq |f|_{{}_{L_1}} \!\cdot a \cdot (4M \,|\, x_{\scriptscriptstyle 0}|)^{_{-1}}$$
 ,

so that Φ is one-to-one and has a continuous inverse.

Now let Ψ be the map of $L_{\infty}(\tau, \mu_2)$ into \mathfrak{X} :

$$arPsi(h)=\int_{ au}[g(\xi)]^{-1}h(\xi)E(d\xi)x_{\scriptscriptstyle 0}$$
 ,

 Ψ is a continuous map with bound no greater than $4M \cdot a^{-1}|x_0|$; we will show that Ψ is one-to-one and bicontinuous. We have

$$egin{aligned} &arPsi_{1} arPsi_{2}(f) arPsi_{2}(h) &= \int_{ au} f(\xi) [g(\xi)]^{-1} h(\xi) [(N^{*})^{m+1} E^{*}(d\xi) x_{0}^{*}] x_{0} \ &= \int_{ au} f(\xi) h(\xi) \mu_{2}(d\xi) \;, \end{aligned}$$

so that

$$\sup_{\|f\|_{L_1^{\leq 1}}} | arPhi(f) arPhi(h) | = \sup_{\|f\|_{L_1^{\leq 1}}} \left| \int_{\pi} f(\xi) h(\xi) \mu_2(d\xi)
ight|$$

= $|h|_{L_{\infty}}$.

But since ϕ is bounded,

$$\begin{split} \sup_{\|f\|_{L_1^{\leq 1}}} |arPsi(f) arPsi(h)| &\leq \sup_{\substack{x^* \in X^* \ \|x^*\| \leq |\Phi|}} |x^* arPsi(h)| \ &= |arPsi| |arPsi(h)| \ , \end{split}$$

so that

$$\|h\|_{L_{\infty}} \leq \|\varphi\| \|\Psi(h)\|;$$

thus Ψ is one-to-one and bicontinuous. The range \mathfrak{Y} of Ψ in \mathfrak{X} is then a closed non weakly complete subspace of \mathfrak{X} . But this is impossible, because every closed subspace of a weakly complete Banach space is again weakly complete; the proof of this last remark is as follows.

Let \mathfrak{X} be a weakly complete Banach space, \mathfrak{Y} a closed subspace. Let y_n be a weakly Cauchy sequence in \mathfrak{Y} , so that y^*y_n is a Cauchy sequence of numbers for every y^* in Y^* . Since any x^* in X^* , when restricted to \mathfrak{Y} , is an element of \mathfrak{Y}^*, x^*y_n is a Cauchy sequence of numbers for every x^* in \mathfrak{X}^* . Since \mathfrak{X} is weakly complete, there is an x_0 in \mathfrak{X} such that $\lim_{n\to\infty} x^*y_n = x^*x_0$ for every x^* in \mathfrak{X}^* ; and since \mathfrak{Y} is strongly closed in \mathfrak{X} , it is weakly closed, so that x_0 must lie in \mathfrak{Y} . Finally since every y^* in \mathfrak{Y}^* is, by the Hahn-Banach theorem, the restriction of an x^* in \mathfrak{X}^* , $\lim y^*y_n = y^*x_0$ for every y^* in \mathfrak{Y}^* , so that \mathfrak{Y} is weakly complete.

THEOREM 3.2. If \mathfrak{X} is weakly complete, then $N^m E(\sigma) = 0$ for every set σ of finite μ_1 -measure.

Proof. Follow exactly the same discussion above, replacing the number m + 1 by m and the measure μ_2 by μ_1 .

Note that Theorems 3.1 and 3.2 also hold if \mathfrak{X} is assume to be separable instead of weakly complete, for the image of the L_{∞} space in \mathfrak{X} would be a nonseparable closed subspace of \mathfrak{X} ; but every closed subspace of a separable space is again separable.

4. Examples. In the following examples we will need two computational lemmas.

Lemma 4.1. For each real number $p \ge 1$ and Borel set σ ,

$$\int_{\tau} |\lambda - \xi|^{-(p+2)} \mu_2(d\xi) \le 8d(\lambda, \sigma)^{-p}, \text{ for all } \lambda \notin \bar{\sigma}.$$

Proof.

$$\begin{split} &\int_{\sigma} |\lambda - \xi|^{-(p+2)} \mu_2(d\xi) \\ &\leq \int_{|\lambda - \xi| \ge d(\lambda, \sigma)} |\lambda - \xi|^{-(p+2)} \mu_2(d\xi) \\ &= \frac{4}{\pi} \int_{0}^{2\pi} d\theta \int_{a(\lambda, \sigma)}^{\infty} r^{-(p+2)} r \, dr \qquad (\lambda - \xi = r e^{i\beta}) \\ &\leq 8 d(\lambda, \sigma)^{-p} \, . \end{split}$$

LEMMA 4.2. For each real number $p \ge 1$ and Borel subset σ of the real line,

$$\int_{\sigma} |\lambda - \xi|^{-(p+1)} \mu_1(d\xi) \leq 2^{p+1} \pi d(\lambda, \sigma)^{-p}$$
 ,

where μ_1 is Lebesgue measure along the line, and λ is any complex number, $\lambda \notin \overline{\sigma}$.

