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Introduction. Riemann surfaces were originally introduced as a tool for the study of multiple valued analytic functions. In Riemann's work they appear as covering surfaces of the complex plane with given branch points. Since then Riemann surfaces have been considered from several different aspects.

Here we shall follow the point of view assumed by Beltrami and Klein, who visualized these surfaces as two-dimensional submanifolds of Euclidean space whose conformal structure is defined by the surrounding metric.

Recent results of J. Nash[1] on isometric imbeddings of Riemannian manifolds assure that all models of Riemann surfaces with the natural Poincaré metric can be C^∞ isometrically imbedded in a sufficiently high (51) dimensional Euclidean space. However, the question still remains open whether or not every Riemann surface has a conformally equivalent representative in the ordinary three-dimmensional space.

Although the dimension requirement seems restrictive, there is reason to believe that, since only conformality is required, at least the compact surfaces can be conformally imbedded. We shall not be directly concerned here with this existence problem; instead, we shall present a family of elementary surfaces which may contain all conformal types and whose conformal structure can be easily characterized.

In the genus one case, the conformal structure is usually described by a complex parameter ν which gives the ratio of two principal periods of an abelian differential of the surface. It is always possible to choose these periods so that their ratio ν lies in the region $\mathfrak{M}$ of the Gauss plane defined by the inequalities:

$$\mathfrak{Im}\nu < 0,\ -\tfrac{1}{2} < \mathfrak{Re}\nu \leqq \tfrac{1}{2}\,;\ |\nu| > 1 \text{ for } \mathfrak{Re}\nu < 0,\ |\nu| \geqq 1 \text{ for } \mathfrak{Re}\nu \geqq 0\,.$$

It is well known that every Riemann surface of genus one has in $\mathfrak{M}$ one and only one representative point.

It is easy to verify that the representative points ν of the tori of revolution lie in the imaginary axis and cover it completely. Thus it seems plausible that the affine images of the tori of revolution should cover all conformal types in the genus one case; however, we have

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[1] "The imbedding problem for Riemannian manifolds". *Annals of Mathematics*, 63 (1956), pp. 20-63.

found no proof of this fact. Indeed the characterization of the parameter ν for an imbedded surface leads in general to rather difficult problems.

For this reason, for quite some time there have been no known examples of surfaces whose representative point in $\mathfrak{M}$ lies off the imaginary axis. In 1944, O. Teichmüller[2] proved the existence of these surfaces by showing that there are small deformations of the tori of revolution for which the variation of ν is not purely imaginary.

Led by these observations we have tried to develop a method of uniformizing a given Riemann surface that could be of practical application for some wide enough family of surfaces. To make our considerations applicable to surfaces of higher genus we needed to introduce some parameters to take the role that ν plays in the genus one case. To this end we have adopted as a canonical form of a Riemann surface the result of the Schottky uniformization. In fact, some imbedded surfaces can be considered topologically "marked" in a natural way, and the Schottky uniformization associates with every marked surface of genus g (>1) a complete set of geometrical invariants which can be expressed by means of $3g-3$ independent complex parameters.

In view of the importance of these parameters we deemed necessary to include in the first section of this paper a description of the Schottky uniformization and some general facts associated with it. In the second section we present a definition of "M-surfaces". These are imbedded surfaces which may have edge type singularities along curves but can be made into Riemann surfaces in a natural way. To generate these surfaces we adopt a process which uses surfaces of genus zero as building blocks to construct surfaces of genus one and surfaces of genus one to construct surface of higher genus.

In the third section we present a method of constructing the Schottky uniformization of a given M-surface. This method is more general than it appears in the context since from the existence of the Schottky uniformization, every marked surface can be considered an M-surface (dropping the condition that the building surfaces of genus zero should be globally imbedded.) As will be shown in the fourth section, this method assumes practical importance when the building blocks of M-surfaces are ordinary spheres. These special M-surfaces we have called "natural".

To present our results in this case we made use of anallagmatic coordinates of spheres as introduced by E. Cartain in [2]; for the sake of completeness a brief introduction to these coordinate is also included.

In the last section a few properties of natural M-surfaces of genus

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[2] "Beweis der analytischen Abhängigkeit des konformen Moduls einer analytischen Ringflächenschar von den Parametern", Deutsche Math. **7** (1944), 309–336.

one are studied, and some of the results are used to construct the Teichmüller models. At the end a process is given by means of which all natural M-surfaces can be made into C^∞ smooth canal surfaces.

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1. A choice of conformal parameters for compact Riemann surfaces.

1.1 Here and in the following Σ shall denote a given 2-sphere; "a coordinate in Σ" shall mean an extended valued complex coordinate introduced by a stereographic projection of Σ upon the Gauss-plane. Let z be such a coordinate. Since z is defined up to a Moebius transformation of Σ onto itself, we can assume that the points $0, 1, \infty$ are situated wherever we may wish. Whenever it does not lead to ambiguities, we shall make use of the same symbol for a point of Σ and its complex coordinate.

If Λ is a Jordan curve and α a point of Σ not lying in Λ, we shall denote by $\Lambda(\alpha)$ the connected component of $\Sigma - \Lambda$ which contains α. $\Lambda(\alpha)$ will be called the interior of Λ with respect to α. If Λ separates α from another point β of Σ we have of course

$$\Sigma = \Lambda(\alpha) + \Lambda + \Lambda(\beta)\,.$$

Let now α_i, β_i $(i = 1, 2, \cdots, g)$ be $2g$ distinct points of Σ and ω_i $(i = 1, 2, \cdots, g)$ given complex numbers of absolute value greater than one. Let τ_i be the Moebius transformation of Σ onto itself defined by the equation

$$\frac{\tau_i z - \alpha_i}{\tau_i z - \beta_i} = \omega_i \frac{z - \alpha_i}{z - \beta_i} \tag{1}$$

We assume for a moment that $\alpha_1 = 0$ and $\beta_1 = \infty$. Under this coordinate system we have

$$\tau_1 z = \omega_1 z\ .$$

Let ρ_1 and ρ_2 be the smallest and the largest of the absolute values

$$|\alpha_i|\,,\ |\beta_i| \qquad\qquad i = 2, 3, \cdots, g\ .$$

If $|\omega_1| > (1/\eta)(\rho_2/\rho_1)$ for some $0 < \eta < 1$, a circle with center at 0 and radius $r = \eta\rho_1$ is transformed by τ_1 onto a concentrical circle of radius

$$r' = |\omega_1| r > \rho_2 .$$

Thus if $|\omega_1| > \rho_2/\rho_1$ there are infinitely many circles Λ such that the points $\alpha_2, \beta_2; \cdots; \alpha_g, \beta_g$ are all interior to the anulus

$$\Lambda(\infty) \cap \tau_1\Lambda(0) .$$

Before expressing this fact in an invariant way we shall introduce a notation. If α and β are two distinct points of Σ by $P(\alpha, \beta)$ we shall denote the pencil of circles which admit α, β as a couple of inverse points.

We have thus shown that:

I. *Provided* $|\omega_1|$ *is sufficiently large we can choose a circle* Λ *in an infinite number of ways so that*

(a) $\Lambda \in P(\alpha_1, \beta_1)$

(b) *the points* $\alpha_2, \beta_2; \cdots: \alpha_g, \beta_g$ *are contained in the domain* $\Lambda(\beta_1) \cap \tau_1\Lambda(\alpha_1)$.

Let Λ_1 be one of these circles.

We shall show now that:

II. *Provided the* $|\omega_i|$*'s are sufficiently large the circles* Λ_i *can be chosen in an infinite number of ways so that*

(a) $\Lambda_i \in P(\alpha_i, \beta_i)$

(b) *the closed disks*

$$\overline{\Lambda_1(\alpha_1)},\ \overline{\tau_1\Lambda_1(\beta_1)},\ \cdots,\ \overline{\Lambda_g(\alpha_g)},\ \overline{\tau_g\Lambda_g(\beta_g)}$$

are exterior to each other.

Because of I we can prove II inductively.

Suppose that the circles $\Lambda_1, \Lambda_2, \cdots, \Lambda_{i-1}$ have been chosen in such a way that

(a) $\Lambda_j \in P(\alpha_j, \beta_j)$ $\qquad (j = 1, 2, \cdots, i-1)$,

(b) the closed disks $\overline{\Lambda_1(\alpha_1)}$, $\overline{\tau_1\Lambda_1(\beta_1)}$; $\cdots$; $\overline{\Lambda_{i-1}(\alpha_{i-1})}$, $\overline{\tau_{i-1}\Lambda_{i-1}(\beta_{i-1})}$ are exterior to each other,

(c) the remaining points α_j, β_j $(j = i, i+i, \cdots, g)$ are contained in the domain

$$\bigcap_{j=1,i-1} \{\Lambda_j(\beta_j) \cap \tau_j\Lambda_j(\alpha_j)\} .$$

We temporarily assume that $\alpha_i = 0$ and $\beta_i = \infty$. We let S be the set consisting of the closed disks

$$\overline{\Lambda_1(\alpha_1)},\ \overline{\tau_1\Lambda_1(\beta_1)},\ \cdots,\ \overline{\Lambda_{i-1}(\alpha_{i-1})},\ \overline{\tau_{i-1}\Lambda_{i-1}(\beta_{i-1})}$$

and (if $i < g$) the points

$$\alpha_{i+1},\ \beta_{i+1},\ \cdots,\ \alpha_g,\ \beta_g .$$

Under this coordinate system let ρ_1 and ρ_2 be the minimum and the maximum value assumed by $|z|$ as z varies in S. Clearly the argument can be completed since, for the same reasons as before, if $|\omega_i| > \rho_2/\rho_1$, the circle Λ can be chosen in an infinite number of ways so that

(a) $\Lambda \in P(\alpha_i, \beta_i)$

(b) the set S is exterior to $\overline{\Lambda(\alpha_i)}$ and $\overline{\tau_i\Lambda(\beta_i)}$. Let Λ_i be one of these circles.

A further investigation on the nature of the inequalites to which the $|\omega_i|$'s are to be subjected, for such a construction to be possible would be of some interest, but for our immediate purposes it is not needed.

We would like to point out, however, that if for a given set of complex numbers $\{\alpha_1, \beta_1, \omega_1; \cdots; \alpha_g, \beta_g, \omega_g\}$ the construction in II is possible, then it is also possible for any other set $\{\alpha_1, \beta_1, \omega_i'; \cdots; \alpha_g, \beta_g, \omega_g'\}$ such that

$$|\omega_i'| \geqq |\omega_i| \qquad\qquad i = 1, 2, \cdots, g\ .$$

1.2. Let $\mathfrak{M}_g$ be the subset of the $3g$-dimensional complex cartesian space composed of those points

$$m \sim \{\alpha_1, \beta_1, \omega_1; \cdots; \alpha_g, \beta_g, \omega_g\}$$

for which it is possible to choose g Jordan curves $\Lambda_1, \Lambda_2, \cdots, \Lambda_g$ of Σ such that

(a) each Λ_i separates α_i from β_i,

(b) the closed sets $\overline{\Lambda_1(\alpha_1)}, \overline{\tau_1\Lambda_1(\beta_1)}, \cdots, \overline{\Lambda_g(\alpha_g)}, \overline{\tau_g\Lambda_g(\beta_g)}$[3] are exterior to each other.

III. *The points of $\mathfrak{M}_g$ give rise to compact Riemann surfaces of genus g.*[4]

If $m \sim \{\alpha_1, \beta_1, \omega_1; \cdots; \alpha_g, \beta_g, \omega_g\}$ and $\Lambda_1, \Lambda_2, \cdots, \Lambda_g$ are chosen to satisfy (a) and (b), we set

$$R = \bigcap_{i=1,g} \{\overline{\Lambda_i(\beta_i)} \cap \overline{\tau_i\Lambda_i(\alpha_i)}\}\ .$$

We then identify the points of the boundaries Λ_i and $\tau_i\Lambda_i$ of R by means of the transformation τ_i. In other words we set $Q \sim \tau_i Q$ for each $Q \in \Lambda_i$. We do this for $i = 1, 2, \cdots, g$. Let X denote the resulting space.

We shall make X into a Riemann surface introducing local uniformizers.

[3] Here the τ_i's are again given by (1).

[4] The construction presented here is to some extent contained in a paper of Schottky published in Crelle's Journal (1887, cfr. **[8]**). See also Hurwitz-Courant **[5]**, p. 462.

If P is a point of X which is interior to R and N is a neighborhood of P contained in R we take as a local uniformizer any coordinate in Σ which does not attain the value ∞ within N.

If P is a point of X which lies on one of the Λ's, say Λ_i, we have to proceed in a different way.

First we take a neighborhood N of P *in* Σ which is so small that it is contained in the set

$$R \cup \tau_i^{-1} R .$$

Then we define a corresponding neighborhood N^* of P *in* X by setting

$$N^* = \{\overline{\Lambda_i(\beta_i)} \cap N\} + \tau_i \{\overline{\Lambda_i(\alpha_i)} \cap N\} = R \cap (N + \tau_i N) .$$

If $z(p)$ is a coordinate in Σ which does not attain the value ∞ in N, we introduce as a local uniformizer in N^* the function on X which takes the value $z(p)$ for $p \in R \cap N$ and the value $z(\tau_i^{-1} p)$ for a point p of $R \cap \tau_i N$.

We proceed in a similar way for each of the curves Λ_i. The resulting manifold is a Riemann surface of genus g; it will be denoted by $\Gamma(m; \Lambda_1, \Lambda_2, \cdots, \Lambda_g)$ and referred to as a "Schottky model".

1.3. We shall give statement III a more precise meaning by showing that

IV. *Any two surfaces $\Gamma(m; \Lambda_1, \Lambda_2, \cdots, \Lambda_g)$ and $\Gamma'(m; \Lambda'_1, \Lambda'_2, \cdots, \Lambda'_g)$ (same m), are conformally equivalent.*

Let G be the group of Moebius transformations generated by the τ_i's. G constitutes what is usually called a "Schottky group".

We shall denote by $\hat{\Gamma}(m)$ the set obtained from Σ by deleting the limit points of G.

The following properties of G are well known (cfr. for instance [4] pages 37 to 66), and can be easily established:

(a) The group G is free.

(b) The sets $D = \bigcap_{i=1,g} \{\overline{\Lambda_i(\beta_i)} \cap \tau_i \Lambda_i(\alpha_i)\}$ and $D' = \bigcap_{i=1,g} \{\overline{\Lambda'_i(\beta_i)} \cap \tau_i \Lambda'_i(\alpha_i)\}$ are fundamental regions of G.

(c) The images of D (as well as those of D') decompose and cover completely the set $\hat{\Gamma}(m)$, i.e. $\hat{\Gamma}(m) = \sum_{\tau \in G} \tau D = \sum_{\tau \in G} \tau D'$[5].

These relations yield

$$D = \sum_{\tau \in G} D \cap \tau D' \tag{3}$$

$$D' = \sum_{\tau \in G} D' \cap \tau D ; \tag{4}$$

[5] We should emphasize that $\hat{\Gamma}(m)$ is a *disjoint* union of the images of D and D'.

since D and D' are bounded away from the limit points of G[6] both these sums, after a finite number of terms, terminate with a string of empty sets. The equality in (3) is also equivalent to

$$D = \sum_{\tau \in G} D \cap \tau^{-1}D' \tag{5}$$

and (4) can be written in the form

$$D' = \sum_{\tau \in G} \tau(D \cap \tau^{-1}D') . \tag{6}$$

We define a mapping[7] φ: $D \leftrightarrow D'$ by setting

$$\varphi p = \tau p \text{ for } p \in D \cap \tau^{-1}D' .$$

Since the unions on the right hand sides of (5) and (6) are disjoint φ is well defined. Clearly φ preserves the identification of points in Γ and Γ' and thus defines a topological mapping of Γ onto Γ', in addition it maps every sufficiently small neighborhood of Γ conformally onto neighborhood of Γ'.

From this the assertion follows.

1.4. The abstract Riemann surface represented by any one of the surfaces $\Gamma(m; \Lambda_1, \Lambda_2, \cdots, \Lambda_g)$ shall be denoted by $\Gamma(m)$; it shall be referred to as "the Schottky model corresponding to m."

Suppose now that there exists a Moebius transformation of Σ onto itself which sends the points $\alpha_1, \beta_1; \cdots; \alpha_g, \beta_g$ respectively onto the points $\alpha'_1, \beta'_1; \cdots; \alpha'_g, \beta'_g$ and assume that the parameters $\omega_1, \omega_2, \cdots, \omega_g$ have been chosen in such a way that both $m \sim (\alpha_1, \beta_1, \omega_1; \cdots; \alpha_g, \beta_g, \omega_g)$ and $m' \sim (\alpha'_1, \beta'_1, \omega_1; \cdots; \alpha'_g, \beta'_g, \omega_g)$ lie in $\mathfrak{M}_g$. Then the corresponding models $\Gamma(m)$ and $\Gamma(m')$ are conformally equivalent. Under these circumstances, it is natural to identify any two points m and m' of $\mathfrak{M}_g$ for which we have

$$\begin{aligned} &\omega_i = \omega'_i , \\ &\text{if } g \geqq 2 \ (\beta_i, \alpha_1, \alpha_2, \beta_1) = (\beta'_i, \alpha'_1, \alpha'_2, \beta'_1)^{8} \qquad i = 2, \cdots, g , \\ &\text{if } g \geqq 3 \ (\alpha_i, \alpha_1, \alpha_2, \beta_1) = (\alpha'_i, \alpha'_1, \alpha'_2, \beta'_1) \qquad i = 3, \cdots, g . \end{aligned} \tag{7}$$

If Γ is a Riemann surface of genus g, the Jordan curves $\Lambda_1, \Lambda_2, \cdots, \Lambda_g$ will be said to form a "canonical semi-basis" if they can be completed to a canonical basis for the cycles of Γ.

