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AN ALGEBRAIC CRITERION FOR IMMERSION

BARRETT O'NEILL

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Let R be the curvature tensor of a simply connected d -dimensional ($d \geq 4$) Riemannian manifold M . T. Y. Thomas [2] has proved that if the rank of R is not too small, there exist conditions expressed in terms of polynomials in the coordinates of R which are satisfied if and only if M can be immersed in the Euclidean space R^{d+1} . The proof is existential; the polynomials are not all given explicitly. Using the notion of Grassmann algebra we shall find a single, rather simple condition on R necessary and sufficient for the existence of an immersion $i: M \rightarrow \bar{M}(K)$ with second fundamental form of rank at least four, where $\bar{M}(K)$ is a complete $(d + 1)$ -dimensional Riemannian manifold of constant curvature K . If coordinates are introduced this condition can be expressed algebraically in terms of polynomial equations and inequalities in the coordinates of R . The case $K = 0$ yields an explicit variant of Thomas' result.

1. A differential criterion for immersion. Following [1] we fix the following notation for the structural elements associated with a d -dimensional C^∞ Riemannian manifold M : $F(M)$, the bundle of frames on M ; R_a , right-multiplication of $F(M)$ by $a \in O(d)$, the group of $d \times d$ orthogonal matrices; φ , the 1-form of the Riemannian connection. Thus $\varphi = (\varphi_{ij})$ is a vertical equivariant 1-form on $F(M)$ with values in the Lie algebra of $d \times d$ skew-symmetric matrices. (We assume throughout that $1 \leq i, j, k \leq d$.) Let $\omega = (\omega_i)$ be the usual horizontal equivariant R^a -valued 1-form on $F(M)$ defined by $\omega_i(x) = \langle d\pi(x), f_i \rangle$, where x is in the tangent space $F(M)_f$ to $F(M)$ at $f = (f_1, \dots, f_d)$ and π is the natural projection. The curvature form $\Phi = (\Phi_{ij})$ is by definition $D\varphi$, the horizontal part of $d\varphi$. In the case of 1-forms or 1-vectors we write xy , rather than $x \wedge y$, for the Grassmann product.

THEOREM 1. *Let M be a simply connected d -dimensional Riemannian manifold, \bar{M} a complete $(d + 1)$ -dimensional Riemannian manifold of constant curvature K . Then M can be immersed in \bar{M} if and only if there exists a horizontal equivariant R^a -valued 1-form $\sigma = (\sigma_i)$ on $F(M)$ such that*

$$(1) \quad \begin{cases} \sum_k \sigma_k \omega_k = 0 \\ \Phi_{ij} = \sigma_i \sigma_j + K \omega_i \omega_j & \text{(Gauss equation)} \\ D\sigma_i = 0 & \text{(Codazzi equation).} \end{cases}$$

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Proof. Suppose there exists an immersion $i: M \rightarrow \bar{M}$. Since M is simply connected, there is a unit normal vector field on the immersed manifold, N being a differentiable ($= C^\infty$) map from M to the tangent of \bar{M} . Then the formula $\psi(m, f_1, \dots, f_a) = (i(m), di(f_1), \dots, di(f_a), N(m))$ defines a differentiable map $\psi: F(M) \rightarrow F(\bar{M})$. (Denote by $\bar{R}_a, \bar{\varphi}, \dots$ the structural elements of \bar{M} .) Note that $\psi \circ R_a = \bar{R}_a \circ \psi$ if $a \in O(d) \subset O(d+1)$. This fact plus the uniqueness of the Riemannian connection of M are used in the proof that

$$(2) \quad \begin{cases} \omega_i = \bar{\omega}_i \circ d\psi \\ 0 = \bar{\omega}_{a+1} \circ d\psi \\ \varphi_{ij} = \bar{\varphi}_{ij} \circ d\psi . \end{cases}$$

Furthermore, the R^d -valued 1-form defined by (3) $\sigma_i = \varphi_{i,a+1} \circ \psi$ satisfies the conditions stated in the theorem. This form is, of course, one expression for the second fundamental form of the immersed manifold.

