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**Introduction.** Lucas, Carmichael [1] and others have given tests for primality of the Fermat and Mersenne numbers which utilize divisibility properties of the Lucas sequences $(U)$ and $(V)$; in this paper we are concerned only with the first sequence;

$$(U)\colon\ U_0,\ U_1,\ U_2,\ \cdots,\ U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ \cdots .$$

Here $\alpha$ and $\beta$ are the roots of a suitably chosen quadratic polynomial $x^2 - Px + Q$, with $P$ and $Q$ coprime integers. (For an account of these tests, generalizations and references to the early literature, see Lehmer's Thesis [2]).

I develop here a test for primality of a less restrictive nature which utilizes a divisibility property of the Sylvester cyclotomic sequence [3]:

$$(Q)\colon Q_0 = 0,\ Q_1 = 1,\ Q_2,\ \cdots,\ Q_n = \prod_{\substack{1\le r\le n\\(r,n)=1}} (\alpha - e^{\frac{2\pi i r}{n}}\beta),\ \cdots$$

Here $\alpha$ and $\beta$ have the same meaning as before. $(U)$ and $(Q)$ are closely connected [4]; in fact

$$U_n = \prod_{d\mid n} Q_d\ . \tag{1.1}$$

The divisibility property is expressed by the following theorem proved in § 3 of this paper.

THEOREM. *If $m$ is an odd number dividing some cyclotomic number $Q_n$ whose index $n$ is prime to $m$, then every divisor of $m$ greater than one has the same rank of apparition $n$ in the Lucas sequence $(U)$ connected with $(Q)$.*

Here the rank of apparition or rank, of any number $d$ in $(U)$ means as usual the least positive index $x$ such that $U_x \equiv 0 \pmod{d}$.

The following primality test is an immediate corollary.

*Primality test. If $m$ is odd, greater than two, and divides some cyclotomic number $Q_n$ whose index $n$ is both prime to $m$ and greater than the square root of $m$, then $m$ is a prime number except in two trivial cases: $m = (n-1)^2$, $n-1$ a prime greater than 3, or $m = n^2 - 1$ with $n-1$ and $n+1$ both primes.*

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The primality tests of Lucas and Carmichael are the special case when $n = m \pm 1$ is a power of two (which allows $Q_n$ to be expressed in terms of $V_n$) with $X^2 - Px + Q$ suitably specialized.

**2. Notations.** We denote the rational field by $R$, and the ring of rational integers by $I$. The polynomial

$$f(x) = x^2 - Px + Q\,, \qquad P,\ Q,\ \text{in } I \text{ and co-prime} \tag{2.1}$$

is assumed to have distinct roots $\alpha$ and $\beta$.

We denote the root field of $f(x)$ by $\mathscr{A}$ and the ring of its integers by $\mathscr{F}$. Thus $\mathscr{A}$ is either $R$ itself, or a simple quadratic extension of $R$.

Let $p$ be an odd prime of $I$, and $\mathfrak{p}$ a prime ideal factor of $p$ in $\mathscr{F}$. Every element $\rho$ of $\mathscr{A}$ may be put in the form $\rho = \alpha/a$ with $\alpha$ in $\mathscr{F}$ and $a$ in $I$. The totality of such $\rho$ with $(a, p) = 1$ forms a subring $\mathscr{F}_p$ of $\mathscr{A}$. Evidently $\mathscr{A} \supset \mathscr{F}_p \supset \mathscr{F} \supseteq I$. If we extend $\mathfrak{p}$ into $\mathscr{F}_p$ in the obvious way, we obtain a prime ideal $\mathfrak{P}$. The homomorphic image of $\mathscr{F}_p$ modulo $\mathfrak{P}$ is a field, $\mathscr{F}_p$. We denote the mapping of $\mathscr{F}_p$ onto $\mathscr{F}_p$ by $(\mathfrak{P})$.

Let $F_n(z)$ denote the cyclotomic polynomial of degree $\phi(n)$. $F_n(z)$ has coefficients in $I$, and if $n$ is greater than one, then (Lehmer [2], Carmichael [1])

$$Q_n = \beta^{\phi(n)} F_n\left(\frac{\alpha}{\beta}\right), \tag{2.2}$$

Furthermore

$$z^n - 1 = \prod_{d \mid n} F_n(z)\,. \tag{2.3}$$

**3. Proof of theorem.** Let $m$ be an odd number greater than one which divides some term of $(Q)$ whose index $n$ is prime to $m$, so that

$$Q_n \equiv 0 \pmod{m}\,, \qquad (n,\ m) = 1. \tag{3.1}$$

Throughout the next three lemmas, $p$ stands for a fixed prime factor of $m$.

LEMMA 1. *If $\mathfrak{p}$ is any ideal factor of $p$ in $\mathscr{F}$, then*

$$(Q,\ p) = (\alpha,\ \mathfrak{p}) = (\beta,\ \mathfrak{p}) = (1)\,. \tag{3.2}$$

*Proof.* It suffices to prove that $(Q, p) = (1)$. Assume the contrary. Then $(p, P) = 1$. Since $U_1 = 1$ and $U_{x+2} = PU_{x+1} - QU_x \equiv PU_{x+1} \pmod{p}$, it follows by induction that $U_n \not\equiv 0 \pmod{p}$. Then by (1.1), $Q_n \not\equiv 0$

(mod $p$). But $p$ divides $m$ so that by (3.1) $Q_n \equiv 0 \pmod{p}$ a contradiction.