The Riemann surface Γ will be said "marked" if a canonical semi-

[6] The limit points of G are contained in the sets $\tau_i^{-1}\Lambda_j(\alpha_j)$, $\tau_i^{-1}\tau_j\Lambda_j(\beta_j)$ and $\tau_i\Lambda_j(\alpha_j)$ $(i, j = 1, 2, \cdots, g)$.

[7] Here and in the following a "mapping" shall mean a "one-to-one mapping".

[8] By the symbol (x, y, z, w) where x, y, z, w are given distinct complex numbers we mean the cross-ratio $(x-y)(z-w)/(x-w)(z-y)$.

basis has been chosen in Γ. The surface Γ marked by $A_1, A_2, \cdots, A_g$ shall be denoted by the symbol $\Gamma(A_1, A_2, \cdots, A_g)$.

We shall consider two marked surfaces $\Gamma(A_1, A_2, \cdots, A_g)$ and $\Gamma'(A'_1, A'_2, \cdots, A'_g)$ as the same object whenever $\Gamma \sim \Gamma'$ (conformally) and A_i is homotopic to A'_i (for $i = 1, 2, \cdots, g$). With these identifications the following theorem holds:

V. *The points of $\mathfrak{M}_g$ are in a one-to-one correspondence with the marked Riemann surfaces of genus g.*

Proof. Clearly, every Schottky model $\Gamma(m; A_1, A_2, \cdots, A_g)$ can be considered a marked surface by the choice of $A_1, A_2, \cdots, A_g$ as a canonical semi-basis.

But the converse is also true: namely, to each marked surface $\Gamma(A_1, A_2, \cdots, A_g)$ there corresponds a Schottky model, uniquely defined up to a Moebius transformation, and thereby a point of $\mathfrak{M}_g$. This correspondence is easily established after constructing the so-called "Schottky covering surface" of each marked surface. This concept is well known (see for instance [4], pp. 256–257), but for the sake of completeness, we shall sketch its definition.

Let $\Gamma(A_1, A_2, \cdots, A_g)$ be a given marked surface.

Let $M_1, M_2, \cdots, M_g$ be a completion of $A_1, A_2, \cdots, A_g$ to a canonical basis, and $\mathscr{M}$ denote the free group generated by the cycles $M_1, M_2, \cdots, M_g$.

We imagine the surface $\Gamma(A_1, A_2, \cdots, A_g)$ cut along the curves A_i to yield a planar region X bounded by the $2g$ Jordan curves $A_1, A_2, \cdots, A_g$; $A_1^{-1}, A_2^{-1}, \cdots, A_g^{-1}$[9] of Γ. We then reproduce an infinite number of exact replicas X_M of X, one for each $M \in \mathscr{M}$. The closed sets $\overline{X}_M$ are then glued together according to the following rules:

(i) If $M = M_i M^*$ (and the first factor of M^* is not M_i^{-1}) then the points of the curve A_i^{-1} of $\overline{X}_{M^*}$ are identified with the corresponding ones in the curve A_i of $\overline{X}_M$.

(ii) If $M = M_i^{-1} M^*$ (and the first factor of M^* is not M_i) then the points of the curve A_i of $\overline{X}_{M^*}$ are identified with the corresponding ones in the curve A_i^{-1} of $\overline{X}_M$.

With these identifications the set $\sum_{M \in \mathscr{M}} \overline{X}_M$ becomes a covering surface of Γ. We shall denote it $\hat{\Gamma}_A$ and call it the "Schottky covering surface" of $\Gamma(A_1, A_2, \cdots, A_g)$.

What then remains to be proved is a consequence of the following well known properties of the surface $\hat{\Gamma}_A$. (cfr. for instance [5] pp. 483-484 or [4] Chapter X).

[9] We tacitly assume, without restriction, that the curves A_i do not intersect each other.

(a) $\hat{\Gamma}_A$ is of planar character, it can be conformally mapped into the sphere Σ.

(b) The mapping μ_i of $\hat{\Gamma}_A \leftrightarrow \hat{\Gamma}_A$ which sends each region X_M of $\hat{\Gamma}_A$ onto the adjacent region $X_{M_i M}$ is a cover transformation of $\hat{\Gamma}_A$.

(c) The group of cover transformations of $\hat{\Gamma}_A$ is free and admits the mappings $\mu_1, \mu_2, \cdots, \mu_g$ as generators.

(d) If φ is any conformal mapping of $\hat{\Gamma}_A$ into Σ, the cover transformations of $\hat{\Gamma}_A$ induce in Σ, through the mapping φ, a set G of Moebius transformations which is a Schottky group. The generators of G are given by the Moebius transformations

$$\tau_1 = \varphi\mu_1\varphi^{-1},\ \tau_2 = \varphi\mu_2\varphi^{-1},\ \cdots,\ \tau_g = \varphi\mu_g\varphi^{-1}\ .$$

(e) The image φX_E of X_E (where by E we mean the identity in $\mathscr{M}$) constitutes a fundamental region for G; its boundary consists of the curves $\varphi A_1, \varphi A_2, \cdots, \varphi A_g$; $\varphi A_1^{-1}, \varphi A_2^{-1}, \cdots, \varphi A_g^{-1}$, and φA_i^{-1} is the image of φA_i under the transformation τ_i for each i.

Thereby φX_E and $\tau_1\tau_2, \cdots, \tau_g$ originate a Schottky model which is conformally equivalent to $\Gamma(A_1, A_2, \cdots, A_g)$.

(f) If φ' is any other conformal mapping of $\hat{\Gamma}_A$ into Σ, $\varphi'\varphi^{-1}$ induces a Moebius transformation of Σ; thus, if we set

$$\tau_i = \varphi\mu_i\varphi^{-1},\ \tau_i' = \varphi'\mu_i\varphi'^{-1}$$

and

$$\frac{\tau_i z - \alpha_i}{\tau_i z - \beta_i} = \omega_i \frac{z - \alpha_i}{z - \beta_i},\quad \frac{\tau_i' z - \alpha_i'}{\tau_i' z - \beta_i'} = \omega_i' \frac{z - \alpha_i'}{z - \beta_i'}$$

(under some coordinate system in Σ), the corresponding points

$$m \sim (\alpha_1,\ \beta_1,\ \omega_1;\ \cdots;\ \alpha_g,\ \beta_g,\ \omega_g)$$
$$m' \sim (\alpha_1',\ \beta_1',\ \omega_i';\ \cdots;\ \alpha_g',\ B_g',\ \omega_g')$$

of $\mathfrak{M}_g$ are to be considered the same since the equalities in (7) will necessarily be satisfied.

1.5. After Statement V it is natural to adopt the following:

DEFINITION. If $\Gamma(A_1, A_2, \cdots, A_g)$ is a given marked Riemann surface and $m \sim \{\alpha_1, \beta_1, \omega_1; \cdots; \alpha_g, \beta_g, \omega_g\}$ is the point of $\mathfrak{M}_g$ corresponding to it, the complex numbers

$$\begin{aligned} &\omega_1,\ \omega_2,\ \cdots,\ \omega_g \\ &\omega_{i+g-1} = (\beta_i,\ \alpha_1,\ \alpha_2,\ \beta_1) \quad (i = 2, \cdots, g \text{ if } g \geqq 2) \\ &\omega_{i+2g-3} = (\alpha_i,\ \alpha_1,\ \alpha_2,\ \beta_1) \quad (i = 3, \cdots, g \text{ if } g \geqq 3) \end{aligned} \tag{8}$$

will be called *"the conformal parameters"* of $\Gamma(\Lambda_1, \Lambda_2, \cdots, \Lambda_g)$.

In the following we shall say that a marked Riemann surface $\Gamma(\Lambda_1, \Lambda_2, \cdots, \Lambda_g)$ has been "uniformized" if the mapping of $\hat{\Gamma}_\Lambda$ into Σ and the conformal parameters of $\Gamma(\Lambda_1, \Lambda_2, \cdots, \Lambda_g)$ have been characterized.

It is interesting to note that Schottky in [8] expressed the abelian differentials and their periods as analytic functions of the parameters $\alpha_1, \beta_1, \omega_1; \cdots; \alpha_g, \beta_g, \omega_g$; unfortunately, there are some restrictive hypotheses in his proofs, and the results, although explicit, assume formidable expressions.

2. Some special models of compact Riemann surfaces.

2.1. The three-dimensional Euclidean space shall be denoted by E_3. Any smooth (four times continuously differentiable), non self-intersecting surface of E_3, homeomorphic to a sphere, shall be called a p-sphere.

A p-sphere shall always be assumed to have been assigned a specific orientation.

Let Λ be a Jordan curve of a p-sphere Γ. If α is a point of Γ not lying in Λ, as before, we shall denote by $\Lambda(\alpha)$ the connected component of $\Gamma - \Lambda$ which contains α.

We can define an orientation of Λ by specifying which of the two connected components of $\Gamma - \Lambda$ is to be the interior or the exterior of Λ; conversely if Λ has been oriented, we can accordingly speak of the interior and the exterior of Λ in Γ. To this end we shall adopt the following convention:

If Q is a point of Λ, $\boldsymbol{t}$ and $\boldsymbol{b}$ are unit vectors having respectively the direction of the positive tangent to Λ and the positive normal to Γ at Q, and if the unit vector $\boldsymbol{n}$, normal to Λ and tangent to Γ at Q, points towards the interior of Λ, then the ordered triplet $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ should form a left handed frame.

Any oriented surface of E_3 can be made into a Riemann surface in a natural way by means of the conformal structure induced by the surrounding metric. In this fashion every p-sphere can be considered a compact Riemann surface of genus zero, and therefore it can be mapped conformally onto a sphere.

2.2. Let Σ be a sphere, and z a complex coordinate in Σ. If Γ is a p-sphere, let $z = \varphi p$ be a conformal mapping of Γ onto Σ. By means of φ we can transfer to Γ several conformally invariant properties of Σ. We shall define the cross-ratio of any four points $\alpha, \beta, \gamma, \delta$ of Γ by setting

$$(\alpha, \beta, \gamma, \delta) = (\varphi\alpha, \varphi\beta, \varphi\gamma, \varphi\delta) . \tag{1}$$

The right hand side of (1) is independent of the mapping φ. In fact, if ψ is any other conformal mapping of Γ onto Σ, the mapping $\tau = \psi\varphi^{-1}$ of Σ onto itself is conformal and necessarily a Moebius transformation.

A Jordan curve Λ of Γ will be called a p-circle if the cross ratio of any four points of Λ is real; i.e., if the curve $\varphi\Lambda$ is a circle in Σ.

If Λ is a p-circle of Γ and α, β, γ are distinct points of Λ by an "inversion with respect to Λ" we shall mean the transformation σ defined by the equation

$$(\sigma p,\ \alpha,\ \beta,\ \gamma) = \overline{(p,\ \alpha,\ \beta,\ \gamma)} \tag{2}$$

the bar meaning complex conjugation. Clearly $\varphi\sigma\varphi^{-1}$ is in Σ an inversion with respect to the circle $\varphi\Lambda$.

The most general conformal mapping τ of Γ onto itself is determined by the images α', β', γ' of any three distinct points α, β, γ of Γ, and its equation can be written in the form

$$(\tau p,\ \alpha',\ \beta',\ \gamma') = (p,\ \alpha,\ \beta,\ \gamma)\ .$$

Such a mapping will be referred to as "a Moebius transformation of the p-sphere Γ".

We will find it convenient, in order to avoid having to refer back to the sphere Σ, to consider Schottky models imbedded in a p-sphere. Indeed, the construction of these models can be carried out for p-spheres in exactly the same way it was done in the last section for ordinary spheres; thus we shall not repeat it.

2.3. Let Γ_1 and Γ_2 be two p-spheres which intersect along a Jordan curve Λ. Suppose that there exists a conformal mapping φ of Γ_1 onto Γ_2 which leaves fixed the points of the intersection Λ.

The mapping φ is unique.

In fact, if ψ is another conformal mapping of Γ_1 onto Γ_2 which leaves the points of Λ fixed, then the mapping $\psi\varphi^{-1}$: $\Gamma_2 \leftrightarrow \Gamma_2$ leaves more than three points fixed and must necessarily be the identity.

This shows that φ is completely determined by the conditions imposed on it by three distinct points of the curve Λ, hence φ may not exist if the intersection of Γ_1 and Γ_2 is arbitrary.

A class of examples of couples of intersecting p-spheres for which such a mapping exists can be obtained by constructing surfaces which have a common axis of revolution and intersect along a common parallel, then taking their images under arbitrary Moebius transformations of space.

Suppose now that the finite ordered set of p-spheres $\Gamma_0, \Gamma_1, \cdots, \Gamma_{n-1}$ is such that for each $i = 1, 2, \cdots, n$:

(a) The surface Γ_{i-1} intersects the successive one Γ_i along a Jordan curve Λ_i which we shall suppose sufficiently well behaved. (We set $\Lambda_n = \Lambda_0$, $\Gamma_n = \Gamma_0$).

(b) There exists a conformal mapping Δ_i of Γ_{i-1} onto Γ_i which leaves fixed the points of the curve Λ_i.

(c) Λ_{i-1} has on points in common with Λ_i.

Let each Λ_i be oriented in such a way that the interior of Λ_i in Γ_{i-1} contains the curve Λ_{i-1}. Let Λ_i^- and Λ_i^+ denote respectively the interior of Λ_i in Γ_{i-1} and the exterior of Λ_i in Γ_i. With this notation we have

$$\Delta_i\Lambda_i^- + \Lambda_i + \Lambda_i^+ = \Gamma_i \ .$$

The ordered set of p-spheres $\Gamma_0, \Gamma_1, \cdots, \Gamma_{n-1}$ will be said to generate "a link of M-surface", if in addition to (a), (b), (c) it satisfies the following conditons:

(d) The exterior Λ_{i-1}^+ of Λ_{i-1} in Γ_{i-1} contains the curve Λ_i.

(e) No two of the sets $\Lambda_{i-1}+\Lambda_{i-1}^+ \cap \Lambda_i^-$ have any points in common.

These conditions being satisfied, the set

$$L = \Lambda_0 + \Lambda_0^+ \cap \Lambda_1^- + \Lambda_1 + \Lambda_1^+ \cap \Lambda_2^- + \cdots + \Lambda_{n-1} + \Lambda_{n-1}^+ \cap \Lambda_0^-$$

constitutes a compact, piece wise smooth, surface of genus one. We shall make L into a Riemann surface.

For each $i = 0, 1, \cdots, n-1$[10] let φ_i be a conformal mapping of Γ_i onto a given sphere Σ.

Let $\varphi_n = \varphi_0$, $\Delta_n = \Delta_0$, $\Gamma_{-1} = \Gamma_{n-1}$, $\Lambda_{-1} = \Lambda_{n-1}$, etc...

If p_0 is a point of $\Lambda_i^+ \cap \Lambda_{i+1}^-$ and N a neighborhood of p_0 in Γ_i, small enough to be contained in $\Lambda_i^+ \cap \Lambda_{i+1}^-$, we take as local uniformizer in N the function $z = \varphi_i p$, where z is any coordinate in Σ which does not assume the value ∞ in $\varphi_i N$.

If p_0 is a point of Λ_i, let N be a neighborhood of p_0 in Γ_i small enough to be contained in the domain $\{\Delta_i\Lambda_{i-1}^+\} \cap \Lambda_{i+1}^-$. We take as a neighborhood of p_0 in L the set

$$N^* = \{\Delta_i^{-1}N\} \cap \Lambda_i^- + N \cap \Lambda_i + N \cap \Lambda_i^+ \ .$$

We introduce as local uniformizer in N^* the function defined by setting

$$z = \varphi_i\Delta_i p \text{ for } p \in \{\Delta_i^{-1}N\} \cap \Lambda_i^-$$

and

$$z = \varphi_i p \quad \text{for } p \in N \cap \{\Lambda_i + \Lambda_i^+\}$$

Again, z is any coordinate in Σ which does not assume the value infinity in $\varphi_i N$.

[10] Here and in the following we shall assume a link to consist of at least 3 p-spheres.

The conformal structure thus introduced in L agrees in a natural way with that induced by the surrounding metric of E_3. Of course, in general along the curves Λ_i there will be discrepancies between angles measured in E_3 and angles measured in L.

