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I. PRESENTATION OF THE PROBLEM

1. Definitions.

1. A *boundary component* of a plane region $D \subset (|z| \leqq \infty)$ is a component of the boundary ∂D of D, i.e., a connected subset of ∂D which is not a proper subset of any connected subset of ∂D.

There is an alternate definition. Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of subregions of D such that

(i) $\Omega_1 \supset \Omega_2 \supset \cdots$,

(ii) the relative boundary $\partial\Omega_n \cap D$ consists of one closed analytic curve in D,

(iii) $\bigcap_{n=1}^{\infty} \Omega_n = \phi$. Two sequences $\{\Omega_n\}$ and $\{\Omega'_n\}$ are said to be equivalent if, for any n, there exists m such that $\Omega_m \subset \Omega'_n$ and $\Omega'_m \subset \Omega_n$. A *boundary component* of D is an equivalence class of $\{\Omega_n\}$.

These two definitions are equivalent in the following sense:

(i) Given a sequence $\{\Omega_n\}$, the set $\bigcap_{n=1}^{\infty} \bar{\Omega}_n$ is a component of ∂D and, for two sequences, these sets coincide if and only if the sequences are equivalent.

(ii) Given a component Γ of ∂D, there exists a sequence such that $\Gamma = \bigcap_{n=1}^{\infty} \bar{\Omega}_n$.

For a boundary component Γ, the sequence $\{\Omega_n\}$ such that $\Gamma = \bigcap_{n=1}^{\infty} \bar{\Omega}_n$ is called a *defining sequence of* Γ.

Let $w = f(z)$ be a topological mapping of D onto a plane region D'. Then we can immediately see from the second definition that f gives a one-to-one correspondence between the boundary components of D and D'. We shall speak of the *image of a boundary component* Γ *under* f in this sense and denote it by $f(\Gamma)$.

2. Let D^c denote the complement of D with respect to the extended plane $|z| \leqq \infty$. For a boundary component Γ, there exists a uniquely determined component of D^c whose boundary coincides with Γ. We call it the *component of* D^c *corresponding to* Γ and denote it by Γ^*.

If D does not contain the point $z = \infty$, the boundary component Γ

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such that $\infty \in \Gamma^*$ is called the *outer boundary of D.*

3. We call a region D a *circular* (or *radial*) *slit disk* if $0 \in D$, $D \subset (|z| < R < \infty)$, the outer boundary is $|z| = R$, and every other boundary component is either a point or an arc on $|z| = \text{const.}$ (or a line segment on $\arg z = \text{const.}$).

2. The stability problem of boundary components.

4. Let D be a plane region and let Γ be a boundary component. Sario [16, 17] gave the following classification:

(a) If $f(\Gamma)$ is a point for every univalent function $w = f(z)$ on D, then Γ is said to be *weak.*

(b) If $f(\Gamma)$ is a continuum, i.e., a connected closed set containing more than one point, for every f, then Γ is said to be *strong.*

(c) If Γ is neither weak nor strong, it is said to be *unstable.*

Weak boundary components were first investigated by Grötzsch in connection with the so-called "Kreisnormierungsproblem" (Grötzsch [7]; see also Denneberg [5] and Strebel [21]). He called them vollkommen punktförmig. Regions of class $O_{SB} = O_{SD}$ introduced by Ahlfors and Beurling [2] coincide with those possessing merely weak boundary components. Sario [16] has generalized the concept weak boundary components for open Riemann surfaces. It has been discussed also by Savage [19] and Jurchescu [10].

We are now lead to the following natural problems:

PROBLEM A. *Given a boundary component consisting of a single point, determine whether it is weak or unstable.*

PROBLEM B. *Given a boundary component consisting of a continuum, determine whether it is strong or unstable.*

We shall attempt to obtain concrete tests with practical applicability.

3. Related extremal problems.

5. Let D be a region containing the point $z = 0$. Let $\mathfrak{P}$ be the family consisting of all functions $w = \varphi(z)$ which are regular and univalent in $D - \{0\}$, and have the expansion $1/z + cz + \cdots$ near $z = 0$.

Consider, with Grötzsch [6], the diameter of the image $\varphi(\Gamma)$ of the boundary component Γ. It is quite easy to see that *Γ is weak if and only if* $\sup_{\varphi \in \mathfrak{P}} \operatorname{diam} \varphi(\Gamma) = 0$, *and Γ is* strong *if* $\inf_{\varphi \in \mathfrak{P}} \operatorname{diam} \varphi(\Gamma) > 0$.

6. Let $\mathfrak{F}_\Gamma$ be the family consisting of functions $w = f(z)$ such that

(i) regular and univalent in D,

(ii) $f(0) = 0$ and $f'(0) = 1$,

(iii) $f(\Gamma)$ is the outer boundary of $f(D)$.

Rengel [14] introduced the following functionals on $\mathfrak{F}_\Gamma$:

$$M(f) = \max_{w \in f(\Gamma)} |w| = \sup_{z \in D} |f(z)| ,$$

$$m(f) = \min_{w \in f(\Gamma)} |w| ,$$

and considered the quantities

$$R(\Gamma) = R(\Gamma;\ D) = \sup_{f \in \mathfrak{F}_\Gamma} m(f)$$

and

$$r(\Gamma) = r(\Gamma;\ D) = \inf_{f \in \mathfrak{F}_\Gamma} M(f) .$$

From the definition we have immediately the basic

THEOREM 1. *Γ is strong if $R(\Gamma) < \infty$. Γ is weak if and only if $r(\Gamma) = \infty$.*

These criteria are equivalent to those in No. 5, since

$$R(\Gamma) = 2/\inf_{\varphi \in \mathfrak{B}} \operatorname{diam} \varphi(\Gamma) ,$$

$$r(\Gamma) = 4/\sup_{\varphi \in \mathfrak{B}} \operatorname{diam} \varphi(\Gamma) .$$

In fact, for an arbitrary function $f(z) \in \mathfrak{F}_\Gamma$, the functions

$$\varphi_f(z) = \frac{1}{f(z)} + \frac{f''(0)}{2}$$

and

$$\psi_f(z) = \varphi_f(z) + \frac{1}{M(f)^2} \cdot \frac{1}{\varphi_f(z)}$$

belong to $\mathfrak{B}$, and

$$m(f) \leqq 2/\operatorname{diam} \varphi_f(\Gamma)$$

$$M(f) \geqq 4/\operatorname{diam} \varphi_f(\Gamma) .$$

On the other hand, for $\varphi(z) \in \mathfrak{B}$, let $F(w)$ be the function which maps $(\varphi(\Gamma)^*)^c$ conformally onto the exterior of a disk with the center at the origin. Assume further that $F(w) = w + c + c'/w + \cdots$ near $w = \infty$. Then $f_\varphi(z) = 1/F \circ \varphi(z) \in \mathfrak{F}_\Gamma$ and

$$2/\operatorname{diam} \varphi(\Gamma) \leqq M(f_\varphi) = m(f_\varphi) \leqq 4/\operatorname{diam} \varphi(\Gamma) .$$

The proof of the above equalities is hereby complete.

7. Whether or not $R(\Gamma) < \infty$ is necessary for strength is still an open problem. We shall discuss this problem in No. 24.

We shall see in No. 17 that $1/r(\Gamma)$ equals the "capacity" of the boundary component Γ introduced by Sario [16] (it is not necessarily equal to the logarithmic capacity of the closed set Γ), and, therefore, that the latter half of Theorem 1 is equivalent to Sario's result ([17], Theorem 6). Jurchescu [10] showed that the "capacity" coincides with the "perimeter" introduced by Ahlfors and Beurling [2].

It will be shown in No. 22 that $R(\Gamma)$ coincides with the quantity which Strebel [22] called "extremal Durchmesser". Finally, Theorem 4 in No. 21 shows that the first half of the above theorem coincides with Sario's result ([17], Theorem 4).

II. Preliminaries

In this chapter, we collect a number of known results which will be needed later.

4. Extremal length.

8. A curve γ considered here is either a closed rectifiable curve or a curve of the form $z = z(t)$ $(0 < t < 1)$ every subarc of which is rectifiable. If $\lim_{t\to 0} z(t)$ or $\lim_{t\to 1} z(t)$ exists, it is called an end point.

Let D be a reginon and let $\{\gamma\}$ be a family of curves $\gamma \subset D$. Let $\{\rho\}$ be the collection of functions ρ which are $\geqq 0$ and lower semi-continuous in D. With the understanding that $0/0 = \infty/\infty = 0$, take

$$\lambda\{\gamma\} = \sup_{\rho} \frac{\left(\inf_{\gamma} \int_{\gamma} \rho\, ds\right)^2}{\iint_D \rho^2\, dxdy}.$$

It is called the *extremal length of* $\{\gamma\}$ (Ahlfors and Beurling [2], Ahlfors and Sario [3]).

9. The following properties (I)–(V) are well known; for the proofs the reader is referred to, e.g., Hersch [8][1]:

(I) $\lambda\{\gamma\}$ is independent of the choice of D.

(II) $\lambda\{\gamma\}$ is conformally invariant.

(III) $\lambda\{\gamma'\} \leqq \lambda\{\gamma\}$ if every γ contains a γ'.

(IV) For $\{\gamma_1\}$ and $\{\gamma_2\}$, assume the existence of disjoint regions D_1 and D_2 such that $\gamma_\nu \subset D_\nu$ $(\nu = 1, 2)$. If, for any γ of the third family

[1] His definition is different from ours, but his proofs remain valid.

$\{\gamma\}$, there exist γ_1 and γ_2 such that $\gamma_1 \cup \gamma_2 \subset \gamma$, then

$$\lambda\{\gamma_1\} + \lambda\{\gamma_2\} \leqq \lambda\{\gamma\} .$$

(V) Let $\{\gamma_1\}$ and $\{\gamma_2\}$ be the same as above. If $\{\gamma_1\} \cup \{\gamma_2\} \subset \{\gamma\}$, then

$$\frac{1}{\lambda\{\gamma_1\}} + \frac{1}{\lambda\{\gamma_2\}} \leqq \frac{1}{\lambda\{\gamma\}} .$$

(VI) (Hersch [8][1]). For three families with $\{\gamma\} = \{\gamma_1\} \cup \{\gamma_2\}$,

$$\frac{1}{\lambda\{\gamma\}} \leqq \frac{1}{\lambda\{\gamma_1\}} + \frac{1}{\lambda\{\gamma_2\}} .$$

(VIII) Let $\{\gamma_1\}$ be the subfamily of $\{\gamma\}$ consisting of γ having both end points and such that $z(t)$ $(0 \leqq t \leqq 1)$ is rectifiable. Then $\lambda\{\gamma\} = \lambda\{\gamma_1\}$.