Proof. Let $\lambda = \alpha + i\beta$, α , β real. Then either, (i), $d(\alpha, \sigma) \ge d(\lambda, \sigma)/2$ or, (ii) $|\beta| \ge d(\lambda, \sigma)/2$. In case (i) we have

$$\begin{split} \int_{\sigma} |\lambda - \xi|^{-(p+1)} \mu_{\mathbf{i}}(d\xi) &\leq \int_{a(\lambda,\sigma)/2}^{\infty} \eta^{-(p+1)} d\eta \qquad (\lambda - \xi = \eta) \\ &\leq 2^{p+1} p^{-1} d(\lambda, \sigma)^{-p} \; . \end{split}$$

In case (ii) we have

$$\begin{split} \int_{\sigma} |\lambda - \xi|^{-(p+1)} \mu_1(d\xi) &\leq \int_{-\infty}^{\infty} |\xi - i\beta|^{-(p+1)} d\xi \\ &\leq \int_{-\infty}^{\infty} (\xi^2 + \beta^2)^{-\frac{1}{2}(p+1)} d\xi \\ &\leq 2^{p+1} \pi d(\lambda, \sigma)^{-p} \;. \end{split}$$

EXAMPLE 4.3. Let Σ be a disc in the plane with μ_2 -measure 1. Let

$$x = L_{\infty}(\Sigma) \oplus L_2(\Sigma) \oplus \cdots \oplus L_2(\Sigma) \oplus L_1(\Sigma)$$
,

where m copies of $L_2(\Sigma)$ are taken. Let T be the operator S + N where S and N are defined as

$$\begin{split} S[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)] \\ &= [\xi f(\xi) \oplus \xi g_1(\xi) \oplus \cdots \oplus \xi g_m(\xi) \oplus \xi h(\xi)] ,\\ N[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)] \\ &= [0 \oplus f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi)] . \end{split}$$

Since Σ has measure 1, any function in L_r is in L_s for all $s \leq r$, and the L_s norm is no greater than the L_r norm; thus N is a bounded operator with norm 1. Also N is a nilpotent for which $N^{m+1} \neq 0$. The operator T is a spectral operator with resolution of the identity

$$E(\sigma)[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)] \\= [f(\xi)\chi_{\sigma}(\xi) \oplus g_1(\xi)\chi_{\sigma}(\xi) \oplus \cdots \oplus g_m(\xi)\chi_{\sigma}(\xi) \oplus h(\xi)\chi_{\sigma}(\xi)].$$

The resolvent function is

$$R(\lambda, T)E(\sigma)[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)] = \left[\frac{f(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} \oplus \left(\frac{g_1(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} + \frac{f(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^2}\right) \oplus \cdots \oplus \left(\frac{g_m(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} + \cdots + \frac{g_1(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^m} + \frac{f(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^{m+1}}\right) \oplus \left(\frac{h(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} + \frac{g_m(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)} + \cdots + \frac{g_1(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^{m+1}} + \frac{f(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^{m+2}}\right)\right].$$

All the terms are clearly of mth order rate of growth except possibly for

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(a)
$$\left| \frac{f(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^{m+1}} \right|_{L_2}$$
, (b) $\left| \frac{f(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^{m+2}} \right|_{L_1}$, and (c) $\left| \frac{g_1(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^{m+1}} \right|_{L_1}$

For (a) we have

$$egin{aligned} &\left\{ \int_{\sigma} |f(\xi)(\lambda-\xi)^{-(m+1)}|^2 \mu_2(d\xi)
ight\}^{1/2} &\leq |f|_{L_{\infty}} igg\{ \int_{\sigma} \lambda-\xi|^{-2m-2} \mu_2(d\xi) igg\}^{1/2} \ &\leq |f|_{L_{\infty}} \sqrt{8} \, d(\lambda,\,\sigma)^{-m} \,\,, \end{aligned}$$

for (b) we have

$$egin{aligned} &\int_{\sigma} |f(\xi)(\lambda-\xi)^{-(m+2)} \,|\, \mu_2(d\xi) \leq |f|_{L_{\infty}} \int_{\sigma} |\lambda-\xi|^{-(m+2)} \mu_2(d\xi) \ &\leq |f|_{L_{\infty}} \cdot 8d(\lambda,\sigma)^{-m} \;, \end{aligned}$$

and for (c) we have

$$\begin{split} \int_{\sigma} &|g_1(\xi)(\lambda-\xi)^{-(m+1)}|\,\mu_2(d\xi) \leq \left\{ \int_{\sigma} &|g_1(\xi)|^2 \mu_2(d\xi) \right\}^{1/2} \left\{ \int_{\sigma} &|\lambda-\xi|^{-2m-2} \mu_2(d\xi) \right\}^{1/2} \\ &\leq &|g_1|_{L_2} \cdot \sqrt{8} \cdot d(\lambda,\sigma)^{-m} \;. \end{split}$$

Thus each term of the resolvent, and hence the resolvent itself satisfies the mth order rate of growth condition; this shows that Corollary 2.3 cannot be improved.