The surface L will be referred to as a "link of M-surface" or briefly a "link". It will be denoted by $L(\Gamma_0, \Gamma_1, \cdots, \Gamma_{n-1})$.

2.4. We shall now construct surfaces of higher genus by putting together several links. There are several ways to achieve this. For our purposes it will be sufficient to construct only surfaces which consist of a p-sphere Γ_0 with many handles, each handle being part of a link containing Γ_0.

Let $L_1, L_2, \cdots, L_g$ be the links

$$\begin{array}{c} L_1(\Gamma_{1,0}, \Gamma_{1,1}, \cdots, \Gamma_{1,n_1-1}) \\ L_2(\Gamma_{2,0}, \Gamma_{2,1}, \cdots, \Gamma_{2,n_2-1}) \\ \cdots \quad \cdots \quad \cdots \\ L_g(\Gamma_{g,0}, \Gamma_{g,2}, \cdots, \Gamma_{g,n_g-1}) \,. \end{array}$$

With the same notations as before we shall use the symbols $\Lambda_{i,j}$, $\Delta_{i,j}$, $\varphi_{i,j}$ where the first index will denote which link the object represented belongs to, and the second index, which position it occupies in the link itself.

Suppose that $L_1, L_2, \cdots, L_g$ satisfy the following conditions:

(f) The initial surfaces $\Gamma_{1,0}, \cdots, \Gamma_{g,0}$ are all the same p-sphere Γ_0.

(g) No two of the sets $L_i - \Gamma_0$ have any point in common.

(h) The closed sets $\Gamma_0 - \Lambda^+_{i,0}$, $\Gamma_0 - \Lambda^-_{j,1}$ $(i, j = 1, 2, \cdots, g)$ are all exterior to each other.

Then the set Ξ defined by

$$\Xi = L_1 \cap L_2 \cap \cdots \cap L_g + \sum_{1,g} {}_i (L_i - \Gamma_0) \,,$$

or, which is the same, by

$$\Xi = \sum_{1,g} {}_i (\Lambda_{i,0} + \Lambda_{i,1}) + \bigcap_{i=1,g} \{\Lambda^+_{i,0} \cap \Lambda^-_{i,1}\} + \sum_{1,g} {}_i (L_i - \Gamma_0)$$

shall be called an "M-surface".

Ξ can be made into a Riemann surface using the same local uniformizers which were introduced for the L_i's themselves.

However, some care has to be applied in the choice of permissible neighborhoods, and this is solely for points of the surface Γ_0.

We shall illustrate the situation with representative cases:

Suppose that P is a point of Ξ that is in Γ_0.

If $P \in \bigcap_{i=1,g} \{\Lambda^+_{i,0} \cap \Lambda^-_{i,1}\}$, then we can take as a neighborhood of P in

Ξ any neighborhood of P in Γ_0 which is small enough to be contained in $\bigcap_{i=1,g} \{\Lambda_{i,0}^+ \cap \Lambda_{i,1}^-\}$.

If $P \in \Lambda_{j,0}$, we choose first a neighborhood N of P in Γ_0 which is small enough to be contained in the domain

$$\Delta_{j,n_j}\{\Lambda_{j,n_j-1}^+ \cap \Lambda_{j,0}^-\} + \Lambda_{j,0} + \bigcap_{i=1,g} \{\Lambda_{i,0}^+ \cap \Lambda_{i,1}^-\} ,$$

then we take as a neighborhood of P in Ξ the set

$$N^* = \{\Delta_{j,n_j}^{-1} N\} \cap \Lambda_{j,0}^- + N \cap \Lambda_{j,0} + N \cap \Lambda_{j,0}^+ .$$

If $P \in \Lambda_{j,1}$, we choose a neighborhood N of P in $\Gamma_{j,1}$ so small that

$$N \subset \Delta_{j,1}\{\bigcap_{i=1,g} (\Lambda_{i,0}^+ \cap \Lambda_{i,1}^-)\} + \Lambda_{j,1} + \Lambda_{j,1}^+ \cap \Lambda_{j,2}^- .$$

We then take as a neighorhood of P in Ξ the set

$$N^* = \{\Delta_{j,1}^{-1} N\} \cap \Lambda_{j,1}^- + N \cap \Lambda_{j,1} + N \cap \Lambda_{j,1}^+$$

3. Characterization of the conformal parameters.

3.1. Let $\Xi \sim (L_1, L_2, \cdots, L_g)$ be a given M-surface, and $\Xi(\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1})$ denote the surface Ξ marked by the set of curves

$$\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1} .$$

We shall now present a construction of the Schottky model corresponding to $\Xi(\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1})$.

Let us first take under consideration the case that Ξ consists of a single link $L(\Gamma_0, \Gamma_1, \cdots, \Gamma_{n-1})$.

We imagine to have cut L along the curve Λ_1

Using the mapping Δ_2 we can collapse the portion $\Lambda_1 + \Lambda_1^+ \cap \Lambda_2^-$ of L into the p-sphere Γ_2. The new set

$$X_1 = \Delta_2\{\Lambda_1 + \Lambda_1^+ \cap \Lambda_2^-\} + \Lambda_2 + \Lambda_2^+ \cap \Lambda_3^- + \cdots + \Lambda_{n-1} + \Lambda_{n-1}^+ \cap \Lambda_0 + \Lambda_0 + \Lambda_0^+ \cap \Lambda_1^-$$

with the points of its boundaries Λ_1 and $\Delta_2\Lambda_1$ identified by the transformation Δ_2, can also be considered a Riemann surface.

We shall briefly describe the neighborhoods and the local uniformizers at the points of the set $\Delta_2\{\Lambda_1 + \Lambda_1^+ \cap \Lambda_2^-\} + \Lambda_2$.

If $p \in \Delta_2\Lambda_1$, we choose $N \ni p$ in Γ_2 so that

$$\Delta_2^{-1} N \subset \Delta_1(\Lambda_0^+ \cap \Lambda_1^-) + \Lambda_1 + \Lambda_1^+ \cap \Lambda_2^- ,$$

we then take

$$N^* = \{\Delta_1^{-1}\Delta_2^{-1} N\} \cap \Lambda_1^- + N \cap \Delta_2\{\Lambda_1 + \Lambda_1^+\} .$$

As a uniformizer in N^* we take the function

$$z = \varphi_2 p \qquad \text{for } p \in N \cap \varDelta_2\{A_1 + A_1^+\}$$
$$z = \varphi_2 \varDelta_2 \varDelta_1 p \text{ for } p \in \{\varDelta_1^{-1}\varDelta_2^{-1}N\} \cap A_1^-$$

(provided that $z \neq \infty$ in N).

If $p \in \varDelta_2\{A_1^+ \cap A_2^-\}$ we choose $N \ni p$ so that

$$N \subset \varDelta_2\{A_1^+ \cap A_2^-\}$$

then set $N^* = N$ and $z = \varphi_2 p$ (assuming $z \neq \infty$ in N).

If $p \in A_2$ we choose $N \ni p$ so that

$$N \subset \varDelta_2\{A_1^+ \cap A_2^-\} + A_2 + A_2^+ \cap A_3^- ,$$

then set $N^* = N$ and $z = \varphi_2 p$ (assuming $z \neq \infty$ in N).

L and X_1 are conformally equivalent.

In fact, the function ψ_1 defined by

$$\psi_1 p = p \quad \text{for } p \in A_2 + A_2^+ \cap A_3^- + \cdots + A_n + A_n^+ \cap A_1^-$$
$$\psi_1 p = \varDelta_2 p \text{ for } p \in A_1 + A_1^+ \cap A_2^-$$

induces a conformal mapping of L onto X_1.

We proceed in a similar way, and collapse the subset

$$\varDelta_2\{A_1 + A_1^+ \cap A_2^-\} + A_2 + A_2^+ \cap A_3^-$$

of Γ_2 into Γ_3 by means of the mapping $\varDelta_3$, the subset

$$\varDelta_3\varDelta_2\{A_1 + A_1^+ \cap A_2^-\} + \varDelta_3\{A_2 + A_2^+ \cap A_3^-\} + A_3 + A_3^+ \cap A_4^-$$

of Γ_3 into Γ_4 by means of the mapping $\varDelta_4$, etc..., the subset

$$\varDelta_{k-1} \cdots \varDelta_2\{A_1 + A_1^+ \cap A_2^-\} + \cdots + \varDelta_{k-1}\{A_{k-2} + A_{k-2}^+ \cap A_{k-1}^-\} + A_{k-1} + A_{k-1}^+ \cap A_k^-$$

of Γ_{k-1} into Γ_k by means of the mapping $\varDelta_k$, and set

$$\begin{aligned} X_{k-1} = \varDelta_k \cdots \varDelta_2\{A_1 + A_1^+ \cap A_2^-\} + \cdots + \varDelta_k\{A_{k-1} + A_{k-1}^+ \cap A_k^-\} \\ + A_k + A_k^+ \cap A_{k+1}^- + \cdots + A_n + A_n^+ \cap A_1^- + A_1 . \end{aligned}$$

Again, X_{k-1} is made into a Riemann surface, by introducing local uniformizers in such a way that the function ψ_{k-1} defined by

$$\begin{aligned} &\psi_{k-1} p = p \quad \text{for } p \in A_k + A_k^+ \cap A_{k+1}^- + \cdots + A_n + A_n^+ \cap A_1^- , \\ &\psi_{k-1} p = \varDelta_k p \text{ for } p \in A_{k-1} + A_{k-1}^+ \cap A_k^- , \\ &\qquad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &\psi_{k-1} p = \varDelta_k \varDelta_{k-1} \cdots \varDelta_2 p \text{ for } p \in A_1 + A_1^+ \cap A_2^- \end{aligned}$$

induces a conformal mapping between L and X_{k-1}.

In this fashion, at each step of the process L and X_{k-1} are kept conformally equivalent, in particular for $k = n$ we obtain that L is conformally equivalent to the subset

$$X_{n-1} = \Delta_n \Delta_{n-1} \cdots \Delta_2 \{\Lambda_1 + \Lambda_1^+ \cap \Lambda_2^-\} + \cdots$$
$$+ \Delta_n \{\Lambda_{n-1} + \Lambda_{n-1}^+ \cap \Lambda_n^-\} + \Lambda_n + \Lambda_n^+ \cap \Lambda_1^- + \Lambda ,$$

of the p-sphere Γ_0. Of course the points of the boundaries Λ_1 and $\Delta_n \Delta_{n-1} \cdots \Delta_2 \Lambda_1$, of X_{n-1} are to be considered identified by the mapping $\Delta_n \Delta_{n-1} \cdots \Delta_2$ or, which is the same[11], by the transformation $\tau = \Delta_n \Delta_{n-1} \cdots \Delta_2 \Delta_1$.

3.2. We shall now prove that

I. *X_{n-1} is a Schottky model in Γ_0.*

Since τ is necessarily a Moebius transformation of Γ_0, all we have to show, to justify our assertion, is that τ is hyperbolic or loxodromic, that it has two fixed points $\alpha \in \Gamma_0 - \Lambda_1^-$ and $\beta \in \tau \Lambda_1^-$, and that

$$\tau \Lambda_1(\alpha) \supset \overline{\Lambda_1(\alpha)} .$$

Now for each k we have

$$\Delta_k \{\Gamma_{k-1} - \Lambda_k^-\} = \Lambda_k + \Lambda_k^+$$

and since

$$\Lambda_{k-1}^+ \supset \Gamma_{k-1} - \Lambda_k^- ,$$

we have

$$\Delta_k \Lambda_{k-1}^+ \supset \Lambda_k^+ . \tag{1}$$

Thus if

$$\Delta_{k-1} \cdots \Delta_1 \{\Gamma_0 - \Lambda_1^-\} \supset \Lambda_{k-1}^+ ,$$

because of (1) it will follow that

$$\Delta_k \cdots \Delta_1 \{\Gamma_0 - \Lambda_1^-\} \supset \Lambda_k^+ . \tag{2}$$

However, we have $\Delta_1 \{\Gamma_0 - \Lambda_1^-\} = \Lambda_1 + \Lambda_1^+ \supset \Lambda_1^+$; hence (2) is true and for $k = n$ we have

$$\tau \{\Gamma_0 - \Lambda_1^-\} \supset \Lambda_0^+ \supset \Gamma_0 - \Lambda_1^- . \tag{3}$$

Since $\Gamma_0 - \Lambda_1^-$ is closed and Λ_0^+ is open, the boundaries Λ_1 and $\tau \Lambda_1$ of $\Gamma_0 - \Lambda_1^-$ and $\tau \{\Gamma_0 - \Lambda_1^-\}$ cannot have any point in common. Therefore, if α^* and β^* are two points of Γ_0 such that $\alpha^* \in \Gamma_0 - \bar{\Lambda}_1^-$ and $\beta^* \in \Lambda_1^-$, otherwise arbitrary, from (3) follows:

[11] $\Delta_n \Delta_{n-1} \cdots \Delta_2$ and $\Delta_n \Delta_{n-1} \cdots \Delta_2 \Delta_1$ agree along Λ_1.

$$\tau^{-1}\overline{\Lambda_1(\alpha^*)} \subset \Lambda_1(\alpha^*)$$

and

$$\tau\overline{\Lambda_1(\beta^*)} \subset \Lambda_1(\beta^*) .$$

From these inclusions we can deduce that τ is neither parabolic nor elliptic:

In fact, if τ were parabolic with γ as a fixed point, then

$$\gamma = \lim_{n\to\infty} \tau^{-n}\alpha^* = \lim_{n\to\infty} \tau^n\beta^* .$$

But this would imply that

$$\gamma \in \Lambda_1(\alpha^*) \cap \Lambda_1(\beta^*)$$

which is absurd.

If τ were elliptic and $p \in \overline{\Lambda_1(\alpha^*)}$, then $\tau^{-1}p \in \Lambda_1(\alpha^*)$ and thus $\tau^{-1}p$ would be contained in an open set $D \subset \Lambda_1(\alpha^*)$; consequently $\tau^{-n}D \subset \Lambda_1(\alpha^*)$ for all $n \geqq 1$; but for a suitable value of n $\tau^{-n}D$ would cover p. This would imply that every point of $\overline{\Lambda_1(\alpha^*)}$ is interior to $\Lambda_1(\alpha^*)$ which is absurd.

Thus τ is hyperbolic or loxodromic and its fixed points are determined by the limits

$$\alpha = \lim_{n\to\infty} \tau^{-n}\{\Gamma_0 - \Lambda_1^-\}$$

$$\beta = \lim_{n\to\infty} \tau^n \Lambda_1^- .$$

With this notation under any coordinate system in Γ_0 the equation of τ takes the form

$$\frac{\tau z - \alpha}{\tau z - \beta} = \omega \frac{z - \alpha}{z - \beta}$$

with $|\omega| > 1$. Finally, since $\alpha \in \Gamma_0 - \overline{\Lambda_1^-}$, from (3) we obtain

$$\tau\Lambda_1(\alpha) \supset \overline{\Lambda_1(\alpha)} .$$

3.3. We shall now consider the general case.

Let $\Xi \sim (L_1, L_2, \cdots, L_g)$, imagine $\Xi(\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1})$ cut along the curves

$$\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1} .$$

We then apply to each link L_i the previous construction. Each handle

$$\Lambda_{i,1} + \Lambda_{i,1}^+ \cap \Lambda_{i,2}^- + \cdots + \Lambda_{i,n_i-1} + \Lambda_{i,n_i-1}^+ \cap \Lambda_{i,0}^- \quad (i = 1, 2, \cdots, g)$$

of Ξ, is flattened into Γ_0 by means of the mapping ψ_i defined by the equalities:

$$(4)\quad \begin{array}{ll} \psi_i p = \Delta_{i,n_i} \cdots \Delta_{i,3}\Delta_{i,2} p & \text{for } p \in \Lambda_{i,1} + \Lambda^+_{i,1} \cap \Lambda^-_{i,2} \\ \psi_i p = \Delta_{i,n_i} \cdots \Delta_{i,3} p & \text{for } p \in \Lambda_{i,2} + \Lambda^+_{i,2} \cap \Lambda^-_{i,3} \\ \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots \\ \psi_i p = \Delta_{i,n_i} p & \text{for } p \in \Lambda_{i,n_i-1} + \Lambda^+_{i,n_i-1} \cap \Lambda^-_{i,0} \,. \end{array}$$

The resulting subregion X of the p-sphere Γ_0 can be considered to be the intersection

$$X = X_{n_1-1} \cap X_{n_2-1} \cap \cdots \cap X_{n_g-1}$$

of the Schottky models X_{n_i-1} corresponding to each link of Ξ.

The pairs of boundaries $\Lambda_{i,1}$ and $\Delta_{i,n_i} \cdots \Delta_{i,3}\Delta_{i,2}\Lambda_{i,1}$ of X should be considered identified by the mapping $\Delta_{i,n_i} \cdots \Delta_{i,3}\Delta_{i,2}$ or, which is the same thing, by the mapping $\tau_i = \Delta_{i,n_i} \cdots \Delta_{i,2}\Delta_{i,1}$. Furthermore:

II. *X is a Schottky model conformally equivalent to Ξ.*

Proof. As a by-product of the proof of Statement I we obtain that

(a) Each mapping τ_i $(i = 1, 2, \cdots, g)$ is a hyperbolic or loxodromic Moebius transformation of Γ_0.