In fact, since the extremal length of $\{\gamma_2\} = \{\gamma\} - \{\gamma_1\}$ is infinite, (VI) shows that $\lambda\{\gamma_1\} \leqq \lambda\{\gamma\}$, and $\lambda\{\gamma\} \leqq \lambda\{\gamma_1\}$ by (III).

(VIII) For a curve $\gamma : z = z(t)$ $(0 < t < 1)$, let $\bar{\gamma}$ be the curve $z = \overline{z(t)}$ $(0 < t < 1)$. If $z(0) = \lim_{t\to 0} z(t)$ exists and is real, put $\hat{\gamma} = \gamma \cup \bar{\gamma} \cup \{z(0)\}$. Let $\{\gamma_0\}$ be a family of curves which are contained in the upper half-plane and have the end points $z(0)$ on the real axis. Let $\{\gamma\}$ be a family which contains all $\hat{\gamma}_0$ and $\bar{\gamma}$. Furthermore it is assumed that, for any γ, there exist γ_0 and γ_0' in $\{\gamma_0\}$ such that $\bar{\gamma}_0 \cup \gamma_0' \subset \gamma$. Then

$$\lambda\{\gamma\} = 2\lambda\{\gamma_0\} .$$

In fact, to define $\lambda\{\gamma\}$, we may restrict $\{\rho\}$ to the subfamily consisting of functions symmetric about the real axis. Since $2 \inf_{\gamma_0}\int_{\gamma_0} \rho\, ds = \inf_{\gamma}\int_{\gamma} \rho\, ds$ for such ρ, we conclude that $\lambda\{\gamma\} = 2\lambda\{\gamma_0\}$.

(IX) Let A be the annulus $1 < |z| < q$ or a region obtained by deleting a finite number of circular slits from this annulus. Let $\{\gamma\}$ be the family of all closed rectifiable curves in A separating $|z| = 1$ from $|z| = q$. Then $\lambda\{\gamma\} = 2\pi/\log q$. This is true even if each γ is restricted to a concentric circle in A.

The proof is found, e.g., in Hersch [8][1].

10. Let D be a region, and let E_0 and E_1 be compact sets such that $E_\nu \cap \bar{D} \neq \phi$ $(\nu = 0, 1)$. Let $\{\gamma\}$ be the family consisting of γ: $z = z(t)$ $(0 < t < 1)$ such that $\gamma \subset D$, $\bigcap_{\varepsilon>0}\overline{\{z(t);\ 0 < t < \varepsilon\}} \subset E_0$, and $\bigcap_{\varepsilon>0}\overline{\{z(t);\ 1 - \varepsilon < t < 1\}} \subset E_1$. Then $\lambda\{\gamma\}$ is called the *extremal distance* $\delta_D(E_0, E_1)$ *between* E_0 *and* E_1 *with respect to* D.

By (VII), $\delta_D(E_0, E_1)$ coincides with the extremal length of the family of rectifiable curves in D whose end points are on E_0 and E_1 respectively. Under a certain restriction of the configuration, it is also equal to that of a subfamily consisting of analytic curves (Wolontis [25]).

From this consideration, we get

(X) If no point of E_1 is accessible from D by a rectifiable curve, then $\delta_D(E_0, E_1) = \infty$.

(XI) (Pfluger [12][1]). If cap $E_1 = 0$, then $\delta_D(E_0, E_1) = \infty$. For $D = (|z| = 1)$, $E_0 = (|z| = \varepsilon < 1)$, and $E_1 \subset (|z| = 1)$, $\delta_D(E_0, E_1) = \infty$ if and only if cap $E_1 = 0$.

Combining (VI), (X), and (XI), we get

(X′) If no point on E_1, except for a set of capacity zero, is accessible from D by a rectifiable curve, then $\delta_D(E_0, E_1) = \infty$.

(XII) Let D, E_0, and E_1 be contained in the closed upper half-plane. Let $\hat{D}$ be the region which is the union of D, the reflection of D across the real axis, and the part of ∂D on the real axis. Let $\hat{E}_0$ and $\hat{E}_1$ have analogous meanings. If $\delta_{\hat{D}}(\hat{E}_0, \hat{E}_1)$ is expressed in terms of the extremal length of a family consisting of analytic curves[2], then

$$\delta_{\hat{D}}(\hat{E}_0, \hat{E}_1) = \frac{1}{2}\delta_D(E_0, E_1) .$$

Proof. Let $\delta_{\hat{D}}(\hat{E}_0, \hat{E}_1) = \lambda\{\gamma\}$ where γ is an analytic curve and let $\delta_D(E_0, E_1) = \lambda\{\gamma'\}$. Using the notation in (VII), we see immediately that $\{\gamma'\}$ and $\{\bar{\gamma}'\}$ are contained in $\{\gamma\}$. Since $\lambda\{\gamma'\} = \lambda\{\bar{\gamma}'\}$, we find, on applying (V), that $\lambda\{\gamma\} \leqq \lambda\{\gamma'\}/2$.

In order to prove the inequality in the opposite direction, we first remark that, to define $\lambda\{\gamma\}$, we may restrict ρ to a function symmetric about the real axis. For a curve $\gamma : z = z(t)$ $(0 < t < 1)$, let γ^* be

$$z = \begin{cases} z(t) & \text{if } \Im z(t) \geqq 0 \\ \overline{z(t)} & \text{if } \Im z(t) \leqq 0 . \end{cases}$$

Evidently $\int_\gamma \rho\, ds = \int_{\gamma^*} \rho\, ds$ for a symmetric ρ.

Since it is assumed that γ is an analytic curve, γ^* intersects the real axis at only a finite number of points $z_1, z_2, \cdots, z_k$. Let $\varDelta_\nu$ be the punctured disk $0 < |z - z_\nu| < r$ $(\nu = 1, 2, \cdots, k)$, where r is taken so small that the $\varDelta_\nu$ are mutually disjoint. The extremal length of the family of curves in $\varDelta_\nu$ separating z_ν from $|z - z_\nu| = r$ is, by (IX), equal to infinite. Therefore, for arbitrary $\varepsilon > 0$ and ρ, there exists a closed curve $\gamma_\nu \subset \varDelta_\nu$ encircling z_ν and such that $\int_{\gamma_\nu} \rho\, ds < \varepsilon/k$. On replacing a part of $\gamma^* \cap \varDelta_\nu$ by a part of $\gamma_\nu (\nu = 1, 2, \cdots, k)$, we obtain from γ^* a

[2] This restriction is satisfied in our subsequent applications. It is perhaps superfluous. However, the author has not succeeded in furnishing the proof without it.

curve γ' belonging to the family $\{\gamma'\}$ and such that $\int_{\gamma'}\rho\,ds - \varepsilon < \int_{\gamma}\rho\,ds$. Since γ and ε are arbitrary, we get $\inf_{\gamma'}\int_{\gamma'}\rho\,ds \leqq \inf_{\gamma}\int_{\gamma}\rho\,ds$ for every symmetric ρ. Since $\iint_{\hat{D}}\rho^2\,dxdy = 2\iint_{D}\rho^2\,dxdy$, we conclude that $\lambda\{\gamma'\} \leqq 2\lambda\{\gamma\}$.

(XIII) Let A be the annulus $1 < |z| < q$ or a region obtained by deleting a finite number of radial slits from this annulus. Let $E_0 = (|z| = 1)$ and $E_1 = (|z| = q)$. Then $\delta_A(E_0, E_1) = (\log q)/2\pi$, and it is also equal to the extremal length of the family of all radials from E_0 to E_1 in A.

For the proof, the reader is referred to, e.g., Strebel [20].

5. Teichmüller's extremal region.

11. Let D be a doubly connected region and let $\{\gamma\}$ be the family of all closed rectifiable curves in D separating the boundary components. The quantity $2\pi/\lambda\{\gamma\}$ is called the *modulus of* D and is denoted by mod D. As is well known, D can be mapped conformally onto an annulus $1 < |z| < q$ where $\log q = \text{mod } D$.

For $P > 0$, the doubly connected region

$$D_P = \{[-1,\ 0] \cup [P,\ \infty]\}^c$$

where the brackets express a closed interval on the real axis, is called *Teichmüller's extremal region.* It has the following extremal property (Teichmüller [23]): *Let D be a doubly connected region such that one component of D^c contains the point $z = 0$ as well as a point on $|z| = 1$ and the other contains the point $z = \infty$ as well as a point on $|z| = P$. Then* $\text{mod } D \leqq \text{mod } D_P$ *and the equality holds if and only if D is a region obtained by rotating D_P about the origin.*

12. It was proved by Teichmüller [23] that $\Psi(P) = \exp(\text{mod } D_P)$ is a continuous function of P such that

$$\lim_{P\to\infty}\frac{\Psi(P)}{P} = 16\ . \tag{1}$$

It is easy to see that

$$\log\Psi\left(\frac{1}{P}\right) = \frac{\pi^2}{\log\Psi(P)}\ . \tag{2}$$

On combining (1) and (2), we have

$$\log\Psi(P) \sim \frac{\pi^2}{\log\dfrac{1}{P}} \qquad \text{for } P \to 0\ . \tag{3}$$

13. The following result will be used later:

LEMMA 1. *Let*

$$A = (1 < |z| < q)\ ,$$
$$\Gamma = (|z| = 1)\ ,$$

and

$$E_\theta = \{z;\ |z| = q,\ |\arg z| \leqq \theta\}\ .$$

Then

$$\delta_A(\Gamma,\ E_\theta) \sim \frac{1}{\pi}\log\frac{1}{\theta} \qquad \textit{for } \theta \to 0\ .$$