Conversely, given a form σ on $F(M)$ with the stated properties we must produce an immersion $i: M \rightarrow \bar{M}$. To do this we first find a differentiable map $\varphi: F(M) \rightarrow F(\bar{M})$ satisfying the differential equations (2) and (3). Consider the 1-forms $\bar{\omega}_i - \omega_i, \bar{\omega}_{a+1} - \varphi_{i,a+1}, \bar{\varphi}_{ij} - \varphi_{ij}, \bar{\varphi}_{i,a+1} - \sigma_i$ on $F(M) \times F(\bar{M})$, where we use the same notation for a form on one factor and that form pulled back to the product manifold by a projection. We want to apply the Frobenius theorem to these forms. Taking account of the structural equations one sees that its hypothesis holds provided $\sum_k \sigma_k \omega_k = 0; d\varphi_{ij} = -\sum_k \varphi_{ik} \varphi_{kj} + \sigma_i \sigma_j + K\omega_i \omega_j$; and $d\sigma_i = -\sum_k \varphi_{ik} \sigma_k$. But these conditions follow from the corresponding equations in (1)—in the case of the last one because for σ (or any other R^d -valued horizontal equivariant 1-form on $F(M)$) we have $d\sigma_i = -\sum \varphi_{ik} \sigma_k + D\sigma_i$. Then if $(g, \bar{g}) \in F(M) \times F(\bar{M})$, an integral manifold through (g, \bar{g}) given by the Frobenius theorem is the graph of a differentiable function ψ' defined on a neighborhood U of $g \in F(M)$, carrying g to \bar{g} , and satisfying (2) and (3). Subject to these conditions φ' is unique, except for the size of its domain. Further, one can show that φ' commutes with right-multiplication in the sense that, where meaningful, $\varphi' \circ R_a$ and $\bar{R}_a \circ \varphi'$ agree. This fact permits us to extend the local solution ψ' by right-multiplication (in an obvious way) to a solution $\varphi: \pi^{-1}(V) \rightarrow F(\bar{M})$, where $V = \pi(U) \subset M$. Thus there exists a unique differentiable map $j: V \rightarrow \bar{M}$ such that $j \circ \pi = \bar{\pi} \circ \psi$ on $\pi^{-1}(V)$. We claim that j is an immersion: In fact, suppose $f \in F(M)$ projects to $m \in V$, and let $\psi(f) = \bar{f} \in F(\bar{M})$. Now if $y \in F(M)_f$ projects to $x \in M_m$ we have

$$\begin{aligned} \langle x, f_i \rangle &= \omega_i(y) = \bar{\omega}_i(d\psi(y)) = \langle d\pi(d\psi(y)), \bar{f}_i \rangle \\ &= \langle dj(x), f_i \rangle, \text{ and } \langle dj(x), f_{a+1} \rangle = \bar{\omega}_{a+1}(d\psi(x)) = 0 . \end{aligned}$$

This proves $j: V \rightarrow \bar{M}$ is an immersion; similarly one checks that its second fundamental form is $\sigma|\pi^{-1}(V)$. But an immersion is controlled by its second fundamental form; explicitly in the case at hand, if j' is another such immersion of V in \bar{M} with $j(m) = j'(m)$ and $dj_m = dj'_m$ for some one $m \in V$, then $j = j'$. This uniqueness property, the simple connectedness of M , and the special character of \bar{M} are the essential points in a proof (which we omit) that out of local immersions as above a global immersion $i: M \rightarrow \bar{M}$ can be constructed of which σ is the second fundamental form.

2. The Gauss equation. Of the conditions (1) imposed on σ , the crucial one is the Gauss equation. Under the usual translation [1] of horizontal equivariant objects on $F(M)$ into objects on M , the curvature form becomes a function which to each $x, y \in M_m$ assigns a linear transformation $R_{xy}: M_m \rightarrow M_m$. Then the equation $\langle R_{xy}(u), v \rangle = \langle R_m(xy), uv \rangle$ defines the curvature transformation R_m as a linear operator on the Grassmann space $\wedge^2 M_m$. The function $m \rightarrow R_m$ is for our purposes the most convenient form of the curvature tensor R of M . The form σ translates to a function S on M with S_m a linear operator on M_m , and the Gauss equation becomes $R = S \wedge S + K$, where K denotes scalar multiplication by the constant curvature K of \bar{M} .