(b) The fixed points α_i, β_i of τ_i are respectively contained in $\Gamma_0 - \overline{\Lambda^-_{i,1}}$ and $\Lambda^-_{i,1}$.

(c) In any coordinate system in Γ_0 the equation of τ_i writes

$$(5)\qquad \frac{\tau_i z - \alpha_i}{\tau_i z - \beta_i} = \omega_i \frac{z - \alpha_i}{z - \beta_i}$$

with $|\omega_i| > 1$.

(d) Each τ_i satisfies the inclusions (see (3))

$$\tau_i\{\Gamma_0 - \Lambda^-_{i,1}\} \supset \Lambda^+_{i,0} \supset \Gamma_0 - \Lambda^-_{i,1}$$

or, changing notation:

$$(6)\qquad \tau_i\overline{\Lambda_{i,1}(\alpha_i)} \supset \Lambda_{i,0}(\alpha_i) \supset \overline{\Lambda_{i,1}(\alpha_i)} \,.$$

Since $\Lambda_{i,0}(\alpha_i)$ is open we can safely conclude that (6) implies

$$\tau_i\Lambda_{i,1}(\alpha_i) \supset \overline{\Lambda_{i,1}(\alpha_i)} \,.$$

Condition (c) in the definition of a M-surface requires the closed sets

$$\Gamma_0 - \Lambda^+_{i,0} = \overline{\Lambda_{i,0}(\beta_i)},\ \Gamma_0 - \Lambda^-_{j,1} = \overline{\Lambda_{j,1}(\alpha_j)} \quad (i, j = 1, 2, \cdots, g)$$

to be disjoint. However, the inclusions

$$\tau_i\overline{\Lambda_{i,1}(\alpha_i)} \supset \Lambda_{i,0}(\alpha_i)$$

imply

$$\tau_i\Lambda_{i,1}(\beta_i) \subset \overline{\Lambda_{i,0}(\beta_i)}$$

hence we must have

$$\tau_i\overline{\Lambda_{i,1}(\beta_i)} \subset \overline{\Lambda_{i,0}(\beta_i)}\ .$$

Therefore also the closed sets

$$\tau_i\overline{\Lambda_{i,1}(\beta_i)},\ \overline{\Lambda_{j,1}(\alpha_j)} \qquad (i, j = 1, 2, \cdots, g)$$

are disjoint. With this, the conditions for X to be a Schottky model are all satisfied.

The conformal equivalence of X to Ξ is a consequence of the fact that the function ψ defined by the equalities

$$\psi p = p \quad \text{for } p \in L_1 \cap L_2 \cap \cdots \cap L_g - \sum_{i=1,g} \Lambda_{i,1}$$

and (see (4))

$$\psi p = \psi_i p \quad \text{for } p \in L_i - \Gamma_0 + \Lambda_{i,1} \qquad (i = 1, 2, \cdots, g\,)$$

induces a conformal mapping of Ξ onto X.

3.4. The mapping ψ, or rather its analytic continuation in Ξ, uniformizes the marked surface $\Xi(\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1})$.

Let $\hat{\Xi}_\Lambda$ represent the Schottky covering surface of $\Xi(\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1})$ and X_E the region obtained by cutting Ξ along the curves $\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1}$.

Let the cycles $M_1, M_2, \cdots, M_g$ of a completion of $\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1}$ to a canonical basis of Ξ be chosen in such a way that each M_i intersects the curves $\Lambda_{i,j}\ (j = 1, 2, \cdots, n_i)$ in the order

$$\Lambda_{i,n_i}, \Lambda_{i,n_i-1}, \cdots, \Lambda_{i,2}, \Lambda_{i,1}\ .$$

As before, let $\mathscr{M}$ be the free group generated by the M_i's and X_M for each $M \in \mathscr{M}$ an exact replica of X_E.

Then we have

$$\hat{\Xi}_\Lambda = \sum_{M \in \mathscr{M}} \overline{X}_M$$

where again the boundaries of the $\overline{X}_M$'s are identified according to the rules (i), (ii) stated in § 1.4.

For each $M \in \mathscr{M}$ let $\tau_M \in G$[12] be the Moebius transformation corresponding to M under the isomorphism of $\mathscr{M}$ onto G defined by setting

[12] As before G denotes the group generated by the τ_i's,

$$M_i \longleftrightarrow \tau_i \qquad (i = 1, 2, \cdots, g) .$$

The mapping $\hat{\psi}$ of $\hat{\Xi}_\Lambda$ into Γ_0 is then obtained taking

$$\hat{\psi} p = \tau_M \psi p \quad \text{for } p \in \bar{X}_M - \sum_{i=1,g} \Lambda_{i,1}^{-1} ,$$

and the region of Γ_0 onto which $\hat{\Xi}_\Lambda$ is mapped is given by the union

$$\hat{\psi}\hat{\Xi}_\Lambda = \sum_{\tau \in G} \tau \psi \bar{X}$$

This shows that X is the Schottky model corresponding to $\Xi(\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1})$ and therefore that the conformal parameters of $\Xi(\Lambda_{1,1}, \Lambda_{2,1}, \cdots, \Lambda_{g,1})$ are characterized by the invariants ω_i and the fixed points α_i, β_i of the transformations τ_i.

4. Links of spheres.

4.1. Given two oriented spheres Γ_1 and Γ_2 intersecting along a circle Λ, there always exists a conformal mapping Δ of Γ_1 onto Γ_2 which leaves unchanged the points of Λ.

The mapping Δ can be constructed in the following way:

Let τ be a Moebius transformation of E_3 which sends a point of Λ onto the point at infinity. The circle Λ is taken by τ onto a straight line $\tau\Lambda$ and the spheres Γ_1 and Γ_2 onto two planes $\tau\Gamma_1$, $\tau\Gamma_2$ intersecting along $\tau\Lambda$. If π_1 and π_2 denote the two planes through $\tau\Lambda$ which bisect the dihedral angle formed by $\tau\Gamma_1$ and $\tau\Gamma_2$, the two transformations τ_{π_1} and τ_{π_2} obtained by reflection across π_1 and π_2 respectively, map $\tau\Gamma_1$ onto $\tau\Gamma_2$ with preservation of angles and leave unchanged the points of $\tau\Lambda$.

The corresponding spheres $\tau^{-1}\pi_1$ and $\tau^{-1}\pi_2$ generate the inversions $\tau_1 = \tau^{-1}\tau_{\pi_1}\tau$, $\tau_2 = \tau^{-1}\tau_{\pi_2}\tau$ which map Γ_1 onto Γ_2 with preservation of angles and leave unchanged the points of Λ. These two spheres are called the spheres of antisimilitude of Γ_1 and Γ_2 (see also [3] page 230).

To see which of τ_1 and τ_2 defines the conformal mapping Δ, suppose that we transfer the orientation of Γ_1 and Γ_2 onto $\tau\Gamma_1$ and $\tau\Gamma_2$ by means of τ. The product $R = \tau_{\pi_1}\tau_{\pi_2}$ is a rotation of π radians around $\tau\Lambda$, therefore whatever may be the orientations of $\tau\Gamma_1$ and $\tau\Gamma_2$, R generates a sense reversing transformation of $\tau\Gamma_1$ and $\tau\Gamma_2$ onto themselves. The same will also be true for the product

$$R' = \tau^{-1}R\tau$$

with respect to Γ_1 and Γ_2. Since $\tau_1 = \{\tau^{-1}\tau_{\pi_1}\tau_{\pi_2}\tau\}\{\tau^{-1}\tau_{\pi_2}\tau\} = R'\tau_2$, either τ_1 or τ_2 is orientation preserving (as a transformation of Γ_1 onto Γ_2). But each of them is a sense reversing transformation of E_3, therefore the transformation Δ is given by that one of τ_1 and τ_2 which sends the interior of Γ_1 onto the exterior of Γ_2. The one of $\tau^{-1}\pi_1$ and $\tau^{-1}\pi_2$

which generates $\varDelta$ will be called the "direct" sphere of antisimilitude of $\varGamma_1$ and $\varGamma_2$.

We can thus construct M-surfaces by means of collections of intersecting oriented spheres. Such M-surfaces will be called "natural".

Natural M-surfaces form a wide family for which the canal surfaces[13] are limit elements. It seems reasonable to conjecture that every Riemann surface can be realized as a natural M-surface. We shall later show that every natural M-surface can be deformed into a C^∞ canal surface without altering its conformal structure. For these reasons we found it of some interest to present a brief study of the conformal parameters of natural M-surfaces. This will lead to a few results concerning the conformal imbedding of Riemann surfaces of genus one.

Before presenting these results we need to introduce a few tools.

4.2. The conformal geometry of the 3 dimensional space is simplified by the use of "anallagmatic coordinates". An introduction to these coordinates can be found in a paper by E. Cartan [2] or in a book by R. Lagrange [6]. Here we will give only a brief description of them.

The collection of all planes, properly or improperly real spheres, and points of E_3 shall be called the "3 dimensional anallagmatic space"; we shall denote it by $\mathscr{A}_3$.

A one-to-one correspondence between the points of a 4-dimensional real projective space $\mathscr{P}_4 \sim (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and the elements of $\mathscr{A}_3$ can be generated in the following way:

To each point $\boldsymbol{\alpha} \sim (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of $\mathscr{P}_4$, if x_1, x_2, x_3 denote the cartesian coordinates of a point of E_3, we can associate the equation

$$\alpha_0(x_1^2 + x_2^2 + x_3^2) - 2\alpha_1 x_1 - 2\alpha_2 x_2 - 2\alpha_3 x_3 + \alpha_4 = 0 . \tag{1}$$

If $\alpha_0 = 0$ this equation defines a plane of E_3.

If $\alpha_0 \neq 0$ (1) is equivalent to the equation

$$\left(x_1 - \frac{\alpha_1}{\alpha_0}\right)^2 + \left(x_2 - \frac{\alpha_2}{\alpha_0}\right)^2 + \left(x_3 - \frac{\alpha_3}{\alpha_0}\right)^2 = \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_0\alpha_4}{\alpha_0^2} , \tag{2}$$

which defines a real sphere, a point or an improperly real sphere according as the quadratic form

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_0\alpha_4 \tag{3}$$

is greater, equal or less than zero.

This correspondence between $\mathscr{P}_4$ and $\mathscr{A}_3$ is clearly invertible. The five real numbers $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ (determined up to a common factor of proportionality) thus associated to each element of $\mathscr{A}_3$, are called the "anallagmatic coordinates" of that element. When expressed in anal-

[13] Surfaces which are envelopes of spheres (see [**1**]).

lagmatic coordinates, the Moebius transformations of E_3 become the homographies of $\mathscr{P}_4$ which leave invariant the binary form

$$(4) \qquad (\boldsymbol{\alpha}, \boldsymbol{\beta}) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 - \tfrac{1}{2}(\alpha_0\beta_4 + \alpha_4\beta_0) .$$

This form is assumed as a scalar product in $\mathscr{P}_4$. We have to distinguish it from the Euclidean scalar product

$$(5) \qquad \boldsymbol{x} \cdot \boldsymbol{y} = x_1y_1 + x_2y_2 + x_3y_3 ,$$

which will also figure in our subsequent formulas. To this end vectors with 5 components will be denoted by means of Greek characters and vectors with 3 components by means of Latin characters. We shall always denote (4) by $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and (5) by $\boldsymbol{x} \cdot \boldsymbol{y}$, $\boldsymbol{x} \cdot \boldsymbol{x}$ often by $\boldsymbol{x}^2$, a point $\boldsymbol{\alpha}$ of $\mathscr{P}_4$ briefly

$$\boldsymbol{\alpha} \sim (\alpha_0, \boldsymbol{a}, \alpha_4) ,$$

and the binary form (4)

$$(6) \qquad (\boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{a} \cdot \boldsymbol{b} - \tfrac{1}{2}(\alpha_0\beta_4 + \alpha_4\beta_0) .$$

To represent oriented spheres of E_3 it is convenient to normalize the anallagmatic coordinates by making use of the factor of proportionality so as to express orientations in an invariant way (see [2]). This is achieved by requiring that:

(1) If $\boldsymbol{\alpha} \sim (\alpha_0, \boldsymbol{a}, \alpha_4)$ corresponds to a point of E_3 we should have

$$\alpha_0 + \alpha_4 > 0$$

(2) If $\boldsymbol{\alpha}$ corresponds to a real oriented sphere Γ of E_3 and $\xi \sim (x_0, \boldsymbol{x}, x_4)$ corresponds to an interior point of Γ we should have

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 1$$
$$(\boldsymbol{\alpha}, \boldsymbol{\xi}) > 0 .$$

(3) If $\boldsymbol{\alpha}$ corresponds to an oriented plane π and $\boldsymbol{\xi}$ to a point of the half-space towards which the positive normal of π is directed, we should have

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 1$$
$$(\boldsymbol{\alpha}, \boldsymbol{\xi}) > 0 .$$

(4) If $\boldsymbol{\alpha}$ corresponds to an improperly real sphere, we should have

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = -1$$
$$\alpha_0 + \alpha_4 > 0 .$$

The transition from Euclidean to normalized anallagmatic coordinates can be carried out according to the following rules:

(a) If $\boldsymbol{p} \sim \boldsymbol{\xi}$ is a point of E_3 and $\lambda > 0$ then

$$\boldsymbol{\xi} = \lambda(1, \boldsymbol{p}, \boldsymbol{p}^2) .$$

(b) If $\Gamma \sim \boldsymbol{\alpha}$ is a sphere or radius R and center in $\boldsymbol{c}$, oriented so that $\boldsymbol{c}$ is an interior point

$$\boldsymbol{\alpha} = \frac{1}{R}(1, \boldsymbol{c}, \boldsymbol{c}^2 - R^2) .$$

(c) If $\Gamma \sim \boldsymbol{\alpha}$ has the same center but imaginary radius

$$\boldsymbol{\alpha} = \frac{1}{R}(1, \boldsymbol{c}, \boldsymbol{c}^2 + R^2) .$$

(d) If $\pi \sim \boldsymbol{\alpha}$ is a plane which contains the point Q and has the unit vector $\boldsymbol{n}$ as positive normal

$$\boldsymbol{\alpha} = (0, \boldsymbol{n}, 2\boldsymbol{n} \cdot \boldsymbol{Q}) .$$

By means of these formulas it can be easily verified that:

(i) The cosine of the Euclidean angle formed by two oriented spheres $\Gamma_1 \sim \boldsymbol{\alpha}$ and $\Gamma_2 \sim \boldsymbol{\beta}$ is given by the binary form (6).

(ii) A point $p \sim \boldsymbol{\xi}$ belongs to a sphere $\Gamma \sim \boldsymbol{\alpha}$ if and only if $(\boldsymbol{\alpha}, \boldsymbol{\xi}) = 0$.

(iii) The equation of the inversion Δ generated by a real sphere $\Gamma \sim \boldsymbol{\delta}$ when expressed in normalized anallagmatic coordinates takes the form[14]

$$\Delta\boldsymbol{\xi} = \boldsymbol{\xi} - 2(\boldsymbol{\xi}, \boldsymbol{\delta})\boldsymbol{\delta} , \tag{7}$$

where $\boldsymbol{\xi}$ denotes a variable element of $\mathscr{P}_4$.

The normalization (1) for anallagmatic coordinates of points of E_3 is invariant under products of inversions generated by real spheres. In fact, from (7) follows that if $\boldsymbol{\delta} = 1/R(1, \boldsymbol{c}, \boldsymbol{c}^2 - R^2)$ and $\boldsymbol{\xi} = \lambda(1, \boldsymbol{p}, \boldsymbol{p}^2)$ then

$$\Delta\boldsymbol{\xi} = \lambda\frac{|\boldsymbol{p} - \boldsymbol{c}|^2}{R^2}(1, \boldsymbol{p}', \boldsymbol{p}'^2) \tag{7*}$$

with

$$\boldsymbol{p}' = \boldsymbol{c} + \frac{R^2}{|\boldsymbol{p} - \boldsymbol{c}|^2}(\boldsymbol{p} - \boldsymbol{c}) .$$

Thus $\Delta\boldsymbol{\xi}$ satisfies condition (1) whenever $\boldsymbol{\xi}$ does.

[14] See also **[6]** pages 25–26.