Proof. $\delta_A(\Gamma,\ E)$ is equal to the extremal length $\lambda\{\gamma\}$ where $\{\gamma\}$ is the family of all analytic curves in A connecting Γ with E_θ (cf. Wolontis [25]). By (VIII) and (XIII), it is equal to $\delta_Q(E'_\theta,\ E''_\theta)/4$ where

$$Q = (1/q < |z| < q) \cap (\Im z > 0)\ ,$$
$$E'_\theta = \{z;\ |z| = 1/q,\ 0 \leqq \arg z \leqq \theta\}\ ,$$

and

$$E''_\theta = \{z;\ |z| = q,\ 0 \leqq \arg z \leqq \theta\}\ .$$

Map Q onto the upper half-plane in such a way that $1/q$ and q correspond to 0 and 1, respectively. Let $-\alpha$ and $1 + \beta$ ($\alpha,\ \beta > 0$) be the images of $e^{i\theta}/q$ and $qe^{i\theta}$, respectively. It is not difficult to see that

$$\begin{cases} \alpha \sim c\dfrac{\theta^2}{q} \\ \beta \sim c'q\theta^2 \end{cases} \qquad \text{for } \theta \to 0$$

where c and c' are constants independent of θ. The region obtained by deleting the intervals $[-\infty,\ -\alpha]$, $[0,\ 1]$, and $[1+\beta,\ \infty]$ from the extended plane is conformally equivalent to Teichmüller's extremal region with

$$P = \frac{\alpha\beta}{1 + \alpha + \beta} \sim c''\theta^4 \qquad (\theta \to 0)\ .$$

Therefore, on applying (VIII) again, we get $\delta_A(\Gamma,\ E_\theta) = \pi/(4\log\Psi(P))$ and, by (3),

$$\delta_A(\Gamma,\ E_\theta) \sim \frac{1}{4\pi}\log\frac{1}{P} \sim \frac{1}{\pi}\log\frac{1}{\theta} \qquad \text{for } \theta \to 0\ .$$

6. Koebe's distortion theorem.

14. The following is a slight modification of the original form of Koebe's well-known distortion theorem, which will be used frequently:

Let $\varphi(z)$ be a function which is univalent and regular in $|z| < \varepsilon_0$ with $\varphi(0) = 0$ and $\varphi'(0) = 1$. Then there are numbers $a(\varepsilon)$ and $b(\varepsilon)$ which are independent of φ and have the properties that

$$a(\varepsilon) \leqq |\varphi(z)| \leqq b(\varepsilon) \qquad \text{on } |z| = \varepsilon < \varepsilon_0$$

and

$$\lim_{\varepsilon \to 0} \frac{a(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{b(\varepsilon)}{\varepsilon} = 1 .$$

In fact, we may take $a(\varepsilon) = \varepsilon\varepsilon_0^2/(\varepsilon + \varepsilon_0)^2$ and $b(\varepsilon) = \varepsilon\varepsilon_0^2/(\varepsilon - \varepsilon_0)^2$.

7. Quasi-conformal mappings.

15. In Chapters IV and V, we shall make use of quasi-conformal mappings to illustrate our results by examples. As in the type problem of Riemann surfaces, they are utilized to replace a given region by a simpler one.

A sense-preserving topological mapping $w = T(z)$ of a region D onto another is said to be *quasi-conformal* if there exists a finite number K such that mod $T(Q) \leqq K$ mod Q for any quadrilateral $Q \subset D$ (Ahlfors [1]). Here, mod Q of a quadrilateral Q means the extremal distance between two opposite sides of Q. The minimum value of K is called the *maximal dilatation of* T.

For the proofs of the following properties (I)–(III), the reader is referred to Ahlfors [1]:

(I) If T is quasi-conformal of maximal dilatation K, then mod $T(A) \leqq K$ mod A for any doubly connected region $A \subset D$.

(II) Let E be a set which is contained in a finite number of analytic arcs. Let D be a region containing E, and let T be a topological mapping of D which is quasi-conformal in $D - E$. Then it is quasi-conformal in D with the same maximal dilatation.

(III) If T is a topological mapping of class C^1, then the maximal dilatation is given by $K = \sup_{z \in D} (|T_z| + |T_{\bar{z}}|)/(|T_z| - |T_{\bar{z}}|)$ where T_z and $T_{\bar{z}}$ are complex derivatives.

(IV) Let $\{\gamma\}$ be a family of curves in D. Let T be a quasi-conformal mapping of class C^1 with the maximal dilatation K. Then

$$\lambda\{T(\gamma)\} \leqq K\lambda\{\gamma\} .$$

The proof is found in Hersch [9][1].

REMARK. Even if T is not of class C^1 throughout D, this inequality holds under, e.g., the following restriction: T is of C^1 in D except for a countable number of analytic arcs clustering nowhere in D, i.e., every point of D has a neighborhood intersecting at most a finite number of the arcs, and every γ is the union of a countable number of analytic arcs clustering nowhere in D. This generalization will be needed in No. 35.

III. CIRCULAR AND RADIAL SLIT DISKS

8. Circular slit disks.

16. Let D be a plane region containing the point $z = 0$, and let Γ be a boundary component. The problem of minimizing $M(f)$ in $\mathfrak{F}_\Gamma$ for a region of finite connectivity has been discussed by Rengel [14]. To consider it for a region of arbitrary connectivity, in particular to show the uniqueness of the minimizing function, Sario [16] introduced the functional

$$J(f) = \int_{\partial D} \log|f| \cdot d\arg f \qquad (f \in \mathfrak{F}_\Gamma)\,.$$

Here the line integral means $\lim_{n\to\infty} \int_{\partial D_n} \log|f| \cdot d\arg f$ for an exhaustion $D_n \uparrow D$; the limiting value exists and is independent of the exhaustion. He proved the existence of a function g_0 such that

$$M(g_0) = m(g_0)$$

and

$$2\pi \log M(g_0) = J(f) - D(\log|f| - \log|g_0|)$$

for all $f \in \mathfrak{F}_\Gamma$, where the second term means the Dirichlet integral over D. Evidently g_0 is the unique function which minimizes $J(f)$.

From these relations we can derive the following facts (Sario [16]):

(I) *There exists a function $g_0 \in \mathfrak{F}_\Gamma$ such that $M(g_0) = \min_{f \in \mathfrak{F}_\Gamma} M(f) = r(\Gamma)$. If $r(\Gamma) < \infty$, the minimizing function is determined uniquely. It maps D onto a circular slit disk $|w| < r(\Gamma)$, where the area of slits, i.e., $g_0(\partial D - \Gamma)^*$, vanishes,*

(II) *Let $0 \in D_n \uparrow D$ be an exhaustion and let Γ_n be the component of ∂D_n separating D_n from Γ. Then*

$$r(\Gamma) = \lim_{n\to\infty} r(\Gamma_n)\,.$$

If $r(\Gamma) < \infty$, the sequence $\{g_n\}$ of the minimizing functions on D_n converges to g_0 uniformly on each compact set in D.

17. By making use of this result, we can express $r(\Gamma)$ in terms of extremal length. Let ε_0 be a small number such that $|z| \leqq \varepsilon_0$ is contained in D. For $0 < \varepsilon < \varepsilon_0$, the numbers $a(\varepsilon)$ and $b(\varepsilon)$ were defined in No. 14. The following theorem has been proved, in essence, by Jurchescu [10]:

THEOREM 2. *Let* $\{\gamma\}_\varepsilon$ *be the family of all closed curves in* $D_\varepsilon = D - (|z| \leqq \varepsilon)$ *which separate* Γ *from the point* $z = 0$. *Then*

$$\log \frac{r(\Gamma)}{b(\varepsilon)} \leqq \frac{2\pi}{\lambda\{\gamma\}_\varepsilon} \leqq \log \frac{r(\Gamma)}{a(\varepsilon)}$$

and, therefore,

$$\log r(\Gamma) = \lim_{\varepsilon \to 0} \left(\log \varepsilon + \frac{2\pi}{\lambda\{\gamma\}_\varepsilon} \right).$$

The result remains valid if the γ *are restricted to analytic curves.*

Proof. Consider the metric given by $\rho = |g_0'| / |g_0|$. Since the area of the circular slits is zero, $\iint_{D_\varepsilon} \rho^2 \, dxdy \leqq 2\pi \log (r(\Gamma)/a(\varepsilon))$. Therefore,

$$\lambda\{\gamma\}_\varepsilon \geqq (2\pi)^2/2\pi \log (r(\Gamma)/a(\varepsilon)) .$$

To prove the left inequality, take an exhaustion $D_n \uparrow D$ and consider the family $\{\gamma_n\}_\varepsilon$ of all closed curves γ_n in $D_n - (|z| \leqq \varepsilon)$ separating Γ_n from $z = 0$. Since D_n is of finite connectivity, the proposition (IX), No. 9, shows that $2\pi/\lambda\{\gamma_n\}_\varepsilon \geqq \log (r(\Gamma_n)/b(\varepsilon))$. When we take the limit for $n \to \infty$, we have by virtue of the relation $\lambda\{\gamma\}_\varepsilon \leqq \lambda\{\gamma_n\}_\varepsilon$ that

$$2\pi/\lambda\{\gamma\}_\varepsilon \geqq \log (r(\Gamma)/b(\varepsilon)) .$$

18. The following criterion for weakness due to Grötzsch [7] will be useful in the next chapter:

THEOREM 3. *In order that* Γ *be weak, it is necessary and sufficient that, for an arbitrary positive number* l, *there exist a finite number of doubly connected regions* $A_1, A_2, \cdots A_k$ *in* $D - (|z| \leqq \varepsilon)$ *satisfying the following conditions*:

(i) *The* A_ν *are mutually disjoint,*

(ii) A_ν *separates* Γ *from* $(|z| \leqq \varepsilon)$ $(\nu = 1, 2, \cdots, k)$ *and separates* $A_{\nu-1}$ *from* $A_{\nu+1}$ $(\nu = 2, 3, \cdots, k-1)$,

(iii)

$$\sum_{\nu=1}^{k} \operatorname{mod} A_\nu \geqq l .$$

Proof. Sufficiency: By (V), No. 9, and by Theorem 2, $l \leqq \sum_{\nu=1}^{k} \bmod A_\nu \leqq 2\pi/\lambda\{\gamma\}_\varepsilon \leqq \log(r(\Gamma)/(\varepsilon))$. Therefore, $r(\Gamma) = \infty$ and, by Theorem 1, Γ is weak.