Reversing the process, suppose that S is a differentiable field of linear operators on the tangent spaces of M such that $R = S \wedge S + K$. Let σ be the horizontal, equivariant R^a -valued 1-form on $F(M)$ corresponding to S . Then $\phi_{ij} = \sigma_i \sigma_j + K \omega_i \omega_j$. The other two conditions on σ follow automatically if the rank of $R - K$, that is, the minimum rank of $R_m - K$ for $m \in M$, is not too small. Explicitly:

LEMMA 1. (notation as above) *Let $R = S \wedge S + K$. If rank $(R - K) \geq 3$, then $\sum_k \sigma_k \omega_k = 0$. If rank $(R - K) \geq 4$, then $D\sigma_i = 0$.*

Proof. By a symmetry of R , shared by K , we have $\mathfrak{S} \langle S(x)S(u), yv \rangle = 0$, where \mathfrak{S} denotes the sum over the cyclic permutations of x, u, y . Eliminating v we get $\mathfrak{S} \{ (\langle S(y), x \rangle - \langle y, S(x) \rangle) S(u) \} = 0$. But since rank $S \wedge S \geq 3$, the same is true for S , and it follows that $\langle S(y), x \rangle = \langle y, S(x) \rangle$. But the symmetry of S is equivalent to $\sum_k \sigma_k \omega_k = 0$.

To prove the second assertion (due essentially to T. Y. Thomas), we apply D to the equation $\phi_{ij} = \sigma_i \sigma_j + K \omega_i \omega_j$. Since $D\omega = 0$ and $D\phi = 0$ (Bianchi identity) we get $D\sigma_i \wedge \sigma_j = \sigma_i \wedge D\sigma_j$. The rank condition implies rank $S \geq 4$, hence rank $\sigma \geq 4$. Thus the result is a consequence of the following.

LEMMA 2. *Let $x_1, \dots, x_d \in V$, a finite-dimensional real vector space, and let $w_1, \dots, w_d \in \wedge^2 V$. If $x_i \wedge w_j = w_i \wedge x_j$ for all $1 \leq i, j \leq d$,*

and the vectors x_1, \dots, x_a span a subspace of dimension ≥ 4 , then $w_1 = \dots = w_a = 0$.

Proof. We may suppose that x_1, x_2, x_3, x_4 are the first four elements of a basis e_1, e_2, \dots for V . Let $P = \{1, 2, 3, 4\}$, and fix an index $p \in P$. By a standard Grassmann argument one can show that there is a $y_p \in V$ such that $w_p = y_p e_p$. Then $e_p \wedge w_q = w_p \wedge e_q$ implies $(y_p + y_q)e_p e_q = 0$ for all $q \in P$. Thus $2y_p = (y_p + y_q) + (y_p + y_r) - (y_q + y_r)$ is in the subspace spanned by e_p, e_q, e_r , where q and r are any elements of P such that p, q, r are all different. It follows that y_p is a multiple of e_p , and thus $w_p = 0$. But if $i > 4$, then $e_p \wedge w_i = w_p \wedge e_i = 0$ for all $p \in P$, so that $w_i = 0$ also.

Summarizing, if M and \bar{M} are as in Theorem 1 and rank $(R - K) \geq 4$, then M can be immersed in \bar{M} if and only if $R - K$ is decomposable, i.e. expressible as $S \wedge S$ with S a differentiable field of linear operators on the tangent spaces of M .

In the following section we consider the purely Grassmannian question of the decomposability of $R_m - K$ at a single point of M .

3. Decomposability. Let V and W be finite-dimensional real vector spaces, and let $T: \wedge^2 V \rightarrow \wedge^2 W$ be a linear transformation. To determine whether T is decomposable we use the following definition: Three bivectors are *crossed* if any two, but not all three, are collinear, (a set of bivectors being called collinear if all have a common non-zero divisor, i.e. all are decomposable and the planes of the non-zero ones have a line in common.) One easily proves:

LEMMA 3. *Bivectors w_1, w_2, w_3 are crossed if and only if there exist linearly independent vectors x, y, z and non-zero numbers K, L, M such that*

$$(4) \quad \begin{cases} w_1 = K xy \\ w_2 = L xz \\ w_3 = M yz. \end{cases}$$

If w_1, w_2, w_3 are crossed, then in any expression (4) the sign of the product KLM is always the same. (In fact, the vectors x, y, z are unique up to non-zero scalar multiplication, so we need only check that changing the signs of any subset of $\{x, y, z\}$ does not change the sign of KLM .) In case $KLM > 0$ we say that w_1, w_2, w_3 are *coherently crossed*. Note that if T is decomposable then T carries coherently crossed bivectors to bivectors which are either coherently crossed or coplanar. Our

aim is to prove the converse when $\text{rank } T \geq 4$. (We do not need the easy cases of lower rank.)

