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ON A CRITERION FOR THE WEAKNESS OF AN IDEAL BOUNDARY COMPONENT

KOTARO OIKAWA

## ON A CRITERION FOR THE WEAKNESS OF AN IDEAL BOUNDARY COMPONENT

#### Kôtaro Oikawa

1. Exhaustion. Let F be an open Riemann surface. An exhaustion  $\{F_n\}$  of F is an increasing (i.e.,  $\overline{F_n} \subset F_{n+1}$ ) sequence of subregions with compact closures such that  $\bigcup_{n=1}^{\infty} F_n = F$ . We assume that  $\partial F_n$  consists of a finite number of closed analytic curves and that each component of  $F - F_n$  is noncompact. This is the most common definition used in the theory of open Riemann surfaces. Sometimes, however, we shall add the restriction that each component of  $\partial F_n$  is a dividing cycle; if this is the case we shall call the exhaustion canonical.

2. Weak boundary component. Let  $\gamma$  be an ideal boundary component of F, and let  $\{F_n\}$  be a canonical exhaustion of F. Then there exists a component  $\gamma_n$  of  $\partial F_n$  which separates  $\gamma$  from  $F_n$ . Let  $n_0$  be a fixed number and consider the component  $G_n$  of  $\overline{F}_n - F_{n_0}$   $(n > n_0)$  such that  $\gamma_n \subset \partial G_n$ . There exists a harmonic function  $s_n(p)$  on  $\overline{G}_n$  which satisfies the following conditions:

(i) 
$$s_n = 0$$
 on  $\gamma_{n_0}$  and  $\int_{\gamma_{n_0}} *ds_n = 2\pi$ ,  $(\gamma_{n_0} = \partial F_{n_0} \cap \partial G_n)$   
(ii)  $s_n = \log r_n = \text{const. on } \gamma_n$ ,  
(iii)  $s_n = \text{const. on each component } \beta_{n\nu}$  of  $\partial G_n - \gamma_n - \gamma_{n_0}$  and  $\int_{\beta_{n\nu}} *ds_n = 0$ .

The condition  $\lim_{n\to\infty} r_n = \infty$  depends neither on  $n_0$  nor on the exhaustion. If it is satisfied,  $\gamma$  is said to be *weak*.

Weak boundary components were introduced for plane regions by Grötzch [1] in connection with the so-called Kreisnormierungsproblem. He called them vollkommen punktförmig. They were generalized for open Riemann surfaces by Sario [6] and discussed also by Savage [7] and Jurchescu [2]. The above definition was given by Jurchescu [2].

A noncompact subregion N whose relative boundary  $\partial N$  consists of a finite number of closed analytic curves is called a *neighborhood of*  $\gamma$  if  $\gamma$  is an ideal boundary component of N as well. Let  $\{c\}$  be the family of all cycles c (i.e., unions of finite numbers of closed curves) which are in N and separate  $\gamma$  from  $\partial N$ . Jurchescu [2] showed that  $\lambda\{c\} = 0$  if and only if  $\gamma$  is weak, where  $\lambda\{c\}$  is the extremal length of the family  $\{c\}$ .

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3. Savage's criterion. Let  $\{F_n\}$  be an arbitrary exhaustion. Let  $E_n$  be the smallest union of components of  $F_n - \overline{F}_{n-1}$  such that  $\gamma_{n-1} = \partial E_n \cap \partial F_{n-1}$  is a cycle which separates  $\gamma$  from  $F_{n-1}$   $(n = 2, 3, \dots)$ . Evidently  $\gamma_n \subset \partial E_n$ . If  $\{F_n\}$  is canonical,  $E_n$  is connected and  $\gamma_n$  is a closed analytic curve.

There exists a harmonic function  $u_n(p)$  on  $\overline{E}_n$  such that

(i) 
$$u_n = 0$$
 on  $\gamma_{n-1}$  and  $\int_{\gamma_{n-1}} *du_n = 2\pi$ ,

(ii)  $u_n = \log \mu_n = \text{const.}$  on  $\partial E_n - \gamma_{n-1} = \partial E_n \cap \partial F_n$ .

The quantity log  $\mu_n$  is called the *modulus of*  $E_n$  (cf. Sario [4,5], who called  $\mu_n$  the modulus). It is expressed in terms of extremal length as follows:

$$\log \mu_n = \frac{2\pi}{\lambda\{c\}_n}$$

where  $\{c\}_n$  is the family of cycles in  $E_n$  homologous to  $\gamma_{n-1}$ .

