

# Pacific Journal of Mathematics

**A CLASS OF HYPER-*FC*-GROUPS**

A. M. DUGUID

# A CLASS OF HYPER-FC-GROUPS

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**1. Introduction.** An element  $g$  of an arbitrary group  $G$  is called an *FC* element if it has a finite number of conjugates in  $G$ . The set of all *FC* elements of  $G$  forms a characteristic subgroup  $H$  of  $G$  (see Baer [1]). The *upper FC-series* of  $G$ , introduced by Haimo [4] as the *FC-chain*, may be defined by

$$H_0 = \{1\}, \\ H_{i+1}/H_i = H(G/H_i),$$

the subgroup of all *FC* elements of  $G/H_i$ . The upper *FC-series* is continued transfinitely in the usual way, by defining

$$H_\alpha = \bigcup_{\beta < \alpha} H_\beta,$$

when  $\alpha$  is a limit ordinal. If  $H_\gamma = G$ , but  $H_\delta \neq G$ , for all  $\delta < \gamma$ , we say that the group  $G$  is *hyper-FC of FC-class*  $\gamma$ , following McLain [7].

A group  $G$  in which the transfinite upper central series

$$\{1\} = Z_0 \leq Z_1 \leq \dots \leq Z_\alpha \leq \dots$$

reaches the whole group is called a *ZA-group* (Kurosh [6]), and we may say that  $G$  has class  $\alpha$  if  $Z_\alpha = G$ , but  $Z_\beta \neq G$ , for all  $\beta < \alpha$ . Glushkov [3] and McLain [7] have given constructions for a *ZA-group* of any given class. The main object of this note is to construct groups of given *FC-class*.

## 2. Constructions and proofs.

**DEFINITION.** We say that a group  $G$  is of type  $Q_\alpha$  if

1.  $G$  has *FC-class*  $\alpha$ , with upper *FC-series*

$$\{1\} = H_0 \leq H_1 \leq \dots \leq H_\alpha = G,$$

2.  $H_{\gamma+1}/H_\gamma$  is infinite, for all  $\gamma < \alpha$ , and
3.  $H_{\gamma+1}/H_\gamma$  has the unit subgroup for its centre, for all  $\gamma < \alpha$ .

Thus the group with only one element is of type  $Q_0$ , and, in constructing a group  $G$  of type  $Q_\alpha$ , we may assume the existence of a group  $G_\beta$  of type  $Q_\beta$ , for each  $\beta < \alpha$ . If  $\alpha$  is a limit ordinal, we define  $G$  to be the ordinary (restricted) direct product of the groups  $G_\beta$ , for all  $\beta < \alpha$ . Then  $G$  has the properties 1–3, and thus has type  $Q_\alpha$ . For the case  $\alpha = \beta + 1$  we shall construct  $G$  by ‘wreathing’ the regular

representation of  $G_\beta$  with a certain kind of infinite centreless  $FC$ -group of permutations of the positive integers. (For convenience, we say that a group is *centreless* if its centre consists of the unit element alone.)

DEFINITION. A faithful representation of a group  $G$  by permutations of the positive integers will be called a *special* representation of  $G$  if

- (i) the stabiliser of each integer has finite index in  $G$  and
- (ii) the intersection of the stabilisers of the elements of any set of all but a finite number of these integers is the unit subgroup.

DEFINITION. An infinite centreless  $FC$ -group possessing a special representation will be called a group of type  $F$ .