Using (7) we can readily obtain the anallagmatic coordinates of the direct sphere of antisimilitude of two given intersecting oriented spheres $\Gamma_1 \sim \boldsymbol{\alpha}_1$ and $\Gamma_2 \sim \boldsymbol{\alpha}_2$. According to the considerations in § 4.1, the sphere $\Gamma \sim \boldsymbol{\delta}$ is the direct sphere of antisimilitude of Γ_1 and Γ_2 if and only if the inversion Δ which it generates, transforms the oriented sphere Γ_1 onto the sphere Γ_2 oriented in the opposite way; thus in anallagmatic coordinates we should have

$$\Delta\boldsymbol{\alpha}_1 = -\boldsymbol{\alpha}_2 ,$$

and by (7)

$$\boldsymbol{\alpha}_1 - 2(\boldsymbol{\alpha}_1, \boldsymbol{\delta})\boldsymbol{\delta} = -\boldsymbol{\alpha}_2 , \tag{8}$$

or

$$(\boldsymbol{\alpha}_1, \boldsymbol{\delta})\boldsymbol{\delta} = \frac{\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2}{2} . \tag{9}$$

To find $(\boldsymbol{\alpha}_1, \boldsymbol{\delta})$ we multiply both sides of (9) scalarly by $\boldsymbol{\alpha}_1$ obtaining

$$(\boldsymbol{\alpha}_1, \boldsymbol{\delta}) = \pm\sqrt{\frac{1 + \cos\varphi}{2}} \tag{10}$$

where by φ we indicate the Euclidean angle formed by Γ_1 and Γ_2. Now, $(\boldsymbol{\alpha}_1, \boldsymbol{\delta})$ does not vanish, for otherwise (8) yields $\boldsymbol{\alpha}_1 = -\boldsymbol{\alpha}_2$; and since the orientation of $\boldsymbol{\delta}$ does not affect the outcome of (7) we can choose the positive sign in (10) so that we obtain

$$\boldsymbol{\delta} = \frac{\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2}{2\cos\varphi/2} . \tag{11}$$

4.3. The conformal parameters of natural M-surfaces admit a purely algebraic characterization in terms of the anallagmatic coordinates of the generating spheres.

Suppose first that $L \sim (\Gamma_0, \Gamma_1, \cdots, \Gamma_{n-1})$ is a given natural link, and that $\Gamma_i \sim \boldsymbol{\alpha}_i$ $(i = 0, 1, \cdots, n-1)$. Set $\Gamma_n = \Gamma_0$, $\boldsymbol{\alpha}_n = \boldsymbol{\alpha}_0$ and φ_i equal to the angle formed by Γ_{i-1} and Γ_i $(i = 1, 2, \cdots, n)$.

Let $\Gamma'_i \sim \boldsymbol{\delta}_i$ be the direct sphere of antisimilitude of Γ_{i-1} and Γ_i and Δ_i be the Moebius inversion generated by $\boldsymbol{\delta}_i$. In other words

$$\boldsymbol{\delta}_i = \frac{\boldsymbol{\alpha}_{i-1} + \boldsymbol{\alpha}_i}{2\cos\varphi_i/2} ,$$

$$\Delta_i\boldsymbol{\xi} = \boldsymbol{\xi} - 2(\boldsymbol{\xi}, \boldsymbol{\delta}_i)\boldsymbol{\delta}_i .$$

The results of § 3.2 imply that the Moebius transformation which defines in Γ_0 the Schottky model corresponding to L is given by the product of inversions

$$\tau = \Delta_n \Delta_{n-1} \cdots \Delta_1 .$$

The conformal parameter of L is related in a simple way to the eigenvalues of τ.

The study of this transformation can be simplified if we introduce a complex coordinate in Γ_0 and make use of the results established in § 3.2.

To construct a stereographic projection $p = \varphi z$ of the complex plane π onto the sphere Γ_0 we can proceed in the following way:

We first choose a basis in $\mathscr{P}_4$ which consists of $\boldsymbol{\alpha}_0$ and four other normalized vectors $\boldsymbol{\gamma}_0$, $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\gamma}_1$ representing respectively

(a) $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$: two distinct real points of Γ_0.

(b) $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$: two real spheres containing the points represented by $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$, orthogonal to each other and to the sphere Γ_0.

We then normalize $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ so that

$$(\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1) = -1/2 ,$$ [15]

and set for each $z = x + iy$ of π:

$$\varphi z = \lambda_0(\boldsymbol{\gamma}_0 + x\boldsymbol{\varepsilon}_1 + y\boldsymbol{\varepsilon}_2 + \{x^2 + y^2\}\boldsymbol{\gamma}_1) ,$$

where the indeterminate λ_0 is only restricted to be a positive real number.

Introducing the two complex points

$$\bar{\boldsymbol{\gamma}} = (\boldsymbol{\varepsilon}_1 - i\boldsymbol{\varepsilon}_2)/2, \ \boldsymbol{\gamma} = (\boldsymbol{\varepsilon}_1 + i\boldsymbol{\varepsilon}_2)/2 ,$$

the equation of φ assumes the more suggestive form

$$\varphi z = \lambda_0(\boldsymbol{\gamma}_0 + z\bar{\boldsymbol{\gamma}} + \bar{z}\gamma + z\bar{z}\boldsymbol{\gamma}_1) . \tag{12}$$

To find the inverse of φ, we observe that if

$$\boldsymbol{\xi} = \lambda_0\boldsymbol{\gamma}_0 + \lambda\bar{\boldsymbol{\gamma}} + \lambda'\boldsymbol{\gamma} + \lambda_1\boldsymbol{\gamma}_1$$

represents a real point of Γ_0 we must have $\bar{\boldsymbol{\xi}} = \boldsymbol{\xi}$ and $(\boldsymbol{\xi}, \boldsymbol{\xi}) = 0$; this yields

$$\lambda' = \bar{\lambda}$$

and

$$\lambda_0\lambda_1 = \lambda\bar{\lambda} .$$

This means that such a $\boldsymbol{\xi}$ can always be written in the form

[15] The scalar product of two normalized vectors of $\mathscr{P}_4$ which represent real points of E_3 is always negative.

$$\boldsymbol{\xi} = \lambda_0\left(\boldsymbol{\gamma}_0 + \frac{\lambda}{\lambda_0}\bar{\boldsymbol{\gamma}} + \frac{\bar{\lambda}}{\lambda_0}\boldsymbol{\gamma} + \frac{\lambda}{\lambda_0}\frac{\bar{\lambda}}{\lambda_0}\boldsymbol{\gamma}_1\right).$$[16]

Thus we can set

$$\varphi^{-1}\boldsymbol{\xi} = \frac{\lambda}{\lambda_0}. \tag{13}$$

In view of the results of § 3.2, the mapping $\tau = \Delta_n \cdots \Delta_i$, restricted to Γ_0, is a loxodromic (in particular hyperbolic) Moebius transformation. Let us denote then by $\boldsymbol{A}$ and $\boldsymbol{B}$ its two fixed points in Γ_0 and assume that $\boldsymbol{A}$ is the source and $\boldsymbol{B}$ is the sink.

If we set

$$\boldsymbol{\gamma}_0 = \frac{1}{\boldsymbol{A}\bar{\boldsymbol{B}}}(1,\ \boldsymbol{A},\ \boldsymbol{A}^2),\ \boldsymbol{\gamma}_1 = \frac{1}{\boldsymbol{A}\bar{\boldsymbol{B}}}(1,\ \boldsymbol{B},\ \boldsymbol{B}^2),$$

and take for $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ any two spheres satisfying condition (b), since φ^{-1} maps $\boldsymbol{A}$ onto the origin and $\boldsymbol{B}$ onto the point at infinity of π, the equation of the Moebius transformation $\tau^* = \varphi^{-1}\tau\varphi$ of π will assume the simple form

$$\tau^* z = \rho e^{i\theta} z, \tag{14}$$

with $\rho > 1$ and $-\pi < \theta \leqq \pi$.

Thus for each point

$$\boldsymbol{\xi} = \lambda_0\boldsymbol{\gamma}_0 + \lambda\bar{\boldsymbol{\gamma}} + \bar{\lambda}\boldsymbol{\gamma} + \frac{\lambda\bar{\lambda}}{\lambda_0}\boldsymbol{\gamma}_1,$$

we have (using (13), (14) and (12)):

$$\begin{aligned}\tau\boldsymbol{\xi} &= \varphi\tau^*\varphi^{-1}\boldsymbol{\xi} = \varphi\tau^*\frac{\lambda}{\lambda_0} = \varphi\rho e^{i\theta}\frac{\lambda}{\lambda_0} \\ &= \lambda_0'\left(\boldsymbol{\gamma}_0 + \frac{\rho e^{i\theta}\lambda}{\lambda_0}\bar{\boldsymbol{\gamma}} + \frac{\overline{\rho e^{i\theta}\lambda}}{\lambda_0}\boldsymbol{\gamma} + \rho^2\frac{\lambda\bar{\lambda}}{\lambda_0^2}\boldsymbol{\gamma}_1\right)\end{aligned}$$

or

$$\tau\boldsymbol{\xi} = \frac{\lambda_0'}{\lambda_0}\left(\lambda_0\boldsymbol{\gamma}_0 + \rho e^{i\theta}\lambda\bar{\boldsymbol{\gamma}} + \rho\bar{e}^{i\theta}\bar{\lambda}\boldsymbol{\gamma} + \rho^2\frac{\lambda\bar{\lambda}}{\lambda_0}\boldsymbol{\gamma}_1\right).$$

A priori the indeterminate λ_0' is only restricted to be a positive real number. However, the ratio λ_0'/λ_0 depends solely upon the transformation τ.

[16] By λ/λ_0 we mean an extended valued complex number.

In fact, since τ preserves the scalar product of $\mathscr{P}_4$ we must have

$$(\tau\boldsymbol{\xi},\ \boldsymbol{\gamma}_0) = (\boldsymbol{\xi},\ \tau^{-1}\boldsymbol{\gamma}_0)\ ,$$
$$(15)\qquad (\tau\boldsymbol{\xi},\ \boldsymbol{\gamma}_1) = (\boldsymbol{\xi},\ \tau^{-1}\boldsymbol{\gamma}_1)\ ,$$
$$(\tau_0\boldsymbol{\gamma},\ \tau\boldsymbol{\gamma}_1) = (\boldsymbol{\gamma}_0,\ \boldsymbol{\gamma}_1)\ .$$

Since $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ represent fixed points of τ

$$\tau\boldsymbol{\gamma}_0 = \mu_0\boldsymbol{\gamma}_0\ ,$$
$$\tau\boldsymbol{\gamma}_1 = \mu_1\boldsymbol{\gamma}_1\ ,$$

for some positive real numbers μ_0 and μ_1. Substituting in the equations (15) we obtain

$$\frac{\lambda_0'}{\lambda_0} = \frac{1}{\rho}, \qquad \mu_0 = \frac{1}{\rho}, \qquad \mu_1 = \rho\ .$$

This gives

$$\tau\boldsymbol{\xi} = \frac{\lambda_0}{\rho}\boldsymbol{\gamma}_0 + \lambda e^{i\theta}\bar{\boldsymbol{\gamma}} + \bar{\lambda}\bar{e}^{i\theta}\boldsymbol{\gamma} + \frac{\lambda\bar{\lambda}}{\lambda_0}\rho\boldsymbol{\gamma}_1\ ,$$

and since λ is arbitrary

$$\tau\bar{\boldsymbol{\gamma}} = e^{i\theta}\bar{\boldsymbol{\gamma}}\ ,$$
$$\tau\boldsymbol{\gamma} = e^{-i\theta}\boldsymbol{\gamma}\ .$$

Finally, the relation $\varDelta_i\boldsymbol{\alpha}_{i-1} = -\boldsymbol{\alpha}_i$ for $i = 1, 2, \cdots, n$ implies

$$\tau\boldsymbol{\alpha}_0 = (-1)^n\boldsymbol{\alpha}_0\ .$$

With this we have shown that the eigenvalues of τ are $(-1)^n$, $1/\rho$, ρ, $e^{i\theta}$, $e^{-i\theta}$. Thereby the relation between these eigenvalues and the conformal parameter of L is established.

Only little has to be added concerning the general case.

If $\Xi \sim (L_1, L_2, \cdots, L_g)$ is a given natural M-surface and Γ_0 is the common initial sphere of the L_i's, we operate separately on each link L_i and determine the transformation τ_i generated by the spheres of L_i.

These transformations alone carry complete information regarding the conformal parameters of Ξ.

However, unlike the case of a single link, the eigenvalues of the τ_i's are not sufficient by themselves to characterize the conformal parameters of Ξ, since they yield only the first g of them. The real eigenvectors of these transformations have to be determined also, and among them those representing the fixed points $\boldsymbol{A}_i$, $\boldsymbol{B}_i$ of each τ_i have to be selected. Then, according to the definition (formulas (8) of § 1.5), the remaining parameters are given by the coordinates of the points

$\boldsymbol{B}_2$; $\boldsymbol{A}_3$, $\boldsymbol{B}_3$; $\cdots$; $\boldsymbol{A}_g$, $\boldsymbol{B}_g$ in a coordinate system in Γ_0 for which $\boldsymbol{A}_1\boldsymbol{B}_1$ and $\boldsymbol{A}_2$ have coordinates 0, ∞ and 1 respectively.

4.4. With the same notation as in § 2.3, let $L \sim (\Gamma_0, \Gamma_1, \cdots, \Gamma_{n-1})$ be a natural link and Λ_i be the intersection of the sphere Γ_{i-1} with the sphere Γ_i. If $\omega = \rho e^{i\theta}$ is the conformal parameter of L, we shall say that ρ is the *thinness* and θ the *torsion* of L.

The thinness of the link L can be estimated in terms of the capacities of the annular domains $\Lambda_{i-1}^+ \cap \Lambda_i^-$. In fact, we have the following:

THEOREM. *Suppose that each annulus $\Lambda_{i-1}^+ \cap \Lambda_i^-$ has a capacity c_i satisfying the inequality*

$$c_i \leqq \frac{1}{\log \rho_i} \tag{16}$$

for some $\rho_i > 1$. Then the thinness ρ of L satisfies the inequality

$$\rho \geqq \rho_1\rho_2 \cdots \rho_n , \tag{17}$$

and the equal sign holds if and only if (16) *are equalities and the spheres $\Gamma_0, \Gamma_1, \cdots, \Gamma_{n-1}$ are all orthogonal to the spheres of a hyperbolic pencil.*

To prove this theorem, we need a few preliminary considerations.

If τ is a loxodromic transformation of a sphere Γ; i.e. if for some coordinate system in Γ

$$\frac{\tau z - \alpha}{\tau z - \beta} = \omega \frac{z - \alpha}{z - \beta}$$

the number $|\omega|$ (which can always be supposed greater than one) will be called the "stretching factor" of τ.

Let Λ and Λ' be two circles of Γ having no points in common and suppose that α_0 and β_0 are the two points of Γ which belong to the elliptic pencil generated by Λ and Λ'. Let α_0 and β_0 be ordered in such a way that the disks $\Lambda(\alpha_0)$ and $\Lambda'(\beta_0)$ are exterior to each other.

LEMMA I. *Among all Moebius transformations of Γ which map $\Lambda(\alpha_0)$ onto $\Lambda'(\alpha_0)$ only those which admit α_0 and β_0 as fixed points have the smallest stretching factor.*

Proof. Let us choose a complex coordinate in Γ which is such that $\alpha_0 = 0$, $\beta_0 = \infty$ and Λ has the equation $|z| = 1$. The equation of Λ' will then be

$$|z| = \rho$$

for a suitable $\rho > 1$.

If τ is a Moebius transformation of Γ which sends $\Delta(\alpha_0)$ onto $\Delta'(\alpha_0)$ its equation can be written in the form

$$\frac{\tau z - \alpha}{\tau z - \beta} = \omega \frac{z - \alpha}{z - \beta}$$

with $\alpha \in \Delta(\alpha_0)$, $\beta \in \Delta'(\beta_0)$ and $|\omega| > 1$[17]. Now, τ must send the points $1/\bar{\alpha}$ and $1/\bar{\beta}$ respectively onto the points $\rho^2/\bar{\alpha}$ and $\rho^2/\bar{\beta}$. In other words, we must have

$$\text{(18)a, b} \qquad \frac{\rho^2/\bar{\alpha} - \alpha}{\rho^2/\bar{\alpha} - \beta} = \omega \frac{1/\bar{\alpha} - \alpha}{1/\bar{\alpha} - \beta}, \; \frac{\rho^2/\bar{\beta} - \alpha}{\rho^2/\bar{\beta} - \beta} = \omega \frac{1/\bar{\beta} - \alpha}{1/\bar{\beta} - \beta}$$

and

$$\text{(19)} \qquad (\alpha, \beta, 1/\bar{\alpha}, 1/\bar{\beta}) = (\alpha, \beta, \rho^2/\bar{\alpha}, \rho^2/\bar{\beta}) .$$

Equation (18)a gives

$$\omega = \frac{\rho^2 - \alpha\bar{\alpha}}{1 - \alpha\bar{\alpha}} \cdot \frac{1 - \bar{\alpha}\beta}{\rho^2 - \bar{\alpha}\beta} ,$$

equation (19), after a few eliminations, yields

$$\frac{\rho^2 - \bar{\alpha}\beta}{1 - \bar{\alpha}\beta} \cdot \frac{\rho^2 - \alpha\bar{\beta}}{1 - \alpha\bar{\beta}} = \rho^2 .$$

Therefore we have

$$|\omega| = \frac{1}{\rho}\left|\frac{\rho^2 - \alpha\bar{\alpha}}{1 - \alpha\bar{\alpha}}\right| .$$

But $\alpha\bar{\alpha} < 1$ (since $\alpha \in \Delta(\alpha_0)$), thus

$$|\omega| \geqq \rho$$

and the equality sign holds if and only if $\alpha\bar{\alpha} = 0$. However, when $\alpha = 0$ equations (18)a,b give $\beta = \infty$. This proves the assertion.