LEMMA 4. *The following conditions on T are equivalent:*

- (a) *T carries decomposable bivectors to decomposable bivectors.*
- (b) *T carries two collinear bivectors to two collinear bivectors.*
- (c) *$T(xy) \wedge T(uv) \in \wedge^2 W$ is skew-symmetric in its arguments.*

LEMMA 5. *If rank $T \geq 4$ and T carries crossed to crossed or coplanar bivectors, then R carries collinear to collinear bivectors.*

Proof. It is sufficient to prove collinearity is preserved in the case of three bivectors. Thus we must show that $T(e_1e_2), T(e_1e_3), T(e_1e_4)$ are collinear. Now any two of these bivectors are collinear, hence all three are either crossed or collinear. We assume the former and get a contradiction. If they are crossed there is a unique subspace U of W , with dimension 3, such that the bivectors are in $\wedge^2 U \subset \wedge^2 W$. We may also assume that e_1, e_2, e_3, e_4 are linearly independent for otherwise we can reduce to the case of two collinear bivectors. Thus these vectors are part of a basis for V .

Case I. There is an index i such that $T(e_1e_i) \notin \wedge^2 U$.

Consider $T(e_1e_2), T(e_1e_3), T(e_1(e_4 + \delta e_i))$, where δ is an arbitrarily small non-zero number. Now the last of these three bivectors is not in $\wedge^2 U$, while the union of the planes of the first two spans U . Hence all three are not in the second Grassmann product of any 3-dimensional subspace of W . Thus they are not crossed. On the other hand, any two are collinear, so all three are collinear. But this is a contradiction, for an arbitrarily small change in the crossed bivectors $T(e_1e_2), T(e_1e_3), T(e_1e_4)$ cannot produce collinear bivectors.

Case II. For all $i, T(e_1e_i) \in \wedge^2 U$.

We prove the contradiction $\text{rank } T \leq 3$ by showing that $T(e_p e_q) \in \wedge^2 U$ for all p, q . If $T(e_1e_p)$ and $T(e_1e_q)$ are independent, then by hypothesis, $T(e_p e_q)$ is crossed with these two bivectors, hence is in $\wedge^2 U$. If they are dependent and $T(e_1e_p) \neq 0$, then by hypothesis $T(e_1e_p)$ and $T(e_p e_q)$ are coplanar and $T(e_p e_q) \in \wedge^2 U$. Finally, if $T(e_1e_p) = 0$, then by Lemma 5 $0 = T(e_1e_p) \wedge T(e_r e_q) = T(e_1e_r) \wedge T(e_p e_q)$ for $r = 2, 3, 4$. But since $T(e_1e_2), T(e_1e_3), T(e_1e_4)$ are crossed one easily deduces from these equations that $T(e_p e_q) \in \wedge^2 U$.

THEOREM 2. *Let $T: \wedge^2 V \rightarrow \wedge^2 W$ be a linear transformation of rank ≥ 4 . Then there exists a linear transformation $S: V \rightarrow W$ such that $T = S \wedge S$ if and only if T carries coherently crossed to coherently*

crossed or coplanar bivectors.

Proof. We may choose a basis e_1, \dots, e_d for V such that T is never zero on the corresponding canonical basis for $\Lambda^2 V$. Fix an index $1 \leq i \leq d$. By the preceding lemma there is a non-zero vector $u_i \in W$ such that u_i divides each $T(e_i e_j)$, $j = 1, \dots, d$. Furthermore this vector is unique up to scalar multiplication. To see this we need only show that these bivectors $T(e_i e_j)$ are not all coplanar. But if they were, then $T(e_i e_j)$, $T(e_i e_k)$, $T(e_j e_k)$, since not crossed, would have to be coplanar for all j, k , implying $\text{rank } T \leq 1$.

Now let i, j be different indices. We claim that $T(e_i e_j) = K_{ij} u_i u_j$. In fact, since there is an index k such that the bivectors $T(e_i e_j)$ and $T(e_i e_k)$ are not coplanar, they are crossed with $T(e_j e_k)$. By Lemma 3 and the divisibility properties of u_i, u_j, u_k , it follows that these crossed bivectors may be written as $Ku_i u_j, Lu_i u_k, Mu_j u_k$ respectively.