Since  $\sum_{n=1}^{\infty} 1/\lambda \{c\}_n \leq 1/\lambda \{c\}$ , we get the following criterion:

THEOREM 1 (Savage [7]). If there exists an exhaustion such that  $\prod_{n=2}^{\infty} \mu_n = \infty$ , then  $\gamma$  is weak.

The purpose of the present note is to discuss the converse of this theorem.

4. Jurchescu's criterion. Suppose the exhaustion  $\{F_n\}$  is canonical. There exists a harmonic function  $U_n(p)$  on  $\overline{E}_n$  such that

- (i)  $U_n = 0$  on  $\gamma_{n-1}$  and  $\int_{\gamma_{n-1}} * dU_n = 2\pi$ ,
- (ii)  $U_n = \log M_n = \text{const. on } \gamma_n$ ,
- (iii)  $U_n = \text{const.}$  on each component  $\beta_{n\nu}$  of  $\partial E_n \gamma_n \gamma_{n-1}$  and  $\int_{\beta_{n\nu}} *dU_n = 0.$

Jurchesch's paper [2] contains implicitly the following result:

THEOREM 2 (Jurchescu). A boundary component  $\gamma$  is weak if and only if there exists a canonical exhaustion such that  $\prod_{n=2}^{\infty} M_n = \infty$ .

*Proof.* Sufficiency: Let  $\{c'\}_n$  be the family of cycles in  $E_n$  separating  $\gamma_n$  from  $\gamma_{n-1}$ . It is not difficult to see that  $\log M_n = 2\pi/\lambda \{c'\}_n$ . Since  $\sum_{n=2}^{\infty} 1/\lambda \{c'\}_n \leq 1/\lambda \{c\}$ , we conclude that  $\sum_{n=2}^{\infty} \log M_n = \infty$  implies  $\lambda \{c\} = 0$ .

*Necessity*: Consider a canonical exhaustion  $\{F_n^0\}$ . The desired exhaustion  $\{F_n\}$  is obtained by taking its subsequence as follows:

 $F_1 = F_1^0$ . To define  $F_2$ , consider the quantity  $r_n$  introduced in No. 2 with respect to  $F_n^0 - \overline{F}_1^0$   $(n = 2, 3, \dots)$ . Take  $n_2$  so large that  $r_{n_2} \ge 2$ ,

and put  $F_2 = F_{n_2}^0$ . Evidently  $M_2 = r_{n_2}$ . Similarly,  $F_3 = F_{n_3}^0$  is defined by considering  $F_n^0 - \overline{F}_{n_2}^0$   $(n = n_2 + 1, n_2 + 2, \cdots)$  and by taking  $n_3 > n_2$ so large that  $r_{n_3} \ge 2$  where  $r_{n_2}$  is the quantity  $r_n$  introduced in No. 2 with respect to  $F_n^0 - \overline{F_{n_2}^0}$ . We have  $M_3 = r_{n_3}$ . On continuing this process, we obtain a canonical exhaustion such that  $\sum_{n=2}^{\infty} \log M_n \ge \sum_{n=2}^{\infty} \log 2 = \infty$ . The idea of this proof was first used by Noshiro [3].

5. The converse of Savage's criterion. We shall now show that Savage's criterion in Theorem 1 is also necessary.

THEOREM 3. If  $\gamma$  is weak, then there exists an exhaustion such that  $\prod_{n=2}^{\infty} \mu_n = \infty$ . It is not necessarily canonical.

*Proof.* By Theorem 2 there exists a canonical exhaustion  $\{F_n^0\}$  such that  $\prod_{n=2}^{\infty} M_n^0 = \infty$ . From this we construct a canonical exhaustion  $\{F_n^*\}$  as follows:

 $F_1^* = F_1^0$ . To construct  $F_2^*$ , let  $\partial E_2^0 - \gamma_1^0 - \gamma_2^0 = \beta_{21} \cup \beta_{22} \cup \cdots \cup \beta_{2k_2}$ be the decomposition into components, and let  $H_3^{\nu}$  be the component of  $F_3^0 - F_2^0$  such that  $\partial H_3^{\nu} \cap \overline{F}_2^0 = \beta_{2\nu}$  ( $\nu = 1, 2, \dots, k_2$ ).  $F_2^*$  is the union of  $F_1^*, E_2^0 \cup \gamma_1^0$ , all the other components of  $F_2^0 - F_1^0$ , and  $\bigcup_{\nu=1}^{k_2} H_3^{\nu}$ . In this way,  $F_n^*$  is defined as the union of  $F_{n-1}^*, E_n^0 \cup \gamma_{n-1}^0$ , every component of  $F_{m+1}^0 - F_m^0$  ( $m \ge n$ ) which is adjacent to  $F_{n-1}^*$ , and  $\bigcup_{\nu=1}^{k_n} H_{n+1}^{\nu}$ . By construction,  $E_n^* = E_n^0 \cup \bigcup_{\nu=1}^{k_n} H_{n+1}^{\nu}$ .

The desired exhaustion  $\{F_n\}$  is obtained by taking a refinement of  $\{F_n^*\}$  as follows: Consider  $E_n^0$  and the function  $U_n^0$  for the exhaustion  $\{F_n^*\}$ . Let  $\partial E_n^0 - \gamma_n^0 - \gamma_{n-1}^0 = \beta_{n1} \cup \beta_{n2} \cup \cdots \cup \beta_{nk_n}$  be the decomposition into components and let  $U_n^0 \equiv a_{\nu}$  on  $\beta_{n\nu}$  ( $\nu = 1, 2, \dots, k_n$ ). We may assume, without loss of generality, that the  $a_{\nu}$ 's are different by pairs. We suppose that

$$0 \equiv a_{\scriptscriptstyle 0} < a_{\scriptscriptstyle 1} < \cdots < a_{\scriptscriptstyle k_{\scriptscriptstyle n}} < a_{\scriptscriptstyle k_{\scriptscriptstyle n}+1} \equiv \log M^{\scriptscriptstyle 0}_{\: n}.$$

Take  $a'_{\nu}(a_{\nu-1} < a'_{\nu} < a_{\nu}; \nu = 1, 2, \dots, k_n, a'_{k_n+1} \equiv \log M_n^{\circ})$  and  $a''_{\nu}(a_{\nu} < a''_{\nu} < a_{\nu+1}; \nu = 1, \dots, k_n, a''_{0} \equiv 0)$  so close to  $a_{\nu}$  that

(1) 
$$\sum_{\nu=1}^{k_n+1} (a_{\nu}'-a_{\nu-1}'') \ge \log M_n^0 - 2^{-n}$$

Consider the sets

$$egin{array}{l} D_n^
u = \{p\,; a_{
u-1}^{\prime\prime} < U_n^0(p) < a_
u^{\prime\prime}\}, \ 
u = 1,\,2,\,\cdots,\,k_n+1, \ (a_{k_n+1}^{\prime\prime} \equiv \log M_n^0) \ D_n^{\prime
u} = \{p\,; a_{
u-1}^{\prime\prime} < U_n^0(p) < a_
u^{\prime}\}, \ 
u = 1,\,2,\,\cdots,\,k_n+1 \ . \end{array}$$

The modulus  $\log \mu'^{(\nu)}$  of  $D_n'^{\nu}$  with respect to  $\beta^{\nu} = \{p ; U_n^0(p) = a_{\nu-1}''\}$  and  $\partial D_n'^{\nu} - \beta'$  is equal to  $a_{\nu}' - a_{\nu-1}''$ , since the function  $U_n^0(p) - a_{\nu-1}''$  plays the role of  $u_n(p)$  introduced in No. 3. Let  $\log \mu^{(\nu)}$  be the modulus of  $D_n''$ 

with respect to  $\beta^{\nu}$  and  $\partial D_n^{\nu} - \beta^{\nu}$ . Since  $\mu^{(\nu)} \ge \mu'^{(\nu)}$ , we obtain, by (1),

(2) 
$$\sum_{\nu=1}^{k_n+1} \log \mu^{(\nu)} \ge \log M_n^0 - 2^{-n} .$$

We have decomposed  $E_n^{\circ}$  into  $k_n + 1$  subsets  $D_n^{\vee}$ .  $E_n^* - E_n^{\circ}$  consists of components  $H_{n+1}^{\vee}$  such that  $\beta_{n\nu} = \partial H_{n+1}^{\vee} \cap \partial E_n^{\circ} (\nu = 1, 2, \dots, k_n)$ . By decomposing  $H_{n+1}^{\vee}$  into  $k_n - \nu + 1$  slices, we obtain a decomposition of  $E_n^*$  into  $k_n + 1$  parts. It is possible to divide each of the other components of  $F_n^* - \overline{F}_{n-1}^*$  into  $k_n + 1$  pieces so that we get an exhaustion  $\{F_n\}$  which is a refinement of  $\{F_n^*\}$ .  $D_n^{\vee}$  plays the role of  $E_n$  with respect to this exhaustion. Therefore, by (2), we get

$$\sum\limits_{n=2}^{\infty}\log\mu_n \geqq \sum\limits_{n=2}^{\infty}\log M^{\scriptscriptstyle 0}_n - 1 = \infty$$