Let the Moebius transformation $(\tau z - \alpha)/(\tau z - \beta) = \omega(z - \alpha)/(z - \beta)$ define in Σ a Schottky model $M(\tau)$. Any circle Δ such that the closed disks $\bar{\Delta}(\alpha)$ and $\overline{\tau\Delta(\beta)}$ are mutually exclusive cuts $M(\tau)$ into a region $\Delta(\beta) \cap \tau\Delta(\alpha)$ which is an annulus. As a consequence of the previous lemma we can show that:

[17] This follows from an argument similar to that presented in § 3.2.

LEMMA II. *Among all circles Δ for which $\overline{\Delta(\alpha)}$ and $\overline{\tau\Delta(\beta)}$ are disjoint, only those belonging to the pencil $P(\alpha, \beta)$ cut $M(\tau)$ into an anulus of minimum capacity.*

Proof. Let α_0 and β_0 be the two points belonging to the elliptic pencil generated by Δ and $\tau\Delta$, and assume that $\alpha_0 \in \Delta(\alpha)$ and $\beta_0 \in \tau\Delta(\beta)$.[18] If c denotes the capacity of the anulus $\Delta(\beta) \cap \tau\Delta(\alpha)$ the stretching factor of every Moebius transformation which sends $\Delta(\alpha)$ onto $\tau\Delta(\alpha)$ and admits α_0 and β_0 as fixed points is given by $\rho = e^{1/c}$.

By Lemma I we must have

$$|\omega| \geqq e^{1/c}$$

or, which is the same (since $|\omega| > 1$)

$$c \geqq 1/\log|\omega|$$

with equality possible if and only if $\alpha = \alpha_0$ and $\beta = \beta_0$. Q.E.D.

We can now give a proof of the theorem.

If c denotes the capacity of the annulus

$$\begin{aligned} \Lambda_1^- - \tau\Lambda_1^- = \Delta_n \cdots \Delta_2\{\Lambda_1 + \Lambda_2^+ \cap \Lambda_3^-\} + \cdots \\ + \Delta_n\{\Lambda_{n-1} + \Lambda_{n-1}^+ \cap \Lambda_n^-\} + \Lambda_0 + \Lambda_0^+ \cap \Lambda_1^- , \end{aligned}$$

from a well known inequality of potential theory (cfr. [7]) we obtain

$$\frac{1}{c} \geqq \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n}, \tag{20}$$

and the equality sign holds if and only if the circles $\Delta_n \cdots \Delta_2\Lambda_1$, $\Delta_n \cdots \Delta_3\Lambda_2, \cdots, \Delta_n\Lambda_{n-1}, \Lambda_0, \Lambda_1$ belong to the same pencil. Since the thinness ρ of the link L is equal to the stretching factor of the transformation $\Delta_n\Delta_{n-1} \cdots \Delta_1$, from Lemma II we get

$$\rho \geqq e^{1/c}, \tag{21}$$

thus from (20) and (16) the desired inequality follows.

To prove the last statement of the theorem, we observe that the equal sign will occur in (17) if and only if, (16) being equalities, equality holds simultaneously in (20) and (21). However, this happens if and only if all the circles $\Delta_n \cdots \Delta_2\Lambda_1$, $\Delta_n \cdots \Delta_3\Lambda_2, \cdots, \Delta_n\Lambda_{n-1}, \Lambda_0, \Lambda_1$ belong to the pencil generated by the fixed points α, β of the transformation

$$\tau = \Delta_n\Delta_{n-1} \cdots \Delta_1 .$$

Let then Γ be any sphere orthogonal to Γ_0 and containing α and

[18] This is always the case after a suitable labeling of α_0 and β_0.

β. Since Γ is orthogonal to Λ_0, Γ will be orthogonal to Γ_{n-1} and Γ'_n (the direct sphere of antisimilitude of Γ_{n-1} and Γ_0.)

Therefore $\Delta_n\Gamma = \Gamma$ and consequently Γ is orthogonal to $\Delta_n(\Delta_n\Lambda_{n-1}) = \Lambda_{n-1}$. Γ will then be orthogonal to Γ_{n-2} and to Γ'_{n-1} (the direct sphere of antisimilitude of Γ_{n-2} and Γ_{n-1}). But this implies that $\Delta_{n-1}\Delta_n\Gamma = \Gamma$ and consequently Γ is orthogonal to $\Delta_{n-1}\Delta_n(\Delta_n\Delta_{n-1}\Lambda_{n-2}) = \Lambda_{n-2}$, etc. Proceeding in this fashion we obtain that Γ is also orthogonal to $\Gamma_{n-3}, \Gamma_{n-4}, \cdots, \Gamma_2, \Gamma_1$. The spheres orthogonal to Γ_0 and containing α and β form a hyperbolic pencil.

Conversely if the spheres $\Gamma_0, \Gamma_1, \cdots, \Gamma_{n-1}$ are orthogonal to the spheres of a hyperbolic pencil P, so will also be the spheres of antisimilitude $\Gamma'_2, \Gamma'_2, \cdots, \Gamma'_n$; consequently each sphere of P will be invariant under any of the transformations $\Delta_1, \Delta_2, \cdots, \Delta_n$.

We can then easily deduce that the circles $\Delta_n \cdots \Delta_2\Lambda_1, \cdots, \Delta_n\Lambda_{n-1}, \Lambda_0$ are orthogonal to the spheres of P and thus they all belong to the pencil generated by the two points α and β intersection of Γ_0 and the spheres of P. But α and β are the fixed points of the transformation $\Delta_n\Delta_{n-1} \cdots \Delta_1$.

Our proof is thus complete.

Although it will not be needed in the following we would like to point out that the inequality (17) holds also for general links. In fact, Lemma II is valid in the stronger form:

"*Among all smooth Jordan curves Λ for which $\overline{\Lambda(\alpha)}$ and $\overline{\tau\Lambda(\beta)}$ are disjoint, only the circles of the pencil $P(\alpha, \beta)$ cut $M(\tau)$ into an annulus of minimum capacity.*"

This statement follows from standard potential theoretical considerations.

5. Some special links.

5.1. Let π_1 denote the w-plane and w_1, w_2 two complex numbers for which

$$\mathfrak{Im}\, w_1/w_2 < 0 .$$

Let G denote the group generated by the translations

$$\begin{aligned} \tau_1 w &= w + w_1 \\ \tau_2 w &= w + w_2 . \end{aligned} \tag{1}$$

If we identify the points of π which are images of each other under the transformations of G, we obtain a Riemann surface of genus one $\Gamma(w_1, w_2)$.

The surface $\Gamma(w_1, w_2)$ can also be thought of as the parallelogram

$$\mathscr{P} = \{w : w = \lambda w_1 + \mu w_2;\ 0 \leqq \lambda \leqq 1,\ 0 \leqq \mu \leqq 1\} .$$

with opposite sides identified by the transformations (1).

This standard construction generates every Riemann surface of genus one: as a matter of fact, as w_1 and w_2 vary, $\Gamma(w_1, w_2)$ assumes every conformal type and each an infinite number of times.

It is clear that two Riemann surfaces $\Gamma(w_1, w_2)$ and $\Gamma(w_1', w_2')$ are conformally equivalent if and only if the lattices

$$\begin{matrix} L \sim \{m_1 w_1 + m_2 w_2\} \\ L' \sim \{m_1 w_1' + m_2 w_2'\} \end{matrix} \quad m_1,\ m_2 = 0,\ \pm 1,\ \pm 2, \cdots$$

can be superimposed by a similarity. Now it is well known that this is possible if and only if the two ratios

$$\nu = \frac{w_1}{w_2} \text{ and } \nu' = \frac{w_1'}{w_2'}$$

are images of each other under a transformation of the restricted unimodular group; in other words if and only if there exist integers $a,\ b,\ c,\ d$ for which $ad - bc = 1$ and

$$\nu' = \frac{a\nu + b}{c\nu + d} .$$

The set

$$\mathfrak{M} = \{\nu : \mathfrak{Im}\,\nu < 0;\ -1/2 < \mathfrak{Re}\,\nu \leqq 1/2;\ |\nu| > 1 \text{ for } \mathfrak{Re}\,\nu < 0;\ |\nu| \geqq 1 \text{ for } \mathfrak{Re}\,\nu \geqq 0\}$$

is a fundamental region of the restricted unimodular group; thus two Riemann surfaces $\Gamma(w_1, w_2)$ and $\Gamma'(w_1, w_2)$ will be conformally equivalent if and only if the complex numbers w_1/w_2 and w_1'/w_2' have the same image point in $\mathfrak{M}$.

If we have a Schottky model $M(\tau)$ defined by a Moebius transformation

$$\frac{\tau z - \alpha}{\tau z - \beta} = \rho e^{i\theta} \frac{z - \alpha}{z - \beta} \qquad (\rho > 1;\ -\pi < \theta \leqq \pi)$$

of some sphere Σ, a conformally equivalent model is given by the surface $\Gamma(\log \rho + i\theta,\ 2\pi i)$. In fact, the function $w = \log (z - \alpha)/(z - \beta)$ defines a conformal mapping of $M(\tau)$ onto $\Gamma(\log \rho + i\theta,\ 2\pi i)$.

The point

$$\nu = \frac{\theta}{2\pi} - i\frac{\log \rho}{2\pi}$$

belongs to $\mathfrak{M}$ if

$$(2)\qquad \begin{aligned} &\left(\frac{\theta}{2\pi}\right)^2 + \left(\frac{\log \rho}{2\pi}\right)^2 \geqq 1 \text{ when } \theta \geqq 0 \\ &\left(\frac{\theta}{2\pi}\right)^2 + \left(\frac{\log \rho}{2\pi}\right)^2 > 1 \text{ when } \theta < 0 . \end{aligned}$$

Thus can we conclude that two distinct Schottky models $M(\tau)$ and $M(\tau')$ whose conformal parameters $\rho e^{i\theta}$ and $\rho' e^{i\theta'}$ satisfy the inequalities (2) are never conformally equivalent.

We shall proceed to show that there exist natural links which are not conformally equivalent to any of the models $\Gamma(\log \rho, 2\pi i)$.

5.2. Let $\boldsymbol{\alpha} = 1/R(1, \boldsymbol{c}, \boldsymbol{c}^2 - R^2)$, $\alpha_1 = 1/R(1, \boldsymbol{c}_1, \boldsymbol{c}_1^2 - R^2)$ and $\boldsymbol{\alpha}_2 = 1/R(1, \boldsymbol{c}_2, \boldsymbol{c}_2^2 - R^2)$ be three given spheres[19] of equal radius and suppose that $\overline{\boldsymbol{c}\boldsymbol{c}_1} = \overline{\boldsymbol{c}\boldsymbol{c}_2} = 2\delta$, $\delta < R < \overline{\boldsymbol{c}_1\boldsymbol{c}_2}/2$.

Let Λ_1 and Λ_2 be the circles of intersection of $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}_2$ respectively, π_1 and π_2 be the planes containing Λ_1 and Λ_2, d the intersection of π_1, and π_2 (proper or improper), $\boldsymbol{p}$ the intersection of d with the plane through $\boldsymbol{c}$ perpendicular to d, and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ represent the points of contact of the two planes through d which are tangent to α.

We would like to compute the capacity of the annulus

$$D = \Lambda_1(\boldsymbol{p}_2) \cap \Lambda_2(\boldsymbol{p}_1) .$$

To do this it is sufficient to compute the stretching factor of a Moebius transformation of $\boldsymbol{\alpha}$ which admits $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as fixed points and sends $\Lambda_1(\boldsymbol{p}_1)$ onto $\Lambda_2(\boldsymbol{p}_1)$.

Let π denote the plane through $\boldsymbol{c}$ and d, and $\boldsymbol{\sigma}$ the sphere through Λ_2 which is orthogonal to $\boldsymbol{\alpha}$. Clearly the product

$$\tau = \tau_\sigma \tau_\pi$$

of the inversions τ_π and τ_σ with respect to π and $\boldsymbol{\sigma}$ generates a transformation of $\boldsymbol{\alpha}$ which is of the type requested. We shall compute its equation.

We indicate by $\boldsymbol{a}$ and $\boldsymbol{b}$ two unit vectors with the directions of $\boldsymbol{c}_1\boldsymbol{c}_2$ and $\boldsymbol{cp}$ respectively. Let us assume for simplicity that the origin of the coordinate system of E_3 is at $\boldsymbol{c}$. We then have

$$\boldsymbol{\alpha} = \frac{1}{R}(1, 0, -R^2)$$

$$\pi = (0, \boldsymbol{a}, 0) .$$

[19] Occasionally we shall make use of the same symbol to denote a geometric object and its representative in $\mathscr{S}_4$.

Setting $\varphi = \widehat{pcc_1} = \widehat{pcc_2}$ and $\psi = \widehat{pcp_1} = \widehat{pcp_2}$:

$$\boldsymbol{\pi}_1 = (0,\ -\sin\varphi\boldsymbol{a} + \cos\varphi\boldsymbol{b},\ 2\delta)\ ,$$

$$\boldsymbol{\pi}_2 = (0,\ \sin\varphi\boldsymbol{a} + \cos\varphi\boldsymbol{b},\ 2\delta)\ ,$$

$$\boldsymbol{\gamma}_1 = \frac{1}{\boldsymbol{p}_1\boldsymbol{p}_2}(1,\ \boldsymbol{p}_1^2,\ \boldsymbol{p}_1^2) = \frac{1}{2R\sin\psi}(1,\ -R\sin\psi\boldsymbol{a} + R\cos\psi\boldsymbol{b},\ R^2)\ ,$$

$$\boldsymbol{\gamma}_2 = \frac{1}{\boldsymbol{p}_1\boldsymbol{p}_2}(1,\ \boldsymbol{p}_2,\ \boldsymbol{p}_2^2) = \frac{1}{2R\sin\psi}(1,\ R\sin\psi\boldsymbol{a} + R\cos\psi\boldsymbol{b},\ R^2)\ .$$

By its definition $\boldsymbol{\sigma}$ belongs to the pencil generated by $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, as well as to the pencil generated by $\boldsymbol{\alpha}$ and $\boldsymbol{\pi}_2$.

Thus for suitable values of μ, λ, μ', λ'

$$\boldsymbol{\sigma} = \mu\boldsymbol{\gamma}_1 + \lambda\boldsymbol{\gamma}_2 = \mu'\boldsymbol{\alpha} + \lambda'\boldsymbol{\pi}_2\ . \tag{3}$$

Observing that since $(\boldsymbol{\sigma},\ \boldsymbol{\sigma}) = 1$ and $(\boldsymbol{\gamma}_1,\ \boldsymbol{\gamma}_2) = -1/2$ we must have $\lambda = -1/\mu$, equating the middle components of (3) we obtain

$$\frac{\lambda^2 + 1}{\lambda^2 - 1} = \frac{\tan\psi}{\tan\varphi}\ .$$

Now

$$\tau_\pi\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2,\ \ \tau_\pi\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_1\ ,$$

$\tau_\sigma\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_1 - 2(\boldsymbol{\gamma}_1,\ \boldsymbol{\sigma})\boldsymbol{\sigma} = \lambda^2\boldsymbol{\gamma}_2$ and analogously $\tau_\sigma\boldsymbol{\gamma}_2 = (1/\lambda^2)\boldsymbol{\gamma}_1$.

Thus for the stretching factor ρ of the product $\tau_\sigma\tau_\pi$ we get

$$\rho = \lambda^2 = \frac{\tan\varphi + \tan\psi}{\tan\varphi - \tan\psi} \tag{4}$$

this determines the capacity of D[20].

5.3. It is easy to show that every point of $\mathfrak{M}$ which lies in the imaginary axis can be obtained as an image of an imbedded surface.

In fact, the image of a torus in $\mathfrak{M}$ is always pure imaginary, and as we vary the radius of the generating circle, keeping the center fixed, we can describe the whole imaginary axis.

We shall exhibit a family of natural links with the same property, and at the same time illustrate our way of computing the conformal parameters of natural links.

Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be unit vectors forming a left handed orthogonal triplet and set

[20] The obvious argument based on the fact that the stereographic projection is a cross-ratio-preserving transformation would lead to the same result with more or less the same effort.

$$\boldsymbol{\alpha}_i = \frac{1}{R}\left(1,\ \cos i\frac{2\pi}{n}\,\boldsymbol{a} + \sin i\frac{2\pi}{n}\boldsymbol{b},\ 1 - R^2\right)$$

(assume $n \geqq 3$). It can be readily verified that $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_{n-1}$ define a natural link for every value of R greater than $\sin \pi/n$ and less than one.