EXAMPLE 4.4. Let Σ be as in the previous example and let

$$\mathfrak{X} = L_r(\Sigma) \bigoplus \cdots \bigoplus L_r(\Sigma) \bigoplus L_s(\Sigma)$$

where *m* copies of L_r are taken. *r* and *s* are to satisfy $1 < s < r < \infty$ and $rs \leq 2(r-s)$. Let T = S + N, where *S* and *N* are defined in essentially the same way as in the previous example. The resolvent function is given by

$$R(\lambda, T)E(\sigma)[f_1(\xi) \oplus \cdots \oplus f_m(\xi) \oplus g(\xi)] = \left[\frac{f_1(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} \oplus \cdots \oplus \left(\frac{f_m(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} + \cdots + \frac{f_1(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^m}\right) \\ \oplus \left(\frac{g(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} + \frac{f_m(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^2} + \cdots + \frac{f_1(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^{m+1}}\right)\right].$$

Each of the terms is clearly of *m*th order rate of growth except possibly for the L_s norm of $f_1(\xi)(\lambda - \xi)^{-(m+1)}\chi_{\sigma}(\xi)$, and for this we have

$$\begin{split} \left\{ \int_{\sigma} |f_{1}(\xi)(\lambda - \xi)^{m+1}|^{s} \mu_{2}(d\xi) \right\}^{1/s} \\ &\leq \left\{ \int_{\sigma} |f_{1}(\xi)|^{r} \mu_{2}(d\xi) \right\}^{1/r} \left\{ \int |\lambda - \xi|^{-\frac{(m+1)rs}{r-s}} \mu_{2}(d\xi) \right\}^{\frac{r-s}{rs}} \\ &\leq |f_{1}|_{L_{r}} \cdot 8^{\frac{s-r}{rs}} \cdot d(\lambda, \sigma)^{-m - \left(1 - \frac{2(r-s)}{rs}\right)} \\ &\leq |f_{1}|_{L_{r}} \cdot 8^{\frac{r-s}{rs}} d(\lambda, \sigma)^{-m} \end{split}$$

Thus the resolvent satisfies the *m*th order rate of growth condition, and $N^m = 0$. Since \mathfrak{X} is reflexive, this shows that Theorem 3.1 cannot be improved. Note that \mathfrak{X} is also separable.

EXAMPLE 4.5. Let Σ be the interval [0, 1] endowed with μ_1 -measure, and let

$$\mathfrak{X} = L_{\infty}(\Sigma) \bigoplus \cdots \bigoplus L_{\infty}(\Sigma) \bigoplus L_{1}(\Sigma)$$

where m copies of L_{∞} are taken. Let T = S + N where S and N are defined in essentially the same way as in the previous examples. The resolvent function is given by

$$R(\lambda, T)E(\sigma)[f_1(\xi) \oplus \cdots \oplus f_m(\xi) \oplus g(\xi)] = \left[\frac{f_1(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} \oplus \cdots \oplus \left(\frac{f_m(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} + \cdots + \frac{f_1(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^m}\right) \\ \oplus \left(\frac{g(\xi)\chi_{\sigma}(\xi)}{\lambda - \xi} + \frac{f_m(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^2} + \cdots + \frac{f_1(\xi)\chi_{\sigma}(\xi)}{(\lambda - \xi)^{m+1}}\right)\right].$$

Each of the terms is clearly of *m*th order rate of growth except for the L_1 norm of $f_1(\xi)(\lambda - \xi)^{-(m+1)}\chi_{\sigma}(\xi)$, and for this we have

$$\begin{split} \int_{\sigma} &|f(\xi)(\lambda-\xi)^{-(m+1)}|\,\mu_{\scriptscriptstyle 1}(d\xi) \leq |f|_{L_{\infty}} \int_{\sigma} |\lambda-\xi|^{-(m+1)} \mu_{\scriptscriptstyle 1}(d\xi) \\ &\leq |f|_{L_{\infty}} 2^{m+1} \pi d(\lambda,\sigma)^{-m} \;. \end{split}$$

Thus we have an example of an operator with spectrum in a rectifiable arc which satisfies the *m*th order rate of growth condition, but for which $N^m \neq 0$.

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