To construct an example of a group of type  $F$ , let  $D = B_1 \times B_2 \times \dots$  be the ordinary direct product of an infinite sequence of finite centreless groups  $B_i$ ,  $i = 1, 2, \dots$ . Let  $D_n = B_{n+1} \times B_{n+2} \times \dots$ , let  $k_n$  be the order of  $D/D_n$  and let the elements of  $D/D_n$ , in an arbitrary order, be

$$X_1^n, X_2^n, \dots, X_{k_n}^n.$$

For each element  $g \in D$  and each  $n = 1, 2, \dots$ , define the permutation  $\pi_{gn}$  of the integers  $1, 2, \dots, k_n$  by the rule

$$(1) \quad \pi_{gn}(i) = j \text{ when } gX_i^n = X_j^n.$$

Now, for each  $g \in G$ , define the permutation  $\pi_g$  of the positive integers by the rule

$$(2) \quad \pi_g\left(i + \sum_{j=1}^{n-1} k_j\right) = \pi_{gn}(i) + \sum_{j=1}^{n-1} k_j,$$

for all  $i = 1, 2, \dots, k_n$ , and  $n = 1, 2, \dots$ . The systems of transitivity in this permutation representation of  $D$  are the sets  $T_n$  of integers  $m$  such that  $\sum_{i=1}^{n-1} k_i < m \leq \sum_{i=1}^n k_i$ , for  $n = 1, 2, \dots$ . If  $m \in T_n$ , then the subgroup  $D_n$  of  $D$  is contained in the stabiliser of  $m$ . Hence the stabiliser in  $D$  of each positive integer has finite index in  $D$ . On the other hand, suppose  $g$  is in the stabiliser in  $D$  of all but a finite number of the positive integers. Then there is a number  $n_0$  such that  $g$  is in the stabiliser of each integer of each system  $T_n$  with  $n \geq n_0$ . So if  $i$  is any integer in the range  $1 \leq i \leq k_n$ ,  $n \geq n_0$ , we know that  $g$  is in the stabiliser of  $i + \sum_{j=1}^{n-1} k_j$ , and this means that  $gX_i^n = X_i^n$ . Thus  $g \in D_n$ . But the subgroups  $D_n$ , with  $n \geq n_0$ , intersect in the unit subgroup of  $D$ . So  $g = 1$ . We observe also that the permutation representation of  $D$  defined by (1) and (2) is faithful. Thus we have a special representation of the infinite centreless  $FC$ -group  $D$ , which is therefore a group of type  $F$ .

LEMMA. If  $G_\beta$  is a group of type  $Q_\beta$  and  $J$  is a group of type  $F$ ,

then a group  $G$  formed by wreathing the regular representation of  $G_\beta$  with a special representation  $R$  of  $J$  is a group of type  $Q_{\beta+1}$ .

*Proof.* The wreath group  $G$  may be regarded as a semi-direct product

$$G = KE, \quad K \cap E = 1,$$

where  $K = \prod_{i=1}^\infty A_i$  is the direct product of a sequence of groups, each isomorphic to  $G_\beta$ , and  $E$  is isomorphic to  $J$ . The automorphisms of  $K$  induced by elements of  $E$  permute the subgroups  $A_i, i = 1, 2, \dots$ , realizing the special representation  $R$  of  $J \simeq E$ . Associated with  $G$  is a set of isomorphisms  $\theta_{ij}, i, j = 1, 2, \dots$  such that  $\theta_{ij}(A_i) = A_j$ , and if  $a \in A_i, g \in E$  and  $g^{-1}A_i g = A_j$ , then  $g^{-1}ag = \theta_{ij}(a)$ .  $\theta_{ii}$  is the identity automorphism, for all  $i$ . (A brief general description of wreath groups, and further references, may be found in Hall [5].)

Let  $C_i$  be the set of all elements  $g$  in  $E$  such that  $g^{-1}A_i g = A_i$ . Then  $C_i$  is the centraliser in  $E$  of each element of  $A_i$ . Since the representation  $R$  is special, the subgroup  $C_i$  of  $E$  has finite index in  $E$ , for each  $i$ , and the unit element is the only element of  $E$  common to all the subgroups of any set of all but a finite number of the  $C$ 's.