6. Remark. On a "schlichtartig" surface, every exhaustion is canonical. If F is an arbitrary Riemann surface, the question arises whether or not Savage's criterion is still necessary under the restriction that  $\{F_n\}$  is canonical. The answer is given by

THEOREM 4. There exist a  $\gamma$  of an F which is weak and such that  $\prod_{n=2}^{\infty} \mu_n < \infty$  for every canonical exhaustion.

Construction of F: In the plane  $|z| < \infty$ , consider the closed intervals

$$I_k: [2^{k^2}, 2^{k^2} + 1] \qquad (k = 2, 3, \cdots)$$

on the positive real axis, and the circular arcs

$$lpha_
u: \, |\, z\,| = 
u, \, |rg\, z\,| \leq rac{\pi}{2}$$
 $(
u = 2^{k_2} + 2, \, 2^{k^2} + 3, \, \cdots, \, 2^{(k+1)^2} - 1\,; \, k = 2, \, 3, \, \cdots)$ 

Take two replicas of the slit plane  $(|z| < \infty) - \bigcup_{k=2}^{\infty} I_k$  and connect them crosswise across  $I_k$   $(k = 2, 3, \dots)$ . From the resulting surface, delete all the  $\alpha_{\nu}$ 's on both sheets. This is a Riemann surface F of infinite genus.

F has an ideal boundary component  $\gamma$  over  $z = \infty$ , which is evidently weak.

Let  $\{F_n\}$  be an arbitrary canonical exhaustion. Consider  $E_n$  corresponding to  $\gamma$  (No. 3). The interval  $I_k$  determines a closed analytic curve  $C_k$  on F. Since  $\gamma_{n-1} = \partial E_n \cap \overline{F}_{n-1}$  is a dividing cycle, the intersection number  $\gamma_{n-1} \times C_k$  vanishes and, therefore,  $\gamma_{n-1} \cap C_k$  consists of an even number of points whenever it is not void.\* Take two consecutive points

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<sup>\*</sup> Added in proof. We should have mentioned the case where  $\gamma_{n-1}$  tangents  $C_k$ . The following discussion covers this case if the number of the points of  $\gamma_{n-1} \cap C_k$  is counted with the multiplicity of tangency and case p=q is not excluded.

p and q in  $\gamma_{n-1} \cap C_k$ . There are two possibilities according as the arc  $\widehat{pq} \subset \gamma_{n-1}$  is homotopic to  $\widehat{pq} \subset C_k$  or not. If the latter case happens for at least one pair of p and q, we shall say that  $\gamma_{n-1}$  intersects  $C_k$  properly.

Since  $\gamma_{n-1}$  is a closed curve separating  $\gamma$  from  $F_{n-1}$ , there exists a number k such that  $\gamma_{n-1}$  intersects  $C_k$  properly. If there is more than one k, we take the greatest one and denote it by k(n).

To estimate  $\mu_n$ , let  $\{c\}_n$  be the family of all cycles in  $E_n$  separating  $\gamma_{n-1}$  from  $\partial E_n - \gamma_{n-1}$ . We have mentioned that  $\log \mu_n = 2\pi/\lambda \{c\}_n$ . Let  $C_k$  be a curve for which there are numbers n with k(n) = k. Evidently these n are finite in number and consecutive. Let  $n_k$  be the greatest.