By changing the signs of u_2, \dots, u_d where necessary, we shall now arrange to have the number K_{ij} ($i < j$) all positive. We can certainly get all $K_{ij} > 0$ in this way. Consider $T(e_i e_i), T(e_i e_j), T(e_i e_j)$. If the first two bivectors are not coplanar, then all three are coherently crossed, hence the product $K_{ii} K_{ij} K_{ij}$, and consequently K_{ij} , are positive. If $T(e_i e_i)$ and $T(e_i e_j)$ are coplanar, we argue as follows: Since $\text{rank } T > 1$ there is an index k (say $k > j$) such that u_k is not in the plane spanned by u_i, u_j . Thus $T(e_i e_i)$ and $T(e_i e_k)$ are not coplanar, so $K_{ik} > 0$. Similarly $K_{jk} > 0$. And since u_i, u_j, u_k are independent, it follows that $K_{ij} > 0$.

To complete the proof it will suffice to find numbers $\lambda_1, \dots, \lambda_d$ such that for any $i < j$ we have $K_{ij} = \lambda_i \lambda_j$. For then the equation $T(e_i e_j) = K_{ij} u_i u_j$ becomes $T(e_i e_j) = (\lambda_i u_i)(\lambda_j u_j)$, and by defining $S: V \rightarrow W$ to be the linear transformation such that $S(e_i) = \lambda_i u_i$ we get $T = S \wedge S$.

Call a set i, j, k of indices a *triple* if $i < j < k$ and u_i, u_j, u_k are independent. For each triple consider the equations $K_{ij} = \lambda_i \lambda_j, K_{ij} = \lambda_i \lambda_k, K_{jk} = \lambda_j \lambda_k$. Since the K 's are positive there is a unique positive solution $\lambda_i, \lambda_j, \lambda_k$. Since each index i is in at least one triple we get at least one such value for λ_i . We must show that the values obtained from two different triples containing i are the same. We need only consider triples of the form i, j, p and i, j, q , for it will be clear from the proof in this case that the position of i in a triple is immaterial and that the case where five indices are involved may be reduced to the present one using $\text{rank } T \geq 4$. We know that

$$\begin{aligned} T(e_i e_j) &= \lambda_i \lambda_j u_i u_j & T(e_i e_j) &= \mu_i \mu_j u_i u_j \\ T(e_i e_p) &= \lambda_i \lambda_p u_i u_p & T(e_i e_q) &= \mu_i \mu_q u_i u_q \\ T(e_j e_p) &= \lambda_j \lambda_p u_j u_p & T(e_j e_q) &= \mu_j \mu_q u_j u_q. \end{aligned}$$

First consider the case in which the vectors u_i, u_j, u_p, u_q are linearly independent. By Lemma 4, $T(e_i e_p) \wedge T(e_j e_q) = -T(e_j e_p) \wedge T(e_i e_q)$, but since $u_i u_p u_j u_q \neq 0$ this implies $\lambda_i \mu_j = \mu_i \lambda_j$. But also $\lambda_i \lambda_j = \mu_i \mu_j$, and since the numbers in the last two equations are all positive we get $\lambda_i = \mu_i$. Now suppose u_i, u_j, u_p, u_q are dependent, hence span a 3-dimensional subspace. Since $\text{rank } T \geq 4$ there must exist an index r (say $r > p, q$) such that u_i, u_j, u_p, u_r and u_i, u_j, u_q, u_r are each linearly independent. Thus the values of λ_i determined by i, j, p and i, j, q are the same as that determined by i, j, r .

This shows the existence of S such that $T = S \wedge S$; uniqueness up to sign is implicit in the proof, for the only ultimate element of choice is in the orientation of u_1 , i.e. the use of u_1 rather than $-u_1$.