Necessity: Take an exhaustion $(|z| \leqq \varepsilon) \subset D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots \uparrow D$ and consider the extremal function g_n on D_n. By Koebe's distortion theorem, No. 14, the image of $|z| = \varepsilon$ is contained in $a(\varepsilon) \leqq |w| \leqq b(\varepsilon)$, so that the set $b(\varepsilon) < |w| < r(\Gamma_n)$ minus the circular slits is contained in the image of $D_n - (|z| \leqq \varepsilon)$. From the annulus $b(\varepsilon) < |w| < r(\Gamma_n)$, delete all the concentric circles containing the circular slits. Then we get a finite number of concentric annuli $A'_1, A'_2, \cdots, A'_k$ such that $\sum_{\nu=1}^{k} \bmod A'_\nu = \log\,(r(\Gamma_n)/b(\varepsilon))$. Since $r(\Gamma) = \lim_{n\to\infty} r(\Gamma_n) = \infty$, we can take n so large that the right hand side is greater than the given l. The inverse images $A_1, A_2, \cdots, A_k$ of $A'_1, A'_2, \cdots, A'_k$ are what we desired.

REMARK. We see from this theorem that the weakness of Γ depends merely on the configuration of ∂D near l. Furthermore, by (I), No. 15, the weakness is invariant under quasi-conformal mappings.

9. Radial slit disks for special regions.

19. Unlike the case of the functional $M(f)$, the function maximizing $m(f)$ does not exist in general; by slightly modifying the example given by Strebel [20], we get a region on which $m(f) < R(\Gamma) = \sup_{f \in \mathfrak{F}_\Gamma} m(f)$ for all $f \in \mathfrak{F}_\Gamma$.

Under a restriction, however, we get a result analogous to that of No. 15. Let G be a region containing the point $z = 0$ and such that a component Γ of ∂G consists of a closed analytic curve which is isolated, i.e., $\overline{\partial G - \Gamma} \cap \Gamma = \phi$. Let $\mathfrak{A}_\Gamma$ be the subfamily of $\mathfrak{F}_\Gamma$ consisting of all functions with $M(f) = m(f)$. On this family Sario [17, 18] introduced the functional

$$I(f) = 2\pi \log m(f) - \int_{\partial D - \Gamma} \log |f| \cdot d \arg f$$

and proved the existence of a function $f_0 \in \mathfrak{A}_\Gamma$ such that

$$(4) \qquad 2\pi \log m(f_0) = I(f) + D(\log |f| - \log |f_0|)$$

for all $f \in \mathfrak{A}_\Gamma$. Evidently f_0 is the unique maximizing function of $I(f)$ in $\mathfrak{A}_\Gamma$.

We can derive from this relation the following facts (Sario [18]), which have been obtained by Rengel [14] for a region G of finite connectivity:

(I) *$R(\Gamma)$ is finite. f_0 is the unique function maximizing $m(f)$ in $\mathfrak{A}_\Gamma$. It maps G onto a radial slit disc $|w| < R(\Gamma)$, where the area of slits, i.e., $f_0(\partial G - \Gamma)^*$, vanishes.*

(II) *Let $\{G_n\}$ be a sequence of regions such that $0 \in G_n \uparrow G$ and ∂G_n consists of Γ and a finite number of closed analytic curves. Then*

$$R(\Gamma;\ G) = \lim_{n\to\infty} R(\Gamma_n;\ G_n)$$

and the sequence $\{f_n\}$ of the maximizing functions on G_n converges to f_0 uniformly on each compact set in $G \cup \Gamma$.

20. Let $\{\gamma\}_\varepsilon$ be the family of rectifiable curves which connect $|z| = \varepsilon$ with Γ in $G - (|z| \leqq \varepsilon)$. In a method similar to the proof of Theorem 2 we can obtain the following relations:

$$\frac{\left(\log \dfrac{R(\Gamma)}{b(\varepsilon)}\right)^2}{\log \dfrac{R(\Gamma)}{a(\varepsilon)}} \leqq 2\pi\lambda\{\gamma\}_\varepsilon \leqq \log \frac{R(\Gamma)}{a(\varepsilon)}\ , \tag{5}$$

$$\log R(\Gamma) = \lim_{\varepsilon\to 0} (\log \varepsilon + 2\pi\lambda\{\gamma\}_\varepsilon)\ . \tag{6}$$

Here $\{\gamma\}_\varepsilon$ can be replaced by the subfamily of analytic curves.

10. Characterizations of $R(\Gamma)$.

21. Let D be an arbitrary region containing the point $z = 0$. Let $\{\Omega_n\}_{n=1}^\infty$ be a defining sequence of Γ such that $0 \notin \Omega_n$ $(n = 1, 2, \cdots)$. Then $G_n = D - \Omega_n$ is a region and its boundary component $\Gamma_n - \partial G_n \cap \partial\Omega_n$ satisfies the condition of No. 19.

THEOREM 4. *$\{R(\Gamma_n, G_n)\}_{n=1}^\infty$ is an increasing sequence and $R(\Gamma) = \lim_{n\to\infty} R(\Gamma_n;\ G_n)$.*

Proof. $\{R(\Gamma_n;\ G_n)\}$ is an increasing sequence by (6).

For an arbitrary $\varepsilon > 0$, there exists an $f(z) \in \mathfrak{F}_\Gamma$ such that $m(f) > R(\Gamma) - \varepsilon/2$. Then there exists an n_0 such that the m of this $f(z)$ on G_n (we denote it by $m_n(f)$) has the property that $m_n(f) > m(f) - \varepsilon/2$ whenever $n \geqq n_0$. Therefore, $R(\Gamma_n;\ G_n) \geqq m_n(f) > R(\Gamma) - \varepsilon$ and $\underline{\lim}_{n\to\infty} R(\Gamma_n;\ G_n) \geqq R(\Gamma)$.

Next, let A_n be the doubly connected region bounded by Γ_n and Γ. Then Γ is an isolated boundary component of the region $\tilde{G}_n = G_n \cup A_n \cup \Gamma_n$. Γ is not necessarily a closed analytic curve, but from the result of No. 19 we can see the existence of the function $\tilde{f}_n(z)$ in $\mathfrak{F}_\Gamma$ of $\tilde{G}_n$ such that $m(\tilde{f}_n) = R(\Gamma;\ \tilde{G}_n)$. Evidently $\tilde{f}_n(z)$ belongs to $\mathfrak{F}_\Gamma$ of D. By (6),

$R(\Gamma_n;\ G_n) \leqq R(\Gamma;\ \tilde{G}_n)$. Consequently, $R(\Gamma_n;\ G_n) \leqq R(\Gamma;\ \tilde{G}_n) = m(\tilde{f}_n) \leqq R(\Gamma)$ and $\overline{\lim}_{n\to\infty} R(\Gamma;\ G_n) \leqq R(\Gamma)$.

This reasoning remains valid for the case where $R(\Gamma) = \infty$.

REMARK. Combining Theorem 4 with Theorem 1, we see that $\lim_{n\to\infty} R(\Gamma_n;\ G_n) < \infty$ implies the strength of Γ. This fact was proved by Sario [17].

22. Let $\{\gamma\}_\varepsilon$ be the family of curves $\gamma : z = z(t)$ $(0 < t < 1)$ in $D - (|z| \leqq \varepsilon)$ such that $\bigcap_{\varepsilon>0} \overline{\{z(t);\ 0 < t < \varepsilon\}} \subset (|z| = \varepsilon)$ and $\bigcap_{\varepsilon>0} \overline{\{z(t);\ 1 - \varepsilon < t < 1\}} \subset \Gamma$. Let $\{\gamma_n\}_\varepsilon$ be the corresponding family in G_n. Strebel [22] has proved the relation $\lambda\{\gamma\}_\varepsilon = \lim_{n\to\infty} \lambda\{\gamma_n\}_\varepsilon$. On combining this with (5), (6), and Theorem 4, we have

THEOREM 5.

$$\frac{\left(\log \dfrac{R(\Gamma)}{b(\varepsilon)}\right)^2}{\log \dfrac{R(\Gamma)}{a(\varepsilon)}} \leqq 2\pi\lambda\{\gamma\}_\varepsilon \leqq \log \frac{R(\Gamma)}{a(\varepsilon)} ,$$

$$\log R(\Gamma) = \lim_{\varepsilon\to 0} (\log \varepsilon + 2\pi\lambda\{\gamma\}_\varepsilon) .$$

Here γ can be restricted to the curve which is the union of a countable number of analytic arcs which cluster nowhere in D (cf. No. 15, Remark).

REMARK. The exponential of the right hand side of the second relation was called "extremal Durchmesser" by Strebel [22]. On combining Theorem 5 with Theorem 1, or directly from (XI), No. 10, we see that $\lambda\{\gamma\}_\varepsilon < \infty$ implies the strength of Γ. This result was generalized for open Riemann surfaces by Constantinescu [4].

23. For an exhaustion $D_n \uparrow D$ in the ordinary sense, it has not been proved whether $\lim_{n\to\infty} R(\Gamma_n;\ D_n)$ exists or not. We obtain merely the following

THEOREM 6. *Let Δ be a region such that $0 \in \Delta$, $\bar{\Delta} \subset D$, and bounded by a finite number of closed analytic curves. Denote by Γ_Δ the component of $\partial\Delta$ which separates Δ from Γ. Then*

$$R(\Gamma) = \varinjlim_{\Delta\to D} R(\Gamma_\Delta;\ \Delta) ,$$

where the right hand side is a directed limit.

Proof. For $\varepsilon > 0$, there exists by Theorem 4 an n such that

$R(\Gamma) - \varepsilon < R(\Gamma_n;\ G_n)$. By Theorem 5 $R(\Gamma_n;\ G_n) \leqq R(\Gamma_\Delta;\ \Delta)$ for any $\Delta \supset \Gamma_n \cup \{0\}$. Therefore, $R(\Gamma) \leqq \underline{\lim}_{\Delta\to D} R(\Gamma_\Delta;\ \Delta)$. On the other hand, for $\varepsilon > 0$ and a compact set $K \subset D$, take an n_0 such that $K \subset G_{n_0}$. There exists, by (II), No. 19, a $\Delta \subset G_{n_0}$ such that $R(\Gamma_\Delta;\Delta) \subset R(\Gamma_{n_0}; G_{n_0}) + \varepsilon$, and, therefore, $R(\Gamma_\Delta;\Delta) < R(\Gamma) + \varepsilon$. Consequently $\underline{\lim}_{\Delta\to D} R(\Gamma_\Delta;\Delta) \leqq R(\Gamma)$.