For all  $\gamma \leq \beta$ , put  $H_\gamma = H_\gamma(K)$ , the  $\gamma$ th term of the upper FC-series of  $K$ . If possible, let  $\tau + 1$  be the least such ordinal for which  $H_{\tau+1}(G) \neq H_{\tau+1}$ . Now any element  $k$  of  $K$  can be written as the product of a finite number of elements  $a_{i_\nu} \in A_{i_\nu}, \nu = 1, 2, \dots, n$ , and the subgroup  $C(k) = \bigcap_{\nu=1}^n C_{i_\nu}$  has finite index in  $E$ . But  $C(k)$  is contained in the centraliser of  $k$  in  $E$ , so  $g^{-1}kg$ , with  $g \in E$ , is finite valued. Hence

$$H_{\tau+1}(G) \cap K = H_{\tau+1}.$$

Suppose  $kg \in H_{\tau+1}(G)$ , where  $k \in K$  and  $g \in E, g \neq 1$ . Let  $\sigma + 1$  be the least ordinal in the range  $\tau + 1 \leq \sigma + 1 \leq \beta$  such that  $k \in H_{\sigma+1}$ . Now  $H_\sigma$  is a characteristic subgroup of  $K$ , and hence is normal in  $G$ , and both  $kH_\sigma$  and  $kgH_\sigma$  are FC elements of  $G/H_\sigma$ . Hence  $gH_\sigma$  is FC in  $G/H_\sigma$ .

We can choose an infinite sequence of distinct positive integers,  $\mu_1, \mu_2, \dots$ , such that  $g^{-1}A_{\mu_i} g \neq A_{\mu_i}$ , for all  $i = 1, 2, \dots$ , for otherwise  $g$  would belong to all but a finite number of the  $C$ 's. Moreover, since  $C_i$  has finite index in  $E$ , for each  $i$ , we can choose the sequence  $\mu_1, \mu_2, \dots$  so that distinct terms belong to distinct systems of transitivity in the representation  $R$  of  $E$ . By relabelling the subgroups  $A_i, i = 1, 2, \dots$ , we may arrange that the sequence  $\mu_1, \mu_2, \dots$  is just the sequence of odd positive integers. So if  $n$  is any odd positive integer, and  $g^{-1}A_n g = A_{\dot{n}}$ , then  $\dot{n}$  is even. Since  $\sigma < \beta$ , we can choose

$a_n \in A_n - H_\sigma(A_n)$ , for  $n = 1, 3, \dots$ . Let  $a_n = g^{-1} a_n g$ , and define

$$c_n = g^{-1} g^{a_n} = a_n^{-1} a_n, \quad n = 1, 3, \dots$$

Then

$$c_n^{-1} c_m = (g^{a_n})^{-1} g^{a_m} = a_n^{-1} a_n a_m^{-1} a_m.$$

If  $n \neq m$ , the four integers  $n, \dot{n}, m$  and  $\dot{m}$  are all distinct and thus  $(g^{a_n})^{-1} g^{a_m} \notin H_\sigma$ . Thus  $gH_\sigma$  is not  $FC$  in  $G/H_\sigma$ , contrary to what we have already proved.

It follows that the upper  $FC$ -series of  $G$  is

$$\{1\} = H_0 \leq H_1 \leq \dots \leq H_\beta = K < G,$$

for  $G/K \simeq E \simeq J$ , and  $J$  is an  $FC$ -group. Moreover  $J$  is infinite and centreless, and the factors  $H_{\gamma+1}/H_\gamma$  are infinite and centreless, for all  $\gamma < \beta$ , since  $G_\beta$  is a group of type  $Q_\beta$ , and  $K$  is a direct product of groups isomorphic with  $G_\beta$ . Thus  $G$  is a group of type  $Q_{\beta+1}$ , as required.

We have now shown how to construct a group of type  $Q_\alpha$ , given groups of type  $Q_\beta$  for all  $\beta < \alpha$ , whether  $\alpha$  is a limit ordinal or not. So, by transfinite induction, we have :

**THEOREM.** *There exist groups of type  $Q_\alpha$ , for any ordinal  $\alpha$ .*

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