Let Λ_i denote the intersection of $\boldsymbol{\alpha}_{i-1}$ with $\boldsymbol{\alpha}_i$, and the sets Λ_i^-, Λ_i^+ have the same meaning as in § 2.3. To compute the conformal parameters of the link

$$L(n,\ R) = \Lambda_0 + \Lambda_0^+ \cap \Lambda_1^- + \cdots + \Lambda_{n-1} + \Lambda_{n-1}^+ \cap \Lambda_n^-\ ,$$

according to the results of § 4.3 we should study the transformation τ product of successive inversions with respect to the spheres

$$\boldsymbol{\delta}_i = \frac{R}{2}\frac{\boldsymbol{\alpha}_{i-1} + \boldsymbol{\alpha}_i}{\sqrt{R^2 - \sin^2 \pi/n}} \qquad (i = 1, 2, \cdots, n)\,.$$

This does not present any difficulty. In fact, we observe that each of the $\boldsymbol{\alpha}_i$'s is orthogonal to the plane

$$\boldsymbol{\varepsilon}_1 = (0,\ \boldsymbol{c},\ 0)$$

and the sphere

$$\boldsymbol{\varepsilon}_2 = \left(\frac{1}{\sqrt{1 - R^2}},\ 0,\ -\sqrt{1 - R^2}\right).$$

Thus all the spheres of $P(\boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2)$ (the pencil generated by $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$) are orthogonal to each of the $\boldsymbol{\alpha}_i$'s and therefore also to each of the $\boldsymbol{\delta}_i$'s. This implies that the spheres of $P(\boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2)$ are all invariant under the transformation τ. Consequently also the points $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ which $\boldsymbol{\alpha}_0$ has in common with the spheres of the pencil $P(\boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2)$ are invariant under τ. We can then conclude that τ admits the decomposition

$$\begin{aligned}
\tau\boldsymbol{\gamma}_0 &= \frac{1}{\rho}\boldsymbol{\gamma}_0\\
\tau\boldsymbol{\varepsilon}_1 &= \boldsymbol{\varepsilon}_1\\
\tau\boldsymbol{\varepsilon}_2 &= \boldsymbol{\varepsilon}_2\\
\tau\boldsymbol{\alpha}_0 &= (-1)^n\boldsymbol{\alpha}_0\\
\tau\boldsymbol{\gamma}_1 &= \rho\boldsymbol{\gamma}_1\ ,
\end{aligned}$$

with a suitable $\rho > 0$ (if $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ are properly labeled ρ will result greater than one).

Thus the torsion of $L(n,\ R)$ vanishes independently of n and R. To determine the thinness ρ we use the formula (4) of last section and obtain for the capacity c_i of each anulus $\Lambda_{i-1} + \Lambda_{i-1}^+ \cap \Lambda_i^-$

$$c_i = \left\{\log \frac{R \cos \pi/n + \sin \pi/n \sqrt{1-R^2}}{R \cos \pi/n - \sin \pi/n \sqrt{1-R^2}}\right\}^{-1} .$$

Applying the theorem of § 4.4 we obtain

$$\rho = \left(\frac{R \cos \pi/n + \sin \pi/n \sqrt{1-R^2}}{R \cos \pi/n - \sin \pi/n \sqrt{1-R^2}}\right)^n . \tag{5}$$

Clearly for any given $n > 3$ this function increases from 1 to ∞ as R decreases from 1 to $\sin \pi/n$.

It is interesting to note that if R is kept fixed in (5) and we let n tend to infinity we obtain

$$\lim_{n\to\infty} \rho = e^{2\pi \frac{\sqrt{1-R^2}}{R}} .$$

This result is not surprising since the link $L(n, R)$ then approaches the torus enveloped by a sphere of radius R as its center describes a circle of radius one,

5.4. The fact that each link $L(n, R)$ has torsion zero could have been predicted. We can show that if a natural link admits a plane of symmetry or a sphere of inversion (which amounts to the same thing) then its torsion must vanish.

We shall consider two representative cases.

Case 1. *All the spheres of the link are orthogonal to the sphere of inversion.*

Let $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_{n-1}$ be the generating spheres and $\boldsymbol{\varepsilon}$ be a real sphere such that

$$(\boldsymbol{\alpha}_i, \boldsymbol{\varepsilon}) = 0 \qquad (i = 0, 1, \cdots, n-1) .$$

From this follows that the spheres of antisimilitude $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \cdots, \boldsymbol{\delta}_n$ will also be orthogonal to $\boldsymbol{\varepsilon}$ and therefore

$$\tau\boldsymbol{\varepsilon} = \Delta_n \Delta_{n-1} \cdots \Delta_1 \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} . \tag{6}$$

We suppose that $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}, \bar{\boldsymbol{\gamma}}$ decompose τ, and set (as in § 4.3)

$$\tau\boldsymbol{\gamma}_0 = \frac{1}{\rho}\boldsymbol{\gamma}_0,\ \tau\boldsymbol{\gamma}_1 = \rho\boldsymbol{\gamma}_1,\ \tau\bar{\boldsymbol{\gamma}} = e^{i\theta}\bar{\boldsymbol{\gamma}},\ \tau\boldsymbol{\gamma} = e^{-i\theta}\boldsymbol{\gamma} ,$$

Since $\boldsymbol{\varepsilon}$ is orthogonal to $\boldsymbol{\alpha}_0$ it must be of the form

$$\boldsymbol{\varepsilon} = \lambda_0\boldsymbol{\gamma}_0 + \lambda_1\boldsymbol{\gamma}_1 + \lambda\bar{\boldsymbol{\gamma}} + \bar{\lambda}\boldsymbol{\gamma} ; \tag{7}$$

however, for a natural link $\rho > 1$ (cfr. theorem of § 4.4), and thus the hypothesis $e^{i\theta} \neq 1$ is incompatible with (6) and (7).

Case 2. The spheres of the link are interchanged by the sphere of inversion. By means of two or more additional spheres we can reduce (without altering the conformal structure of the link) every possible situation to the following one:

The spheres $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_{n-1}$ are an even number $n = 2p$ and furthermore the sphere of inversion $\boldsymbol{\varepsilon}$ is such that

$$\tau_\varepsilon \boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_0, \tau_\varepsilon \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_{n-1}, \tau_\varepsilon \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_{n-2}, \cdots, \tau_\varepsilon \boldsymbol{\alpha}_p = \boldsymbol{\alpha}_p.$$ [21]

The spheres of antisimilitude will then be related in the following way

$$\boldsymbol{\delta}_n = \tau_\varepsilon \boldsymbol{\delta}_1, \boldsymbol{\delta}_{n-1} = \tau_\varepsilon \boldsymbol{\delta}_2, \cdots, \boldsymbol{\delta}_{p+1} = \tau_\varepsilon \boldsymbol{\delta}_p .$$

This implies that the transformation $\tau = \Delta_n \Delta_{n-1} \cdots \Delta_1$ can be written in the form

$$\tau = \tau_\varepsilon \Delta_1 \Delta_2 \cdots \Delta_p \tau_\varepsilon \Delta_p \Delta_{p-1} \cdots \Delta_1$$

or, setting $\sigma = \Delta_p \Delta_{p-1} \cdots \Delta_1$:

$$\tau = \tau_\varepsilon \sigma^{-1} \tau_\varepsilon \sigma .$$

Assuming that $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1$ are the source and the sink of the transformation τ, for a suitable $\rho > 1$ we have

$$\tau(\tau_\varepsilon \boldsymbol{\gamma}_0) = \tau_\varepsilon \sigma^{-1} \tau_\varepsilon \sigma \tau_\varepsilon \boldsymbol{\gamma}_0 = \tau_\varepsilon \tau^{-1} \boldsymbol{\gamma}_0 = \rho \tau_\varepsilon \boldsymbol{\gamma}_0 .$$

In view of the unicity of $\boldsymbol{\gamma}_1$ (since $\rho \neq 1$) we must have

$$\tau_\varepsilon \boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_0 - 2(\boldsymbol{\gamma}_0, \boldsymbol{\varepsilon})\boldsymbol{\varepsilon} = \lambda \boldsymbol{\gamma}_1 \tag{8}$$

for some $\lambda > 0$ (cfr. the properties of the normalization in § 4.2). Scalar multiplication of (8) by $\boldsymbol{\gamma}_0$ yields $2(\boldsymbol{\gamma}_0, \boldsymbol{\varepsilon}) = \pm \sqrt{\lambda}$ so that choosing the positive sign (the orientation of $\boldsymbol{\varepsilon}$ is irrelevant) we obtain

$$\boldsymbol{\varepsilon} = \frac{1}{\sqrt{\lambda}} \boldsymbol{\gamma}_0 - \sqrt{\lambda} \boldsymbol{\gamma}_1 . \tag{9}$$

Considering the spheres $\boldsymbol{\alpha}_i$ in the different order

$$\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_{p+1}, \cdots, \boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_{p-1} ,$$

we obtain again the same link; the source and the sink of the corresponding Moebius transformation $\tau^* = \sigma \tau_\varepsilon \sigma^{-1} \tau_\varepsilon$ will then be the points $\boldsymbol{\gamma}'_0 = \sigma \boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}'_1 = \sigma \boldsymbol{\gamma}_1$. Therefore we must also have

[21] By τ_ε we mean the inversion generated by $\boldsymbol{\varepsilon}$.

$$\varepsilon = \frac{1}{\sqrt{\mu}}\gamma_0' - \sqrt{\mu}\gamma_1' \tag{10}$$

for some $\mu > 0$.

We set $\bar{\gamma} = (\varepsilon_1 - i\varepsilon_2)/2$, $\gamma = (\varepsilon_1 + i\varepsilon_2)/2$ where ε_1 and ε_2 are any real spheres containing γ_0 and γ_1 orthogonal to each other and to the sphere α_0; from (9) follows that

$$\tau_\varepsilon\bar{\gamma} = \bar{\gamma},\ \tau_\varepsilon\gamma = \gamma\ .$$

Now $(\sigma\bar{\gamma}, \sigma\gamma_0) = (\bar{\gamma}, \gamma_0) = 0$ and similarly $(\sigma\bar{\gamma}, \sigma\gamma_1) = (\sigma\gamma, \sigma\gamma_0) = (\sigma\gamma, \sigma\gamma_1) = 0$, therefore in view of (10) we deduce

$$\tau_\varepsilon\sigma\tau_\varepsilon\bar{\gamma} = \tau_\varepsilon\sigma\bar{\gamma} = \sigma\bar{\gamma}\ ,\ \tau_\varepsilon\sigma\tau_\varepsilon\gamma = \tau_\varepsilon\sigma\gamma = \sigma\gamma\ ,$$

and

$$\tau\bar{\gamma} = \sigma^{-1}\sigma\bar{\gamma} = \bar{\gamma}\ ,\ \tau\gamma = \sigma^{-1}\sigma\gamma = \gamma\ .$$ [22]

which is what we wanted to show.

Case 2 illustrates the intuitive fact that if a link L admits a plane of symmetry then whatever torsion L might inherit from one of its symmetric parts is taken away by the other. This property is not peculiar to natural links but it holds for all Riemann surfaces of genus one imbedded in E_3.

We shall give only a sketch of the proof for the general case.

If a surface admits a plane of symmetry then it admits an anticonformal (sense-reversing angle-preserving) mapping onto itself. This fact by itself is sufficient to exclude that the corresponding parallelgramm lattice could be a general one, it must have rectangular or rhomboidal generators.[23]

However, the case of rhomboidal generators can be excluded also. The anticonformal mapping generated by a plane of symmetry in E_3 will always leave invariant two distinct closed curves of the surface as loci of fixed points. On the other hand, if a rhomboidal lattice is a general one, the reflections which preserve the identification of points admit also two distinct invariant curves, but only one of them as a locus of fixed points.

5.5. In contrast with the results of the previous section, it is not difficult to construct natural links whose torsion does not vanish. The simplest models of such links can be obtained using five linearly independent spheres.

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[22] A shorter but less illustrative proof could be derived from the fact that the equation $\tau_\varepsilon\gamma = \nu\bar{\gamma}$ together with (8) leads to an absurdity.

[23] We owe this observation to Professor H. Royden.

In fact, we can show that
If a link L is generated by five spheres $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_4$ then its torsion vanishes if and only if the vectors $\boldsymbol{\alpha}_i$ are linearly dependent.

The torsion of L vanishes if and only if there exist vectors which are invariant under the product of inversions $\tau = \varDelta_5\varDelta_4 \cdots \varDelta_1$ generated by the spheres $\boldsymbol{\delta}_i$. Now, the transform of a vector $\boldsymbol{\xi}$ by τ (after a repeated application of formula (7) of § 4.2) can be written in the form

$$\tau\boldsymbol{\xi} = \boldsymbol{\xi} - 2(\boldsymbol{\xi}, \boldsymbol{\delta}_1)\boldsymbol{\delta}_1 - 2(\varDelta_1\boldsymbol{\xi}, \boldsymbol{\delta}_2)\boldsymbol{\delta}_2 - \cdots - 2(\varDelta_4 \cdots \varDelta_1\boldsymbol{\xi}, \boldsymbol{\delta}_5)\boldsymbol{\delta}_5$$

and the equation

$$(\boldsymbol{\xi}, \boldsymbol{\delta}_1)\boldsymbol{\delta}_1 + (\varDelta_1\boldsymbol{\xi}, \boldsymbol{\delta}_2)\boldsymbol{\delta}_2 + \cdots + (\varDelta_4 \cdots \varDelta_1\boldsymbol{\xi}, \boldsymbol{\delta}_5)\boldsymbol{\delta}_5 = 0$$

can be satisfied when and only when the $\boldsymbol{\delta}_i$'s are dependent. On the other hand if we let α denote the matrix whose columns are the vectors $\boldsymbol{\alpha}_i$, δ denote the matrix whose columns are the vectors $\boldsymbol{\delta}_i$, and set $\mu_i = \sqrt{1 + (\boldsymbol{\alpha}_{i-1}, \boldsymbol{\alpha}_i)/2}$[24] we have

$$\delta = \alpha \begin{vmatrix} 1/2\mu_1 & 0 & 0 & 0 & 1/2\mu_5 \\ 1/2\mu_1 & 1/2\mu_2 & 0 & 0 & 0 \\ 0 & 1/2\mu_2 & 1/2\mu_3 & 0 & 0 \\ 0 & 0 & 1/2\mu_3 & 1/2\mu_4 & 0 \\ 0 & 0 & 0 & 1/2\mu_4 & 1/2\mu_5 \end{vmatrix}$$

and

$$\det \delta = \frac{\det \alpha}{2^4\mu_1\mu_2 \cdots \mu_5}\,. \tag{11}$$

Thus the $\boldsymbol{\delta}_i$'s are dependent or independent together with the $\boldsymbol{\alpha}_i$'s. This proves the assertion.

This result does not quite solve our original problem of constructing models whose representative point in $\mathfrak{M}$ is off the imaginary axis, at least as long as we do not know when the point $\theta/2\pi - i(\log \rho)/2\pi$ is contained in $\mathfrak{M}$. We shall get around this difficulty by showing that our models can be made sufficiently thin (cfr. the inequalities (2) of § 5.1). To this end we shall exhibit a family of links within which this deformation is possible.

Let $\boldsymbol{C}_0, \boldsymbol{C}_1, \cdots, \boldsymbol{C}_4$ be points of E_3 and P denote the closed polygonal line $\boldsymbol{C}_0\boldsymbol{C}_1 \cdots \boldsymbol{C}_4\boldsymbol{C}_0$. Suppose that each segment $\overline{\boldsymbol{C}_i\boldsymbol{C}_{i+1}}$ ($i = 0, \cdots, 4$; $\boldsymbol{C}_5 = \boldsymbol{C}_0$) has length equal to twice that of the unit of measure, and set $2\varphi_i =$ angle $\boldsymbol{C}_{i-1}\widehat{\boldsymbol{C}_i\boldsymbol{C}}_{i+1}$. Let α_i be a sphere of radius R and center $\boldsymbol{C}_i$, i.e.,

[24] cfr. (10), (11) of § 4.2.

$$\alpha_i = \frac{1}{R}(1, C_i, C_i^2 - R^2) . \tag{12}$$

In order that the spheres $\boldsymbol{\alpha}_i$ fulfill the conditions (a), (b), (c), (d), (e) of § 2.3, so that they can be used to define a link, it is sufficient to require that for each $i = 1, \cdots, 5$ $\boldsymbol{\alpha}_{i-1}$ intersects $\boldsymbol{\alpha}_i$ and does not intersect $\boldsymbol{\alpha}_{i+1}$ (Set $\boldsymbol{\alpha}_6 = \boldsymbol{\alpha}_1$). We shall thus assume that P is such that

$$\varphi_i > \pi/6 + \sigma, \text{ or } \overline{C_{i-1}C_{i+1}} > 2(1 + \varepsilon) . \tag{13}$$

for some $0 < \sigma < \pi/3, 0 < \varepsilon < 1$, and restrict R to satisfy

$$1 < R < 1 + \varepsilon . \tag{14}$$

Let $L(P, R)$ denote the link defined by such a choice of P and R. From (12) follows that

$$\det \alpha = \frac{1}{R^5} \det \begin{vmatrix} 1 & C_1 & C_1^2 \\ 1 & C_2 & C_2^2 \\ \cdots & \cdots & \cdots \\ 1 & C_5 & C_5^2 \end{vmatrix} , \tag{15}$$

therefore the torsion of $L(P, R)$ vanishes if and only if the vertices of P lie on the same sphere. Now, it is geometrically evident that if we keep P fixed and let R decrease to 1 the capacities of the anuli $\Lambda_{i-1}^{+} \cap \Lambda_i^{-}$ will decrease to zero (see also formula (4) of § 5.2) and thus by the theorem of § 4.4 we can predict that the thinness of $L(P, R)$ will tend to infinity.