REMARK. On combining Theorem 6 with Theorem 1 we see that $\underline{\lim}_{\Delta\to D} R(\Gamma_\Delta;\ \Delta) < \infty$ implies the strength of Γ. Sario [18] has shown that Γ is strong if $\overline{\lim}_{\Delta\to D} R(\Gamma_\Delta;\ \Delta) < \infty$.

11. Unsolved problems.

24. As we pointed out in No. 7, the following problem has not been solved:

(1) Is $R(\Gamma) < \infty$ necessary for the strength of Γ?

Since the maximizing function of $m(f)$ in $\mathfrak{F}_\Gamma$, or equivalently the minimizing function of diam $\varphi(\Gamma)$ in $\mathfrak{B}$, does not exist in general, the case is different from that of a weak boundary component. The example of Strebel [20] stated in No. 19 is for $R(\Gamma) > \infty$, and it does not answer this question.

Let $\{G_n\}_{n=1}^{\infty}$ be the sequence introduced in No. 21 and let $f_n(z)$ be the extremal function on G_n. Since $\{f_n\}_{n=1}^{\infty}$ is a normal family, we may assume that f_n converges to a univalent function $f(z)$. One can imagine that, if $R(\Gamma) = \infty$, then $f(\Gamma)$ would be a point. However, we can only prove that $f(\Gamma)$ consists of the point $w = \infty$ and possibly of radial segments emanating from it whose arguments form a set of measure zero (Strebel [22]). Such line segments appear in our Example 10, Nos. 39, 40. Nevertheless the boundary component of this example is unstable, because we can map it onto a region such that $f(\Gamma)$ is a point and $f(\partial D - \Gamma)$ consists of circles (No. 39).

We have several other unsolved problems as follows:

(2) Is strength a boundary property?

(3) Is $\overline{\lim}_{\Delta\to D} R(\Gamma_\Delta;\ \Delta)$ equal to $\underline{\lim}_{\Delta\to D} R(\Gamma_\Delta;\ \Delta)$?

(4) Is strength preserved under quasi-conformal mappings?

IV. CRITERIA FOR WEAKNESS AND INSTABILITY

In this chapter we consider Problem A presented in No. 4. Several sufficient conditions for weakness have been obtained by Savage [19]. Here we shall consider some special regions and attempt to get more concrete necessary or sufficient conditions.

12. Boundary on the positive real axis.

25. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences of positive numbers such that

$$1 < b_{n-1} \leqq a_n < b_n \qquad (n = 1, 2, \cdots),$$

$$\lim_{n\to\infty} a_n = \infty .$$

Denote by $[a, b]$ the closed interval on the real axis. Then

$$D = (|z| < \infty) - \bigcup_{n=1}^{\infty} [b_{n-1}, a_n]$$

is a region and $\Gamma = \{\infty\}$ is its boundary component. The present section is devoted to discussing the following problem: *When is Γ weak and when is it unstable?*

26. THEOREM 7. (i) *If*

$$\sum_{n=1}^{\infty} \left(\frac{b_n}{a_n} - 1\right) = \infty , \tag{7}$$

then Γ is weak.

(ii) *If*

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 1 \tag{8}$$ [3]

and

$$\sum_{n=1}^{\infty} \frac{1}{\log \dfrac{1}{(b_n/a_n) - 1}} < \infty \tag{9}$$

then Γ is unstable.

Proof. (i) Consider the annuli $A_n = (a_n < |z| < b_n)$ $(n = 1, 2, \cdots)$. Since $\sum \operatorname{mod} A_n = \sum \log (b_n/a_n) = \infty$, Theorem 3 shows that Γ is weak.

(ii) Let $A_1, A_2, \cdots, A_k$ be doubly connected regions satisfying the conditions (i) and (ii) of Theorem 3. For any A_ν, there exists an n such that A_ν passes through the open interval (a_n, b_n) and a component of A_ν contains 0 as well as a_n. The region

$$D^{(n)} = \{[0, a_n] \cup [b_n, \infty]\}^c$$

is conformally equivalent to Teichmüller's extremal region with $P = (b_n/a_n) - 1$. By the extremal property of $D^{(n)}$, No. 11, the sum of the

[3] If $\overline{\lim}_{n\to\infty} b_n/a_n > 1$, then Γ is weak by (i), Theorem 7

moduli of all such A_ν does not exceed mod $D^{(n)} = \log \Psi((b_n/a_n) - 1)$.

$$\sum_{\nu=1}^{k} \operatorname{mod} A_\nu \leqq \sum_{n=1}^{\infty} \log \Psi\left(\frac{b_n}{a_n} - 1\right). \tag{10}$$

By (3), No. 12,

$$\log \Psi\left(\frac{b_n}{a_n} - 1\right) \sim \frac{\pi^2}{\log \dfrac{1}{(b_n/a_n) - 1}}.$$

Therefore, the right hand side of (10) converges and, by Theorem 3, Γ is unstable.

EXAMPLE 1. $a_n = 2n + 1$, $b_n = 2n + 2$. Evidently (7) diverges so that *Γ is weak.*

EXAMPLE 2. $a_n = n^k$, $b_n = n^k + 1$ $(k > 1)$. Since (7) converges and (9) diverges, we cannot decide by Theorem 7 (see also No. 27).

EXAMPLE 3. $a_n = e^n$, $b_n = e^n + 1$. Similarly, we cannot decide (see also No. 27).

EXAMPLE 4. $a_n = e^{n^\alpha}$, $b_n = e^{n^\alpha} + 1$ $(\alpha > 1)$. *Γ is unstable* by (ii).

27. We derive another criterion applicable to Examples 2 and 3. To this end, we first prove

LEMMA 2. *For the doubly connected region*

$$A_h = (1 < |z| < q) - [1 + h,\ q)$$

where $h > 0$ and q is fixed,

$$\operatorname{mod} A_h \sim \frac{\pi^2}{2 \log \dfrac{1}{h}} \qquad \textit{for } h \to 0 .$$

Proof. By (VIII), No. 9, mod $A_h = 4\pi/\lambda\{\gamma\}$ where $\{\gamma\}$ is the family of rectifiable curves in $Q = A_h \cap (\Im z > 0)$ joining $[-q,\ -1]$ with $[1,\ 1 + h]$. Map Q conformally onto the upper half-plane in such a manner that $-q$, -1, 1 correspond to $-\infty$, -1, 0, respectively. The image P of $1 + h$ has the property that

$$P \sim ch^2 \qquad \text{for } h \to 0$$

where c is a constant independent of h. From (VIII), No. 9, we conclude that

$$\text{mod } A_h = \log \Psi(P) \sim \frac{\pi^2}{\log \dfrac{1}{P}} \sim \frac{\pi^2}{2 \log \dfrac{1}{h}} \qquad (h \to 0) .$$

THEOREM 8. *Suppose that* $\lim_{n\to\infty} b_n/a_n = 1$. *If* a_{n+1}/a_n *is bounded away from* 1, *then* Γ *is weak if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\log \dfrac{1}{(b_n/a_n) - 1}} = \infty .$$

Proof. If the series converges, Γ is unstable by (ii) of Theorem 7.

Conversely, suppose that the series diverges. The doubly connected region $A_n = (a_n < |z| < a_{n+1}) - [b_n, a_{n+1})$ is conformally equivalent to the region $A'_n = (1 < |z| < a_{n+1}/a_n) - [b_n/a_n, a_{n+1}/a_n)$. By the assumption $1 < 1 + \delta < a_{n+1}/a_n$ and, therefore, $A''_n = (1 < |z| < 1 + \delta) - [b_n/a_n, 1 + \delta) \subset A'_n$ so that $\text{mod } A''_n \leqq \text{mod } A_n$. By Lemma 2

$$\text{mod } A''_n \sim \frac{\pi^2}{2 \log \dfrac{1}{(b_n/a_n) - 1}} \qquad (n \to \infty) .$$

Consequently, the assumption implies that $\sum \text{mod } A_n = \infty$, and we infer from Theorem 3 that Γ is weak.

EXAMPLE 3 (No. 26). $a_n = e^n$, $b_n = e^n + 1$. By Theorem 8, Γ *is weak.*

EXAMPLE 2 (No. 26). $a_n = n^k$, $b_n = n^k + 1$ $(k > 1)$. Since $a_{n+1}/a_n = (n+1)^k/n^k$ is not bounded away from 1, the above theorem is not applicable. However, we can see as follows that Γ *is weak.* For simplicity, we consider the case $k = 2$; the general case can be treated in a similar fashion. Consider the region $A_n = (a_{2^n} < |z| < a_{2^{n+1}}) - [b_{2^n}, a_{2^{n+1}})$, which is conformally equivalent to $(1 < |z| < 4) - [1 + 2^{-2n}, 4)$. By Lemma 2, $\text{mod } A_n \sim \pi^2/(4n \log 2)$ for $n \leftarrow \infty$ and $\sum \text{mod } A_n = \infty$. It follows from Theorem 3 that Γ is weak.

More generally, this result can be stated as follows:

THEOREM 8′. *Suppose that* $\lim_{n\to\infty} b_n/a_n = 1$ *and that there exists a subsequence* $\{n_i\} \subset \{n\}$ *such that* $a_{n_{i+1}}/a_{n_i}$ *is bounded away from* 1 *and*

$$\sum_{i=1}^{\infty} \frac{1}{\log \dfrac{1}{(b_{n_i}/a_{n_i}) - 1}} = \infty . \tag{12}$$

Then Γ *is weak.*

28. When a_{n+1}/a_n is not bounded away from 1, we may also apply the following criterion:

THEOREM 9. *Suppose* $\lim_{n\to\infty} b_n/a_n = 1$ *and* $\lim_{n\to\infty} a_{n+1}/a_n = 1$. *If*

$$\lim_{n\to\infty} \frac{\log (b_n/a_n)}{\log (a_{n+1}/a_n)} \tag{13}$$

exists, then

$$\sum_{n=1}^{\infty} \frac{\log (a_{n+1}/a_n)}{\log \dfrac{1}{\left(\dfrac{b_n}{a_n}\right)^{1/\log(a_{n+1}/a_n)} - 1}} = \infty \tag{14}$$

implies that Γ is weak.