I. If k(n) = k and  $n < n_k$  then  $\gamma_{n-1}$  and  $\gamma_n$  intersect  $C_k$  properly. Since every  $c \in \{c\}_n$  separates  $\gamma_{n-1}$  from  $\gamma_n$ , it has a component which intersects  $C_k$  and is not completely contained in the doubly connected region  $\Delta_k$  consisting of all points that lie over  $\{z; 2^{k^2} - 1 < |z| < 2^{k^2} + 2, |\arg z| < \pi/2\}$ . Therefore, every c contains a curve in  $\{c'\}^{(k)}$  which is the family of all curves in the right half-plane connecting  $I_k$  with the imaginary axis. Consequently

(3) 
$$\sum_{\substack{k(n)=k\\n\neq n_k}} \frac{1}{\lambda\{c\}_n} \leq \frac{1}{\lambda\{c'\}^{(k)}} .$$

II. k(n) = k and  $n = n_k$ . Consider all the  $\alpha_{\nu}$  ( $\nu \ge 2^{k^2} + 2$ ) on the upper sheet. Let  $G_{n-1}$  be the component of  $F - \overline{F}_{n-1}$  such that  $\partial G_{n-1} =$  $\gamma_{n-1}$ . For a sufficiently large  $\nu$ ,  $\alpha_{\nu}$  is an ideal boundary component of  $G_{n-1}$ . Let  $\nu(k)$  be the least  $\nu$  with this property. If  $\nu(k) = 2^{k^2} + 2$ , then every  $c \in \{c\}_n$  separates  $\gamma_{n-1}$  from  $\alpha_{\nu(k)}$  and, therefore, it has a component intersects either  $C_k$  or one of four line segments over  $[2^{k^2}-1, 2^{k^2}]$  or  $[2^{k^2}+1, 2^{k^2}+2]$ . When  $\nu(k) = 2^{i^2}+2$  for some l > k, then  $\gamma_{n-1}$  separates  $\alpha_{\nu(k)-3}$  from  $\alpha_{\nu(k)}$  and every  $c \in \{c\}_n$  separates  $\gamma_{n-1}$  from  $\alpha_{\nu(k)}$ , so that c has a component with the above property. If  $\nu(k)$  is not of the form  $2^{t^2} + 2$ , then, for the same reason, every  $c \in \{c\}_n$  has a component which intersects the line segment on the upper sheet lying over  $[\nu(k) - 1, \nu(k)]$ , and is not contained in the simply connected region on the upper sheet consisting of all points over  $\{z : \nu(k) - 1 < |z| < \nu(k),$  $|\arg z| < \pi/2$ . In any case, every  $c \in \{c\}_n$  contains a curve in  $\{c''\}^{(k)}$ which is the family of all curves in the right half-plane connecting  $[\nu(k) - 3, \nu(k)]$  with the imaginary axis. Therefore,

$$(4) \qquad \qquad \frac{1}{\lambda\{c\}_n} \leq \frac{1}{\lambda\{c''\}^{(k)}} .$$

By (3) and (4), we obtain

(5) 
$$\sum_{n=2}^{\infty} \log \mu_n = 2\pi \sum_{n=2}^{\infty} \frac{1}{\lambda\{c\}_n} \leq 2\pi \sum_{k=2}^{\infty} \left( \frac{1}{\lambda\{c'\}^{(k)}} + \frac{1}{\lambda\{c''\}^{(k)}} \right).$$

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To show the convergence of  $\sum_{k=2}^{\infty} 1/\lambda \{c'\}^{(k)}$ , we make use of the transformation  $z \to z^2$ . It is immediately seen that  $\lambda \{c'\}^{(k)}$  is equal to the extremal distance between  $[-\infty, 0]$  and  $I'_k = [2^{2k^2}, (2^{k^2} + 1)^2]$  with respect to the region  $A = \{[-\infty, 0] \cup I'_k\}^c$ . Since A is conformally equivalent to Teichmüller's extremal region  $\{[-1, 0] \cup [P, \infty]\}^c$  where

$$P=rac{2^{2k^2}}{(2^{k^2}+1)^2-2^{2k^2}}$$
 ,

we have (Teichmüller [8])

$$\lambda \{c'\}^{(k)} \sim \frac{\log P}{2\pi} \qquad (P \to \infty)$$
  
 $\sim \frac{k^2 \log 2}{2\pi} \qquad (k \to \infty) ,$ 

and, therefore,  $\sum_{k=2}^{\infty} 1/\lambda \{c'\}^{(k)} < \infty$ . Similarly  $\sum_{k=2}^{\infty} 1/\lambda \{c''\}^{(k)} < \infty$  because  $\nu(k) \ge 2^{k^2} + 2$ . We conclude that

$$\sum\limits_{n=2}^{\infty}\log\mu_n<\infty$$
 .

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