LEMMA 2. *The rank of apparition of $p$ in $(U)$ is $n$.*

*Proof.* Since $U_n \equiv 0 \pmod{p}$, $p$ has a positive rank of apparition in $(U)$, $r$ say. Then $r$ divides $n$. But by (1.1), $U_r = \prod_{d|n} Q_d$. Hence $Q_d \equiv 0 \pmod{p}$ for some $d$ dividing both $r$ and $n$. Clearly, if $d = n$, then $r = n$ and we are finished. Assume that $d$ is less than $n$.

The number $\alpha/\beta = \alpha^2/Q$ is in $\mathscr{I}_p$ by Lemma 1. Let $\tau$ be its image in $\mathscr{F}_p$ under the mapping ($\mathfrak{P}$). Then by (2.2) and Lemma 1 $F_n(\tau) = F_d(\tau) = 0$ in $\mathscr{F}_p$. Consequently the resultant of the polynomials $F_n(z)$ and $F_d(z)$ is zero in $\mathscr{F}_p$. Therefore its inverse image under the mapping is in $\mathfrak{P}$. But this resultant is evidently in $I$. Therefore it must be divisible by $p$. But by formula (2.3), since $d < n$ the resultant of $F_n(z)$ and $F_d(z)$ must divide the discriminant $\pm n^{n-1}$ of $z^n - 1$. Thus $n \equiv 0 \pmod{p}$ so that $(n, m) \equiv 0 \bmod p$ which contradicts (3.1) and completes the proof.

LEMMA 3. *The rank of apparition in $(U)$ of any positive power of $p$ which divides $m$ is $n$.*

*Proof.* Let $p^k$ divide $m$, $k \geq 1$ and let the rank of $p^k$ in $(U)$ be $r$. Now $U_n = \prod_{d|n} Q_d \equiv 0 \pmod{p^k}$. But by Lemma 2, each $Q_d$ with $d < n$ is prime to $p$. Hence $r$ must equal $n$.

The theorem proper now follows easily. For let $m'$ be any divisor of $m$ other than one. By Lemma 3, every prime power dividng $m'$ has rank of apparition $n$ in $(U)$. But the rank of apparition of $m'$ in $(U)$ is the least common multiple of the ranks of the prime powers of maximal order diving $m'$. (Carmichael [1]). Hence $m'$ also has rank of apparition $n$ in $(U)$.

**4. Proof of primality test.** Assume that (3.1) holds for some $n$ greater than $\sqrt{m}$. If $m$ is not a prime, it has a prime factor $\leq \sqrt{m}$. Let $p$ be the smallest such factor, and let

$$m = pq\,, \qquad q \geq 3\,. \tag{4.1}$$

Then $p$ has rank $n$ in $(U)$ by Lemma 3. But by a classical result of Lucas, $U_{p\pm 1} \equiv 0 \pmod{p}$. Hence $n$ divides $p \pm 1$. If $n$ is less than $p + 1$, $\sqrt{m} < p \leq \sqrt{m}$, a contradiction. Hence $n = p + 1$. If $p = \sqrt{m}$, then $m = (n - 1)^2$ and $n - 1$ is a prime. Since $m$ is odd, $n \geq 4$. This is the first trivial case.

If $p < \sqrt{m}$, then $q \geq p + 2$ and $m \geq p(p + 2)$. But if $m > p(p + 2)$,

then $n^2 > m \geq (p+1)^2 = n^2$, a contradiction. Hence $m = p(p+2)$ where $p+2$ has no prime factor smaller than $p$. Hence $p+2$ is a prime and $m = n^2 - 1$ with both $n-1$ and $n+1$ primes. This is the second trivial case. In every other case then, $m$ must be a prime.

**5. Conclusion.** The two trivial cases can actually occur. For if $P = 22$ and $Q = 3$, then $Q_6 = \alpha^2 - \alpha\beta + \beta^2 = P^2 - 3Q = 475$. Hence $Q_6 \equiv 0 \pmod{25}$ and $25 = (6-1)^2$. Again, if $P = 17$ and $Q = 3$, then $Q_6 = 280$. Hence $Q_6 \equiv 0 \pmod{35}$ and $35 = 6^2 - 1 = 5 \times 7$. It is worth noting that these trivial cases cannot occur if $\alpha$ and $\beta$ are rational integers. (See [1], Theorem XII and remark.)

To illustrate the theorem, note that if $P = 2$ and $Q = 1$, $Q_9 = 73$. Since $\sqrt{73} < 9$ and $(9, 73) = 1$, 73 is a prime. But for $P = 3$ and $Q = 1$, $Q_9 = 91$. But $9 < \sqrt{91}$ so the test is inapplicable. As a matter of fact, 91 is the product of two primes. Evidently the test may be extended to cover such a case. That is, if $Q_n \equiv 0 \pmod{m}$, $(n, m) = 1$ and $n > \sqrt[3]{m}$, $m$ will usually be either a prime, or the product of two primes.

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