Proof. Consider the doubly connected region $A'_n = (1 < |z| < q_n) - [1 + h_n, q_n)$ $(n = 1, 2, \cdots)$, where $0 < h_n < q_n - 1$ and $\lim_{n\to\infty} q_n = 1$. Map the annulus $1 < |z| < q_n$ onto $1 < |w| < e$ by the quasi-conformal mapping

$$w = T_n(z) = r^{1/\log q_n} e^{i\theta} \qquad (z = re^{i\vartheta}) .$$

Its dilatation equals $1/\log q_n$ provided n is so large that $q_n < e$. The image of A'_n is $A''_n = (1 < |w| < e) - [(1 + h_n)^{1/\log q_n}, e)$. From (I), No. 15, we have

$$\log q_n \cdot \operatorname{mod} A''_n \leqq \operatorname{mod} A'_n . \tag{15}$$

Now suppose that $\lim_{n\to\infty} (\log (1 + h_n))/\log q_n$ exists. If

$$\lim_{n\to\infty} (1 + h_n)^{1/\log q_n} > 1 ,$$

then $\operatorname{mod} A''_n$ and $\log \{1/[(1 + h_n)^{1/\log q_n} - 1]\}$ are bounded and bounded away from zero. Hence the divergence of

$$\sum_{n=1}^{\infty} \frac{\log q_n}{\log \dfrac{1}{(1 + h_n)^{1/\log q_n} - 1}} \tag{16}$$

implies that $\sum_{n=1}^{\infty} \log q_n \cdot \operatorname{mod} A''_n = \infty$ and, by (14), that $\sum_{n=1}^{\infty} \operatorname{mod} A'_n = \infty$. If $\lim_{n\to\infty} (1 + h_n)^{1/\log q_n} = 1$ we obtain by Lemma 2

$$\log A''_n \sim \frac{\pi^2}{2 \log \dfrac{1}{(1 + h_n)^{1/\log q_n} - 1}} \qquad (n \to \infty) .$$

Therefore, the divergence of (16) again implies that of $\sum_{n=1}^{\infty} \operatorname{mod} A'_n$.

In the given region, consider $A_n = (a_n < |z| < a_{n+1}) - [b_n, a_{n+1})$. It is conformally equivalent to the above A'_n for $1 + h_n = b_n/a_n$ and $q_n = a_{n+1}/a_n$. Therefore, $\sum_{n=1}^{\infty} \operatorname{mod} A_n = \infty$ and Γ is weak.

This criterion is applicable to Example 2.

EXAMPLE 5. $a_n = n$, $b_n = n + e^{-n}$. In this case (7) converges and (9) diverges, so that we cannot decide by Theorem 7. Since a_{n+1}/a_n is not bounded away from zero, we cannot apply Theorem 8.[4] For every subsequence such that $\underline{\lim}_{i\to\infty} a_{n_{i+1}}/a_{n_i} > 1$, (12) converges, and we cannot use Theorem 8′. (14) also converges and, therefore 9 is inapplicable. We have not been able to decide whether Γ is weak or unstable. In general, for $a_n = n$, $b_n = n + e^{-n^\alpha}$ $(\alpha > 0)$, Γ is unstable for $\alpha > 1$ but it is unknown if it remains true for $0 < \alpha \leqq 1$.

13. A generalization.

29. Consider the case where the intervals are distributed on the whole real axis. We treat again the simplest case.

PROBLEM. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be the sequence of positive numbers such that*

$$0 < b_{n-1} \leqq a_n < b_n \qquad (n = 1, 2, \cdots)$$

$$\lim_{n\to\infty} a_n = \infty .$$

Consider the region

$$\tilde{D} = (|z| < \infty) - \bigcup_{n=1}^{\infty} [b_{n-1}, a_n] - \bigcup_{n=1}^{\infty} [-a_n, -b_{n-1}] .$$

Under what condition is $\tilde{\Gamma} = \{\infty\}$ a weak boundary component of $\tilde{D}$?

This problem can be reduced to the case which we discussed in the previous section. More precisely, let $\Gamma = \{\infty\}$ be a boundary component of

$$D = (|z| < \infty) - \bigcup_{n=1}^{\infty} [b_{n-1}, a_n] ;$$

then we have

THEOREM 10. *$\tilde{\Gamma}$ is weak if and only if Γ weak.*

Proof. If Γ is unstable, then, since $\tilde{D} \subset D$, $\tilde{\Gamma}$ is unstable by the definition.

[4] The author is indebted to Professor R. Redheffer for the argument that follows in this example.

Suppose that $\tilde{\Gamma}$ is unstable. Since weakness is a boundary property (No. 18), we may assume without loss of generality that $b_0 > 1$. By Theorem 2, $\lambda\{\gamma\} > 0$ where $\{\gamma\}$ is the family of curves in $\tilde{D} - (|z| \leqq 1)$ separating $\tilde{\Gamma}$ from $|z| = 1$. Let $\{\gamma_1\}$ be the family consisting of curves in the upper half of $\tilde{D} - (|z| \leqq 1)$ connecting $(1, \infty) - \bigcup_{n=1}^{\infty} [b_{n-1}, a_n]$ with $(-\infty, -1) - \bigcup_{n=1}^{\infty} [-a_n, -b_{n-1}]$. Let $\{\gamma_1'\}$ be its subfamily consisting of curves whose end points are symmetric with respect to the origin. Then, by (VIII), No. 9,

$$\lambda\{\gamma_1'\} \geqq \lambda(\gamma_1\} = \lambda\{\gamma\}/2 > 0 .$$

Consider the region $\Delta = (|\zeta| < \infty) - \bigcup_{n=1}^{\infty} [b_{n=1}^2, a_n^2]$ and its boundary component $(\zeta = \infty)$. Let $\{\gamma^*\}$ be the family of curves in $\Delta - (|\zeta| \leqq 1)$ separating ∞ from $|\zeta| \leqq 1$. By making use of the mapping $\zeta = z^2$, we can immediately see that $\lambda\{\gamma^*\} = \lambda\{\gamma_1'\}$ and, therefore, $(\zeta = \infty)$ is an unstable boundary component of Δ.

The mapping

$$\zeta = T(z) = r^2 e^{i\theta} \qquad (z = re^{i\theta})$$

is quasi-conformal and maps D onto Δ, $z = \infty$ onto $\zeta = \infty$. Since weakness is preserved under quasi-conformal mappings (No. 18), Γ is unstable.

Remark. Using the same method, we can also prove Theorem 10 when the intervals are distributed on k half-lines $re^{i2\pi\nu/k}$ $(0 \leqq r < \infty)$, $\nu = 0, 1, \cdots, k$.

14. Criteria for arbitrary regions.

30. Let D be a plane region such that $\Gamma = \{\infty\}$ is a boundary component. If D is contained in another region discussed in preceding sections and $\{\infty\}$ is its unstable boundary component, then, by the definition of instability, Γ is an unstable boundary component of D.

If such a condition is not satisfied, the following criterion may be applicable. It is a simple generalization of (ii) of Theorem 7, and we omit the proof.

Theorem 11. *Let D be a region such that $0 \in D$ and $\Gamma = \{\infty\}$ is a boundary component. Γ is unstable if there exists a sequence $\{C_n\}_{n=1}^{\infty}$ of components of $\partial D - \Gamma$ satisfying the following conditions:*

(i) *For a doubly connected region $A \subset D$ separating 0 from ∞, there exists a number n such that A separates C_n from C_{n+1}.*

(ii) *For every n, there exist points $a_n \in C_n$ and $b_n \in C_{n+1}$ such that $|a_n - b_n| = \operatorname{dist}(C_n, C_{n+1})$,*

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 1$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\log \dfrac{1}{|(b_n/a_n) - 1|}} < \infty .$$

31. This criterion is not a necessary condition for instability. This is apparent from the following

EXAMPLE 6. Consider the closed sets

$$E_n = \{z;\ n^2 + 1 \leqq |z| \leqq (n+1)^2,\ |\arg z| \leqq \pi - \varepsilon_n\} ,$$

$$0 < \varepsilon_n < \pi , \qquad n = 1, 2, \cdots .$$

If $\varepsilon_n (n = 1, 2, \cdots)$ are taken sufficiently small, then $\Gamma = \{\infty\}$ is an unstable boundary component of $D = (|z| < \infty) - \bigcup_{n=1}^{\infty} E_n$. It does not satisfy the assumption of Theorem 11.

Proof. For an arbitrary subsequence $\{C_n\}_{n=1}^{\infty}$ of $\{E_n\}_{n=1}^{\infty}$ and every choice of a_n and b_n,

$$\sum_{n=1}^{\infty} \frac{1}{\log \dfrac{1}{|(b_n/a_n) - 1|}} \geqq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\log n} = \infty .$$

Therefore, the assumption of Theorem 11 is not satisfied.

In order to show the instability of Γ, consider the following cross cuts of D:

$$\alpha_n:\ \Re z = 0,\ (n+1)^2 \leqq \Im z \leqq (n+1)^2 + 1 ,$$

$$\beta_n:\ |z| = (n+1)^2,\ |\arg z| \leqq \pi - \varepsilon_n ,$$

$$\beta'_n:\ |z| = (n+1)^2 + 1,\ |\arg z| \leqq \pi - \varepsilon_{n+1} ,$$

$$(n = 1, 2, \cdots) .$$

Let δ_n be the extremal distance between α_n and $\beta_n \cup \beta'_n$ with respect to the region $(n+1)^2 < |z| < (n+1)^2 + 1$. It is possible to take ε_n and ε_{n+1} so small that $\delta_n > n^2$ $(n = 1, 2, \cdots)$. Let $\{\gamma\}_n$ be the family consisting of closed curves in $D - (|z| \leqq 1)$ separating Γ from $|z| \leqq 1$ and passing through α_n. Let $\{\gamma_1\}_n \subset \{\gamma\}_n$ be the subfamily of curves contained in $(n+1)^2 < |z| < (n+1)^2 + 1$ and put $\{\gamma_2\}_n = \{\gamma\}_n - \{\gamma_1\}_n$. By (VI), No. 9,

$$\frac{1}{\lambda\{\gamma\}_n} \leqq \frac{1}{\lambda\{\gamma_1\}_n} + \frac{1}{\lambda\{\gamma_2\}_n}.$$

Since $n^2 < \delta_n \leqq \lambda\{\gamma_2\}_n$ and $2\pi/\lambda\{\gamma_1\}_n = \log(1 + 1/(n+1)^2)$, we get

$$\frac{1}{\lambda\{\gamma\}_n} \leqq \frac{1}{2\pi}\log\left(1 + \frac{1}{(n+1)^2}\right) + \frac{1}{n^2}$$

if n is sufficiently large, and, therefore, $\sum_{n=1}^{\infty} 1/\lambda\{\gamma\}_n$ converges.