This proves the existence of links whose torsion does not vanish and whose representative point is in $\mathfrak{M}$.

More accurate results about the links $L(P, R)$ could be obtained by a direct calculation of the eigenvalues of the corresponding Moebius transformations. However, without going into tedious computations we can show that: *the portion of $\mathfrak{M}$ covered by the images of the links $L(P, R)$ contains a strip of constant width around the imaginary axis.*

It can be shown (see [6] pp. 26–28 and 154–155) that the characteristic polynomial of the Moebius transformation generated by a set of linearly independent spheres $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \cdots, \boldsymbol{\delta}_5$ is given by the expression

$$x(\lambda) = \det \begin{vmatrix} 1 + \lambda & 2(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) & \cdots 2(\boldsymbol{\delta}_1, \boldsymbol{\delta}_5) \\ 2\lambda(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) & 1 + \lambda & \cdots 2(\boldsymbol{\delta}_2, \boldsymbol{\delta}_5) \\ \cdots & \cdots & \cdots \\ 2\lambda(\boldsymbol{\delta}_1, \boldsymbol{\delta}_5) & 2\lambda(\boldsymbol{\delta}_2, \boldsymbol{\delta}_5) & \cdots 1 + \lambda \end{vmatrix} . \tag{16}$$

On the other hand, from the results of § 4.3 we have

$$x(\lambda) = (\lambda^2 - 2\cos\theta\lambda + 1)(\lambda^2 - 2\cosh\sigma\lambda + 1)(\lambda + 1)^{25} . \tag{17}$$

(we have set $\cos h\sigma = 1/2(\rho + 1/\rho)$). Evaluating (16) and (17) for $\lambda = 1$ and equating the results we obtain

$$\sin h^2\sigma/2 \sin^2\theta/2 = -\det \| (\boldsymbol{\delta}_i, \boldsymbol{\delta}_j) \| . \tag{18}$$

If we recall the definition of the scalar product ((4) of § 4.2) we see that it is

$$\| (\boldsymbol{\delta}_i, \boldsymbol{\delta}_j) \| = \delta^T \begin{pmatrix} 0 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 0 & 0 \end{pmatrix} \delta$$

this means that

$$\det \| (\boldsymbol{\delta}_i, \boldsymbol{\delta}_j) \| = -1/4\{\det \delta\}^2 .$$

Substituting in (18) we obtain

$$| \sin\theta/2 | = 1/2 \frac{| \det \delta |}{\sin h\sigma/2} . \tag{19}$$

We now observe that for a link $L(P, R)$ we have $(\boldsymbol{\alpha}_{i-1}, \boldsymbol{\alpha}_i) = 1 - 2/R^2$ and setting $r = \sqrt{R^2 - 1}$, (11) gives

$$\det \delta = \frac{R^5 \det \alpha}{2^4 r^5} ,$$

so that, using (15), (19) yields

$$| \sin\theta/2 | = \frac{\left| \det \begin{vmatrix} 1 & \boldsymbol{C}_0 & \boldsymbol{C}_0^2 \\ \cdots & \cdots & \cdots \\ 1 & \boldsymbol{C}_4 & \boldsymbol{C}_4^2 \end{vmatrix} \right|}{2^5 r^5 \sin h\sigma/2} . \tag{20}$$

We shall get upper and lower bounds for $\sin h\sigma/2$.

Let $\boldsymbol{\gamma}^0$ be the sink of the Moebius transformation corresponding to $L(P, R)$, and set

$$\boldsymbol{\gamma}^0 = (1, \boldsymbol{G}_0, \boldsymbol{G}_0^2),\ \Delta_1\boldsymbol{\gamma}^0 = \lambda_1\boldsymbol{\gamma}^1 = \lambda_1(1, \boldsymbol{G}_1, \boldsymbol{G}_1^2),\ \cdots,\ \Delta_5\boldsymbol{\gamma}^4 = \lambda_5\boldsymbol{\gamma}^5 = \lambda_5(1, \boldsymbol{G}_5, \boldsymbol{G}_5^2) = \rho\boldsymbol{\gamma}^0 .$$

Since $\boldsymbol{\delta}_i = 1/r(1, \boldsymbol{A}_i, \boldsymbol{A}_i^2)$ with $\boldsymbol{A}_i = (\boldsymbol{C}_{i-1} + \boldsymbol{C}_i)/2$, recalling formula (7)* of § 4.2 we obtain

$$\lambda_1 = \frac{\overline{G_0A_1}^2}{r^2} ,\ \lambda_2 = \frac{\overline{G_1A_2}^2}{r^2} ,\ \cdots,\ \lambda_5 = \frac{\overline{G_4A_5}^2}{r^2} ;$$

[25] $(\lambda + 1)$, since the number of spheres is odd.

this gives

$$\rho = \frac{\overline{G_0A_1}^2 \cdot \overline{G_1A_2}^2 \cdots \overline{G_4A_5}^2}{r^{10}} .$$

But each G_i is a point of the corresponding sphere α_i; thus we get

$$\rho \leqq \frac{(1+R)^{10}}{r^{10}} . \tag{21}$$

The theorem of § 4.4 gives a bound from below. Let C_i denote the capacity of the anulus $\Lambda_{i-1}^+ \cap \Lambda_i^-$, using (4) of § 5.2 and some geometrical considerations we obtain

$$C_i = \left\{\log \frac{R \sin \varphi_i + \sqrt{1 - R^2 \cos^2\varphi_i}}{R \sin \varphi_i - \sqrt{1 - R^2 \cos^2\varphi_i}}\right\}^{-1} ,$$

thus

$$\rho \geqq \frac{(R \sin \varphi_1 + \sqrt{1 - R^2 \cos^2 \varphi_1})^2 \cdots (R \sin \varphi_5 + \sqrt{1 - R^2 \cos^2\varphi_5})^2}{r^{10}} ;$$

since we keep $R < 2 \sin \varphi_i$ each of the factors in the numerator of the right hand side is greater than one therefore

$$\rho > \frac{1}{r^{10}} . \tag{22}$$

Finally (21) and (22) used in (20) yield (assuming $r \leqq 1$):

$$\frac{\left|\det \begin{vmatrix} 1 & C_0 & C_0^2 \\ \cdots & \cdots & \cdots \\ 1 & C_4 & C_4^2 \end{vmatrix}\right|}{2^4\{(R+1)^5 - (R-1)^5\}} \leqq |\sin \theta/2| \leqq \frac{\left|\det \begin{vmatrix} 1 & C_0 & C_0^2 \\ \cdots & \cdots & \cdots \\ 1 & C_4 & C_4^2 \end{vmatrix}\right|}{2^4(1 - r^{10})} . \tag{23}$$

These inequalities imply our assertion:

For each polygon $P \sim C_0C_1 \cdots C_4C_0$ let $D(P)$ denote the value of

$$\left|\det \begin{vmatrix} 1 & C_0 & C_4^2 \\ \cdots & \cdots & \cdots \\ 1 & C_4 & C_4^2 \end{vmatrix}\right| .$$

If P_0 is a regular pentagon of side 2 then the link $L(P_0, R)$ is certainly well defined when $1 < R\sqrt{2}$. Simple geometrical considerations together with formula (5) of § 5.3 show that the link $L(P_0, \sqrt{2})$ has a thinness ρ_0 for which $\log \rho_0 < 2\pi$. Let then P vary among the polygonals which satisfy the following conditions.

(1) $D(P) \neq 0$.

(2) The link $L(P, \sqrt{2})$ is well defined.

(3) The point $\nu(P) = (\theta(P)/2\pi) - i(\log \rho(P))/2\pi$ corresponding to $L(P, \sqrt{2})$ is contained in the region $|Re\nu| < 1/2, |\nu| \leqq 1$.

Assume $1 < R < \sqrt{2}$ and set $\nu(P, R) = (\theta(P, R)/2\pi) - i(\log \rho(P, R))/2\pi$ where $\theta(P, R)$ and $\rho(P, R)$ represent the thinness and the torsion of $L(P, R)$.

For every fixed P, as R decreases from $\sqrt{2}$ to 1, the point $\nu(P, R)$ describes a curve $M(P)$ which starts from a point outside $\mathfrak{M}$, enters $\mathfrak{M}$ for a suitably small value of R and tends to infinity from within $\mathfrak{M}$ as $R \to 1$.

The first inequality in (23) shows that each curve $M(P)$ is bounded away from the imaginary axis. Then, if we let P approach P_0, because of the second inequality in (23), $M(P)$ will tend to the imaginary axis and sweep a neighborhood of the type asserted.

A family of polygons satisfying the conditions (1), (2), (3) can be obtained from the following model. Let (x, y, z) be a cartesian coordinate system in E_3. Let

$$\boldsymbol{C}_0 = (1/\sin \pi/5, 0, 0),\ \boldsymbol{C}_1 = (x(\psi), y(\psi), z(\psi)),\ \boldsymbol{C}_2 = (-\cot \pi/5, 1, 0)$$
$$\boldsymbol{C}_3 = (-\cot \pi/5, -1, 0),\ \boldsymbol{C}_4 = (x(\psi), -y(\psi), -z(\psi))$$

with

$$x(\psi) = 1/2\frac{1}{\sin 2\pi/5} + 2\sin \pi/5 \sin \pi/10 \cos \psi$$
$$y(\psi) = 1/2 + 2\sin \pi/5 \cos \pi/10 \cos \psi$$
$$z(\psi) = 2 \sin \pi/5 \sin \psi\ ,$$

and set $P(\psi) \sim \boldsymbol{C}_0\boldsymbol{C}_1(\psi)\boldsymbol{C}_2\boldsymbol{C}_3\boldsymbol{C}_4(\psi)\boldsymbol{C}_0$. The points $\boldsymbol{C}_i$ have been chosen so that $P(0)$ is the regular pentagon of side 2 which lies in the plane x, y, has its center at the origin and a vertex in the positive real axis. When ψ varies $\boldsymbol{C}_1(\psi)$, $\boldsymbol{C}_4(\psi)$ describe the circles H, K loci of points whose distances from $\boldsymbol{C}_0$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_0$, $\boldsymbol{C}_3$ respectively are equal to 2. A short calculation gives (for $\psi < \pi/2$)

$$D(P) = 2^5 \sin \pi/5 \sin \pi/10 \sin \psi(1 - \cos \psi)\ . \tag{24}$$

It can be easily seen that the links $L(P(\psi), \sqrt{2})$ are well defined when defined when $|\psi| < \pi/4$ (the only critical distance in this range is $\overline{\boldsymbol{C}_1\boldsymbol{C}_4}$ and it is well above $2\sqrt{2}$).

Numerical estimates of the width of the strip covered are poor, since (21) is rather crude. Nevertheless using (23) and (24) with $R = 1.2$ and $\rho \geqq 11$ we obtain $|\theta| > 2$ degrees.

5.6. We shall conclude by showing that each natural M-surface can be deformed into a conformally equivalent C^∞ canal surface. Our construction is based on the following observation.

Let Γ be a Riemann surface, N a subregion of Γ and Λ the boundary of N. Let N^* be a Riemann surface with a boundary Λ^* and suppose there exists a conformal mapping Δ of N^* onto N which is defined and continuous up to Λ^*. Then we can make the set

$$\Gamma^* = (\Gamma - N) + N^*$$

into a Riemann surface conformally equivalent to Γ. The proof is immediate. We introduce local uniformizers in Γ^* so that the mapping $\varphi(P)$ of Γ^* onto Γ defined by

$$\begin{aligned} \varphi(P) &= P \quad \text{for} \quad P \in \Gamma^* - N^* \\ \phi(P) &= \Delta P \quad \text{for} \quad P \in N^* \end{aligned}$$

is conformal.[26]

We shall illustrate the use of this observation in a simple case. Suppose Γ is imbedded in E_3. Assume that N is a simply connected piece of a surface of revolution whose boundary is a parallel. Let N^* be any other simply connected piece of surface of revolution which has the same boundary and the same axis as N. The existence of the mapping Δ in this case is trivial. The observation can thus be applied, and we can deduce that Γ and $\Gamma^* = (\Gamma - N) + N^*$ must inherit the same conformal structure from E_3.

If Γ is C^∞ across Λ and we want Γ^* to possess the same property, then we have to restrict N^* to osculate N along Λ to an infinite degree.

Our next application will be the smoothing of natural M-surfaces. Let L be a given natural link and suppose that we want to render smooth the edge formed by the spheres Γ_1 and Γ_2 of L. Let Λ be the circle of intersection of Γ_1 and Γ_2. For simplicity we shall assume that the whole space has been subjected to a Moebius transformation so that Γ_1 and Γ_2 have become spheres of equal radius, their centers being interior points. Let Λ^-, Λ^+ be the portions of Γ_1 and Γ_2 which are exterior to Γ_2 and Γ_1 respectively, π the plane of Λ; π_1 and π_2 two planes parallel to π at a small distance ε from π. Assume that π_1 and π_2 intersect Λ^- and Λ^+ respectively and set

$$\Lambda_1 = \pi_1 \cap \Lambda^- \ , \ \Lambda_2 = \pi_2 \cap \Lambda^+ \ .$$

Let a be the straight line which contains the centers of Γ_1 and Γ_2, ν a half plane bounded by a; k_1 and k_2 the semicircles $\Gamma_1 \cap \nu$, $\Gamma_2 \cap \nu$ respectively. Let

[26] In § 2.3 we have proceeded in a similar way.

$$A_1 = \nu \cap \Lambda_1\ ,\ A = \nu \cap \Lambda\ ,\ A_2 = \nu \cap \Lambda_2\ .$$

Let N be the portion of L generated by the rotation of the arcs A_1k_1A and Ak_2A_2 around a.[27]

We shall choose k to be a curve of ν which joins A_1 to A_2 and fits with k_1 and k_2 at its end points in a C^∞ fashion. Let $N^*(k)$ be the surface of revolution generated by rotation of the arc A_1kA_2 around a. It is easy to see that when the non-Euclidean length of the arc A_1kA_2 in the half-plane ν is equal to the sum of the non-Euclidean lengths of the arcs A_1k_1A and Ak_2A_2 there exists a conformal mapping Δ of $N^*(k)$ onto N which leaves invariants the points of Λ_1 and Λ_2. And then, in view of our observation, $N^*(k)$ can be used to replace N in L. It remains to be shown that such a k can be found.

Let us first choose k to be the semicircle of ν which joins A_1 with A_2 and is orthogonal to a. Since k is then a geodesic, using the triangle inequality, we obtain

$$n.\mathscr{E}.l.A_1kA_2 < n.\mathscr{E}.l.A_1k_1A + n.\mathscr{E}.l.Ak_2A_2\ . \tag{25}$$

Now, k can be deformed at its end points to fit with k_1, and k_2 as smoothly as we please, increasing its length as little as we wish. Thereafter, if necessary, we can increase the length of k to change (25) into an equality.

To complete our argument we must show that L can be rendered smooth without introducing self-intersections. However, it is clear that k can be chosen to be a simple curve contained in the circle of center A and radius the (Euclidean) length of the segment $\overline{AA_1}$, for any given ε.

References

1. W. Blaschke, *Vorlesungen über Differentialgeometrie*, Berlin: Julius Springer, 1929, vol. III.
2. E. Cartan, *OEuvres completes*, Paris: Gauthier-Villars, 1955; vol. 2, part III, pp. 1701–1726.
3. J. L. Coolidge, *A Treatise on the Sphere and the Circle*, London: Oxford U. Press, 1916.
4. L. Ford, *Automorphic Functions*, New York: Chelsea Publ. Co., 1951.
5. A. Hurwitz and R. Courant, *Funktionentheorie*, Berlin: Julius Springer, 1925.
6. R. Lagrange, *Produits d'inversions et metrique conforme*, Cahiers Scientifiques, Fasc. XXIII, Paris: Gauthier-Villars, 1957.
7. G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Annals of of Math. Studies, No. 27, Princeton U. Press, (1951), 51–54.
8. F. Schottky, *Ueber eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Arguments unverändert bleibt*, Crelle's J., **101** (1887), 227–272.

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[27] This sentence is meaningful when ε is sufficiently small.

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