4. Coordinate criteria for decomposability. With notation as in the preceding section, fix bases e_1, \dots, e_a for V and $f_1, \dots, f_{\bar{a}}$ for W . Let $T_{ij} = T(e_i e_j) = \sum_{\alpha < \beta} T_{ij\alpha\beta} f_\alpha f_\beta$. What conditions on T_{ij} are necessary and sufficient for T to be decomposable, or alternatively (if $\text{rank } T \geq 4$) for T to carry coherently crossed to coherently crossed or coplanar bivectors? Necessary is that T carry decomposable to decomposable bivectors, and this is easily proved equivalent to

$$(5) \quad T_{ij} \wedge T_{kl} = T_{kj} \wedge T_{li} \text{ for all } 1 \leq i, j, k, l \leq d$$

This condition as well as the condition $\text{rank } T \geq 4$ are standardly expressible in terms of polynomials in $T_{ij\alpha\beta}$.

LEMMA 6. *Suppose that any two of the bivectors $a, b, c \in \wedge^2 W$ are collinear, and let $a = \sum_{\alpha < \beta} A_{\alpha\beta} f_\alpha f_\beta$, similar expressions for b, c . Then a, b, c are coherently crossed if and only if there exist indices $1 \leq \alpha < \beta < \gamma \leq \bar{d}$ such that*

$$\Delta(\alpha\beta\gamma) = \begin{vmatrix} A_{\alpha\beta} & A_{\alpha\gamma} & A_{\beta\gamma} \\ B_{\alpha\beta} & B_{\alpha\gamma} & B_{\beta\gamma} \\ C_{\alpha\beta} & C_{\alpha\gamma} & C_{\beta\gamma} \end{vmatrix} > 0$$

Proof. The bivectors a, b, c are either crossed or collinear. We show:

- (1) if crossed, then for some α, β, γ we have $\Delta(\alpha\beta\gamma) \neq 0$,
- (2) if coherently crossed, then each such non-zero determinant is positive,
- (3) if collinear, then each such determinant is zero.

For α, β, γ let $x \rightarrow \bar{x}$ be the natural projection of W onto the subspace U spanned by $f_\alpha, f_\beta, f_\gamma$; same notation for the induced projection of $\wedge^2 W$ onto $\wedge^2 U$. In the first two cases above we can write a, b, c in

the form of (4), hence $\bar{a} = K\bar{x}\bar{y}$, $\bar{b} = L\bar{x}\bar{z}$, $\bar{c} = M\bar{y}\bar{z}$. For (1), since x, y, z independent there are indices α, β, γ such that $\bar{x}, \bar{y}, \bar{z}$ are independent, hence $\bar{a}, \bar{b}, \bar{c}$ are independent, and the result follows. For (2), suppose $\Delta(\alpha\beta\gamma) \neq 0$. Using the above notation we have $KLM > 0$. Notice that any two canonical bases (lexicographic order) for $\wedge^2 U$ have the same orientation. Thus $\Delta(\alpha\beta\gamma) > 0$. The proof of (3) is similar.

A further necessary condition for decomposability of T is that T_{ij}, T_{ik}, T_{jk} be coherently crossed or coplanar. Assuming (1), this is equivalent to

(6) If $1 \leq i < j < k \leq d$, then either T_{ij}, T_{ik}, T_{jk} are coplanar or there exist indices $1 \leq \alpha < \beta < \gamma \leq \bar{d}$ such that

$$\begin{vmatrix} T_{ij\alpha\beta} & T_{ij\alpha\gamma} & T_{ij\beta\gamma} \\ T_{ik\alpha\beta} & T_{ik\alpha\gamma} & T_{ik\beta\gamma} \\ T_{jk\alpha\beta} & T_{jk\alpha\gamma} & T_{jk\beta\gamma} \end{vmatrix} > 0.$$

If the basis e_1, \dots, e_d is such that all $T_{ij} \neq 0$, then (5) and (6) are necessary and sufficient for the decomposability of T , for Lemma 5 and Theorem 2 use no more than this. For an arbitrary basis, however, they are not enough, as one can see from simple examples. We must add, say

(7) If $T_{ij} = T_{ik} = 0$, then either $T_{jk} = 0$, or, for all r , $T_{ir} = 0$.

Now one can prove the following lemma by reducing to the case in which all $T_{ij} \neq 0$.

LEMMA 7. Let $T: \wedge^2 V \rightarrow \wedge^2 W$ be a linear transformation with rank $T \geq 4$. Then T is decomposable if and only if, relative to arbitrary canonical bases for $\wedge^2 V$ and $\wedge^2 W$, conditions (5), (6), (7) hold.