To apply Theorem 3, take $A_1, A_2, \cdots, A_k$. Then evidently

$$\sum_{\nu=1}^{k} \operatorname{mod} A_\nu \leqq \sum_{n=1}^{\infty} 1/\lambda\{\gamma\}_n < \infty$$

and we conclude that Γ is unstable.

32. Finally, for the sake of completeness, we shall present a well-known sufficient condition for weakness. For a bounded doubly connected region A, we have that $\operatorname{mod} A \geqq \log(1 + (\pi d/4l))$. Here d is the distance between the components of ∂A and l is the infimum of the lengths of closed curves which separate the components of ∂A and whose distance from ∂A is $\geqq d/2$ (Sario [15], Meschkowsky [11]). Therefore we get immediately from Theorem 3 the following result (Meschkowsky [11], Savage [19]):

THEOREM 12. *Let D be a plane region containing the point $z = 0$ and such that $\Gamma = \{\infty\}$ is a boundary component. Suppose there exists a sequence of doubly connected regions $A_n \subset D - (|z| \leqq \varepsilon)$ $(n = 1, 2, \cdots)$ with the following properties:*

(i) *The A_n are mutually disjoint,*

(ii) *A_n separates Γ from $|z| \leqq \varepsilon$ $(n = 1, 2, \cdots)$ and also separates A_{n-1} from A_{n+1} $(n = 2, 3, \cdots)$,*

(iii)

$$\sum_{n=1}^{\infty} d_n/l_n = \infty\ .$$

Then Γ is a weak boundary component of D.

On applying this theorem, we obtain

EXAMPLE 7 (Denneberg [5]). Let D be a region such that $\Gamma = \{\infty\}$ is the only accumulating boundary component. If there exist numbers $\alpha > 0$ and $\beta < \infty$ such that the distance between every pair components of $\partial D - \Gamma$ is $\geqq \alpha$ and the diameter of every component of $\partial D - \Gamma$ is $\leqq \beta$, then Γ is weak.

EXAMPLE 8 (Cf. Wagner [24]). Let $\mathfrak{G}$ be the group of transformations $z' = z + m\omega + n\omega'$ $(m, n = 0, \pm 1, \pm 2, \cdots)$ and let E_0 be a closed set contained in the interior of the fundamental parallelogram of $\mathfrak{G}$. Then $\Gamma = \{\infty\}$ is a weak boundary component of the region $D = (|z| < \infty) - \bigcup_{T \in \mathfrak{G}} T(E_0)$.

V. Criteria for Strength and Instability

In this chapter we shall discuss Problem B, No. 4. For simplicity we mean by a *boundary continuum* a boundary component of a region which is a continuum containing more than one point.

15. Strong boundary components.

33. If Γ is an isolated boundary continuum of D, i.e., if there exists an open set U such that $\Gamma \subset U$ and $U \cap (\partial D - \Gamma) = \phi$, then Γ is evidently strong. More generally,

Theorem 13. *A boundary continuum Γ of a region D is strong if there exists a disk U such that $U \cap \Gamma \neq \phi$ and $U \cap (\partial D - \Gamma) = \phi$.*

This theorem is also almost trivial. To prove it rigorously, we shall use the following

Lemma 3. *Let Δ be a simply connected region which is a proper subset of $(|\zeta| < 1)$. Map Δ conformally onto the upper half-plane. Then the image E of $\overline{\partial\Delta \cap (|\zeta| < 1)}$ is a set which does not belong to the class N_D.*[5]

The proof is easy and we omit it. It may appear plausible that E contains an interval. That this is however not so has been remarked by Koebe (see Radó [13], p. 2, Bemerkung). We can even see that the condition of Lemma 3 is necessary and sufficient.

Proof of Theorem 13. Map a component Δ of $U \cap D$ onto the upper half-plane by φ and let E be the image of $\Gamma \cap \bar{\Delta}$. By Lemma 3 $E \notin N_D$ and, therefore, E is of positive measure (Ahlfors and Beurling [2]). If Γ is unstable, a univalent function $f(z)$ transforms Γ to a point. Therefore, the univalent function $f \circ \varphi$ on the upper half-plane takes a constant boundary value on E, contrary to the well-known theorem of F. and M. Riesz.

Remark 1. In this case, $R(\Gamma) < \infty$ and we can also use Theorem 1 to conclude that Γ is strong. To prove the finiteness of $R(\Gamma)$, we apply Theorem 5. Take a component V of $U \cap D$. It is easy to find

[5] A compact set E is said to belong to the class N_D if E^c does not admit a function with a finite Dirichlet integral.

a simply connected region Δ such that $\Delta \subset D$, $V \subset \Delta$ and $(|z| \leqq \varepsilon) \subset \Delta$. Since the set $E \notin N_D$ is of positive capacity (Ahlfors and Beurling [2]), $\lambda\{\gamma\}_\varepsilon < \infty$ by Lemma 3 and (XI), No. 10.

REMARK 2. Because of this theorem, we may consider from now on only the case where every point of Γ is an accumulation point of $\partial D - \Gamma$.

34. We shall now give two other kinds of examples of strong boundary components which do not satisfy the condition of Theorem 13.

EXAMPLE 7. Let D be a radial slit disc $|z| < a$ in the sense of No. 3 and let $\Gamma = (|z| = a)$. If the arguments of the slits form a set of measure μ less than 2π, then $R(\Gamma) < \infty$ and, consequently, Γ is strong.

In fact, we can easily obtain the estimate

$$\lambda\{\gamma\}_\varepsilon \leqq \{\log(a/\varepsilon)\}/(2\pi - \mu) < \infty .$$

35. EXAMPLE 8. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of numbers such that $0 < c_n \leqq \pi/2^{n+1}$. Put $r_n = 1 - 1/(n+1)$ and let

$$s_n^k = \left\{z;\ |z| = r_n,\ \frac{\pi(k-1)}{2^n} + c_n \leqq \arg z \leqq \frac{\pi k}{2^n} - c_n\right\}$$
$$(k = 1, 2, \cdots, 2^{n+1};\ n = 1, 2, \cdots) .$$

$\Gamma = (|z| = 1)$ is a boundary continuum of the circular slit disc $D = (|z| < 1) - \bigcup_{n,k} s_n^k$. If $\underline{\lim}_{n\to\infty} c_n 2^n > 0$, then $R(\Gamma) < \infty$ and therefore, Γ is strong.

Proof. Clearly it is sufficient to give the proof for $c_n 2^n = \delta > 0$. For simplicity, we choose $\delta = \pi/4$, i.e., $c_n = \pi/2^{n+2}$. In order to show the finiteness of $R(\Gamma)$, we map D quasi-conformally onto the radial slit disc $\Delta = (|w| < 1) - \bigcup_{n,k} \sigma_n^k$, where

$$\sigma_n^k = \left\{w;\ r_n e^{-c_n/2} \leqq |w| \leqq r_n e^{c_n/2},\ \arg w = \frac{\pi(2k-1)}{2^{n+1}}\right\}$$
$$(k = 1, 2, \cdots, 2^{n+1};\ n = 1, 2, \cdots) .$$

Consider the doubly connected regions

$$A_z = \{z;\ -1 < \Re z < 1,\ -\tfrac{1}{2} < \Im z < \tfrac{1}{2}\}$$
$$- \{z;\ -\tfrac{1}{2} \leqq \Re z \leqq \tfrac{1}{2},\ \Im z = 0\}$$

and

$$A_w = \{w;\ -1 < \Re w < 1,\ -\tfrac{1}{2} < \Im w < \tfrac{1}{2}\}$$
$$- \{w;\ \Re w = 0,\ -\tfrac{1}{4} \leqq \Im w \leqq \tfrac{1}{4}\} .$$

It is not difficult to map A_z quasi-conformally onto A_w by a function which is of class C^1 in A_z and is the identity mapping on the outer periphery of A_z.

In our region D, consider the quadrilaterals

$$Q_n^k = \left\{ z;\ r_n e^{-c_n} < |z| < r_n e^{c_n},\ \frac{\pi(k-1)}{2^n} < \arg z < \frac{\pi k}{2^n} \right\}$$

$$(k = 1, 2, \cdots, 2^{n+1};\ n = 1, 2, \cdots) .$$

They are mutually disjoint and all $Q_n^k - s_n^k$ and $Q_n^k - \sigma_n^k$ are conformally equivalent to A_z and A_w, respectively. Therefore, we can contruct the mapping $w = T_n^k(z)$ of $Q_n^k - s_n^k$ onto $Q_n^k - \sigma_n^k$ which is the identity mapping on ∂Q_n^k and whose maximal dilatation K depends neither on k nor on n. Then

$$w = T(z) = \begin{cases} T_n^k(z) & \text{in } Q_n^k - s_n^k \ (k = 1, 2, \cdots, 2^{n+1};\ n = 1, 2, \cdots) \\ z & \text{in } D - \bigcup_{n,k} Q_n^k \end{cases}$$

is a qussi-conformal mapping of D onto Δ such that $T(T) = (|w| = 1) = \Gamma'$.

Since Δ belongs to the case of Example 7, $R(\Gamma'; \Delta) < \infty$, and, by Theorem 5, $\lambda\{\gamma'\}_\varepsilon < \infty$. Here γ' is a rectifiable curve in $\Delta - (|w| \leqq \varepsilon)$ connecting $|w| = \varepsilon$ with Γ'. It is furthermore assumed that γ' is a union of a countable number of analytic arcs clustering nowhere in Δ (cf. Remark, No. 15). On D, we have the corresponding family $\{\gamma\}_\varepsilon$ and, by (IV), No. 15, $\lambda\{\gamma\}_\varepsilon \leqq K\lambda\{\gamma'\}_\varepsilon < \infty$. Therefore, by Theorem 5, $R(\Gamma) < \infty$ and Γ is strong.

35. We continue to consider Example 8. If c_n decreases sufficiently fast, then $R(\Gamma) = \infty$. In fact, let $\{\gamma_n\}_\varepsilon$ be the subfamily of $\{\gamma\}_\varepsilon$ which consists of curves passing through the arc $\{z;\ z = r_n,\ |\arg z| \leqq c_n\}$. By (VI), No. 9, $\lambda\{\gamma\}_\varepsilon \geqq \lambda\{\gamma_n\}_\varepsilon/2^{n+1}$ and, By Lemma 1, No. 13,

$$\lambda\{\gamma_n\}_\varepsilon \sim \frac{1}{2\pi} \log \frac{1}{c_n} \qquad (n \to \infty) .$$

For this reason $R(\Gamma) = \infty$ if, for instance, $c_n = \exp(-2^{2n})$. However, it is unknown in this case whether Γ is strong or unstable.