5. Summary. Again let R be the curvature transformation of the simply connected manifold M . For simplicity we discuss the case $\bar{M} = R^{d+1}$. Assume that (at each point) rank $R \geq 4$ and R carries coherently crossed to coherently crossed or coplanar bivectors. It is clear that the proof of Theorem 2 applies simultaneously to all R_n with n any point of a convex neighborhood C of $m \in M$. One need only use the simple connectedness of C to choose the orientations of the various choices of u_1 consistently. We thus obtain a differentiable field of linear operators S such that $R = S \wedge S$, first locally, then as usual, globally. When rank $R \geq 3$ we can still prove R decomposable, but the Codazzi equation may fail; thus our criterion for immersion, while always sufficient, is necessary only in the case of immersions for which the second fundamental form S has rank at least four. Call such an immersion 4-regular.

This same argument, with $R - K$ in place of K , proves

THEOREM 3. *Let M be a simply connected d -dimensional manifold ($d \geq 4$) with curvature transformation R . Let $\bar{M}(K)$ be a complete $(d + 1)$ -dimensional manifold of constant curvature K . Then M has a 4-regular immersion in $\bar{M}(K)$ if and only if $\text{rank}(R - K) \geq 4$ and $R - K$ carries coherently crossed to coherently crossed or coplanar bivectors, i.e. conditions (5), (6), (7) hold at each point of M .*

For a given M one may ask for the set \mathcal{K} of numbers K such that M has a regular immersion in an $\bar{M}(K)$. Consider two cases:

(i) If R does not preserve decomposability, say $R(xy)^2 \neq 0$, then M is not immersible in R^{d+1} and \mathcal{K} contains at most the number K determined by the necessary condition $R(xy) = S(x)S(y) + Kxy$. We check as above whether $K \in \mathcal{K}$.

(ii) If R preserves decomposability, so that (5) holds, \mathcal{K} may well be infinite. By studying conditions (6), (7) one can characterize \mathcal{K} in terms of polynomials in an unknown K and the coordinates of R .

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Vol. 9, No. 4

August, 1959

Frank Herbert Brownell, III, <i>A note on Kato's uniqueness criterion for Schrödinger operator self-adjoint extensions</i>	953
Edmond Darrell Cashwell and C. J. Everett, <i>The ring of number-theoretic functions</i>	975
Heinz Otto Cordes, <i>On continuation of boundary values for partial differential operators</i>	987
Philip C. Curtis, Jr., <i>n-parameter families and best approximation</i>	1013
Uri Fixman, <i>Problems in spectral operators</i>	1029
I. S. Gál, <i>Uniformizable spaces with a unique structure</i>	1053
John Mitchell Gary, <i>Higher dimensional cyclic elements</i>	1061
Richard P. Gosselin, <i>On Diophantine approximation and trigonometric polynomials</i>	1071
Gilbert Helmsberg, <i>Generating sets of elements in compact groups</i>	1083
Daniel R. Hughes and John Griggs Thompson, <i>The H-problem and the structure of H-groups</i>	1097
James Patrick Jans, <i>Projective injective modules</i>	1103
Samuel Karlin and James L. McGregor, <i>Coincidence properties of birth and death processes</i>	1109
Samuel Karlin and James L. McGregor, <i>Coincidence probabilities</i>	1141
J. L. Kelley, <i>Measures on Boolean algebras</i>	1165
John G. Kemeny, <i>Generalized random variables</i>	1179
Donald G. Malm, <i>Concerning the cohomology ring of a sphere bundle</i>	1191
Marvin David Marcus and Benjamin Nelson Moysl, <i>Transformations on tensor product spaces</i>	1215
Charles Alan McCarthy, <i>The nilpotent part of a spectral operator</i>	1223
Kotaro Oikawa, <i>On a criterion for the weakness of an ideal boundary component</i>	1233
Barrett O'Neill, <i>An algebraic criterion for immersion</i>	1239
Murray Harold Protter, <i>Vibration of a nonhomogeneous membrane</i>	1249
Victor Lenard Shapiro, <i>Intrinsic operators in three-space</i>	1257
Morgan Ward, <i>Tests for primality based on Sylvester's cyclotomic numbers</i>	1269
L. E. Ward, <i>A fixed point theorem for chained spaces</i>	1273
Alfred B. Willcox, <i>Šilov type C algebras over a connected locally compact abelian group</i>	1279
Jacob Feldman, <i>Correction to "Equivalence and perpendicularity of Gaussian processes"</i>	1295