16. Unstable boundary continua.

37. As in No. 21, let $\{\Omega_n\}_{n=1}^\infty$ be a defining sequence of Γ and let $0 \in G_n = D - \Omega_n \uparrow D$. Consider the function $w = f_n(z)$ maximizing the functional $m(f)$ in $\mathfrak{F}_{\Gamma_n}$ on G_n (No. 19). We may assume that $\{f_n(z)\}_{n=1}^\infty$ converges to a univalent function $w = f(z)$.

In the following case, $R(\Gamma) = \infty$ implies that $f(\Gamma) = \{\infty\}$:

THEOREM 14. *Let D be a region containing $z=0$ and let Γ be a boundary continuum. Suppose that*

(i) *D is symmetric with respect to the lines*

$$l_\nu:\ re^{\nu\pi/2k}\ (-\infty<r<\infty),\ \nu=1,2,\cdots,2^k$$

for some integer $k \geqq 0$, and

(ii) *every component of $\partial D-\Gamma$ intersects at least one l_ν.*

Then Γ is strong if and only if $R(\Gamma)<\infty$.

Proof. We may assume that each G_n is symmetric with respect to all the l_ν. By the uniqueness of $f_n(z)$ (No. 19), we can immediately see that $f_n(z)$ and, *a fortiori*, $f(D)$ are symmetric about these lines. As has been shown by Strebel [22], $f(\partial D-\Gamma)$ consists of radial segments. By the assumption $f(\partial D-\Gamma)$ is contained in $\bigcup_{\nu=1}^{2k} l_\nu$.

Now assume that $f(\Gamma)\neq\{\infty\}$. If $f(\Gamma)\subset\bigcup_{\nu=1}^{2k} l_\nu\cup\{\infty\}$, then $f(\Gamma)\cap l_\nu$ is a line segment which does not meet $\overline{f(\partial D-\Gamma)}$, so that $R(\Gamma)<\infty$ by Remark 1, No. 33. If $f(\Gamma)\not\subset\bigcup_{\nu=1}^{2k} l_\nu\cup\{\infty\}$ there exists a sector S bounded by two neighboring l_ν's such that $S\cap f(\Gamma)$ does not intersect $f(\partial D-\Gamma)$ and we have $R(\Gamma)<\infty$. Consequently, the strength of Γ implies that $R(\Gamma)<\infty$.

38. We can find many examples of unstable boundary continua belonging to this category, e.g., as follows:

EXAMPLE 9. Consider the region

$$D=(|z|\leqq\infty)-\Gamma-\bigcup_{k=1}^{\infty}(s_k^+\cup s_k^-\cup\sigma_k^+\cup\sigma_k^-),$$

where

$$\Gamma=\{z;\,-1\leqq\Re z\leqq1,\ \Im z=0\},$$

$$s_k^+=\left\{z;\ 1+\frac{1}{2k+1}\leqq\Re z\leqq1+\frac{1}{2k},\ \Im z=0\right\},$$

$$s_k^-=\left\{z;\ -1-\frac{1}{2k}\leqq\Re z\leqq-1-\frac{1}{2k+1},\ \Im z=0\right\},$$

$$\sigma_k^\pm=\left\{z;\ -1\leqq\Re z\leqq1,\ \Im z=\frac{\pm1}{k}\right\}.$$

Since every point on Γ, except ±1, is inaccessible, $R(\Gamma)=\infty$ by (X'), No. 10. From this and from Theorem 14, we infer that Γ is an unstable boundary continuum of D.

39. Meschkowsky [11] has proved that a region satisfying certain

metric conditions can be mapped conformally onto a region bounded by circles or points in such a way that the image of a preassigned boundary continuum is a point. This case is also an example of an unstable boundary continuum.

40. The following example belongs to this category but does not necessarily satisfy Meschkowsky's conditions. Moreover, the function $f(z) = \lim_{n\to\infty} f_n(z)$ of No. 37 does not transform Γ to a point.

EXAMPLE 10. Let $I = \{z;\ -1 \leqq \Re z \leqq 1,\ \Im z = 0\}$ and let

$$I' = \{z;\ \Re z = 0,\ -1 \leqq \Im z \leqq 1\} .$$

Choose a sequence $\{c_k;\ k = \pm 1,\ \pm 2, \cdots\}$ such that

$$c_{-k} = -c_k,\ c_1 > c_2 > \cdots \downarrow 0 ,$$

and let

$$s_k^0 : z = re^{ic_k} \qquad (1/|k|! \leqq r \leqq 1) ,$$

$$s_k^{\pi/2} : z = re^{i(c_k+\pi/2)} \qquad (1/|k|! \leqq r \leqq 1) ,$$

$$s_k^{\pi} : z = re^{i(c_k+\pi)} \qquad (1/|k|! \leqq r \leqq 1) ,$$

$$s_k^{-\pi/2} : z = re^{i(c_k-\pi/2)} \qquad (1/|k|! \leqq r \leqq 1) ,$$

where $k = \pm 1,\ \pm 2, \cdots$. Then $\Gamma = I \cup I'$ is an unstable boundary continuum of the region

$$D = (|z| \leqq \infty) - \Gamma - \bigcup_{\substack{k=-\infty \\ k\neq 0}}^{\infty} (s_k^0 \cup s_k^{\pi/2} \cup s_k^{\pi} \cup s_k^{-\pi/2}) .$$

In fact, D can be mapped onto a region such that $f(\Gamma)$ is a point and every component of $f(\partial D - \Gamma)$ is a circle. For the proof, map the region

$$(|z|) \leqq \infty) - \bigcup_{\substack{k=-n \\ k\neq 0}}^{\infty} (s_k^0 \cup s_k^{\pi/2} \cup s_k^{\pi} \cup s_k^{-\pi/2})$$

conformally onto a region bounded by $8n$ circles; we may require that the mapping function $w = f^{(n)}(z)$ has the expansion $z + b_n/z + \cdots$ near $z = \infty$ $(n = 1, 2, \cdots)$. The existence and the uniquess of such a mapping are well known. A suitable subsequence of $\{f^{(n)}(z)\}_{n=1}^{\infty}$ converges to a univalent function $w = f(z)$. We can easily prove that every component of $f(\partial D - \Gamma)$ is a circle (see, e.g., Meschkowsky [11]). In what follows we shall show that $f(\Gamma) = \{0\}$.

First we remark that $R(\Gamma) = \infty$, because every point on Γ, except $0,\ \pm 1,\ \pm i$, is inaccessible (cf. (X'), No. 10). Second, D and, therefore,

$f(D)$ are symmetric with respect to the following four lines: $l_0 =$ (real axis), $l_{\pi/4} = (\Re z = \Im z)$, $l_{\pi/2} =$ (imaginary axis), and $l_{-\pi/4} = (\Re z = -\Im z)$.

The component $f(\Gamma)^*$ of $f(D)^c$ corresponding to $f(\Gamma)$ is a compact connected set which contains the point $w = 0$ and is symmetric about these four lines.

The component $f(s_k^\beta)^*$ of D^c ($\beta = 0, \pm\pi/2, \pi;\ k = \pm 1, \pm 2, \cdots$) is a disk, which we denote by

$$\Delta_k^\beta :\ | w - a_k^\beta | \leqq \rho_k .$$

The radius ρ_k does not depend on β because of the symmetry. Furthermore,

$$\lim_{k\to\infty} \rho_k = 0; \tag{17}$$

in fact, all the Δ_k^β cluster to $f(\Gamma)^*$, so that the sum $8\pi \sum_{k=1}^{\infty} \rho_k^2$ of their areas converges.

Consider a quadrilateral

$$Q_k = \left\{ z;\ \frac{1}{k!} < |z| < \frac{1}{(k-1)!},\ c_k < \arg z < \frac{\pi}{2} - c_k \right\},$$

which connects s_k^0 with $s_{-k}^{\pi/2}$ ($k = 1, 2, \cdots$). The extremal distance between s_k^0 and $s_{-k}^{\pi/2}$ with respect to D does not exceed

$$\operatorname{mod} Q_k = \frac{(\pi/2) - 2c_k}{\log k} .$$

Let L_k be the infimum of lengths of curves in $f(D)$ connecting Δ_k^0 with $\Delta_{-k}^{\pi/2}$. Then

$$\frac{L_k^2}{\mu U} \leqq \frac{(\pi/2) - 2c_k}{\log k} \to 0 \qquad (k \to \infty) \tag{18}$$

where μU expresses the area of a bounded open set U containing $f(\Gamma)^*$. For this reason and by virtue of (17) and (18), we have

$$\lim_{k\to\infty} | a_k^0 - a_{-k}^{\pi/2} | \leqq \lim_{k\to\infty} (L_k + 2\rho_k) = 0 .$$

It follows, by symmetry, that $\{a_k^0\}_{k=1}^{\infty}$ and $\{a_{-k}^{\pi/2}\}_{k=1}^{\infty}$ cluster to $l_{\pi/4}$ in the first quadrant. From this and again from the symmetry, we see that the set H of all accumulation points of a_k^β ($\beta = 0, \pm\pi/2, \pi;\ k = \pm 1, \pm 2, \cdots$) is contained in $l_{\pi/4} \cup l_{-\pi/4}$. Evidently it is symmetric about l_0 and $l_{\pi/2}$, and $H \subset f(\Gamma)^*$.

Next we shall show that $H = \{0\}$. Suppose that H contains a point $w_0 = pe^{i\pi/4}$ ($p > 0$). Then there must exist a point $qe^{i\pi/4} \in H$ ($0 \leqq q < p$). For otherwise H would consist of four points: $H = \{pe^{i\theta};\ \theta = \pm\pi/4, \pm 3\pi/4\}$.

Then all but a finite number of components of $f(\partial D-\Gamma)$ in the first quadrant would be contained in $|w-pe^{i\pi/4}|<p/4$. Since w_0 and 0 are contained in $f(\Gamma)^*$ and $f(\Gamma)^*$ is a continuum, $f(\Gamma)$ would have a "free" subset as in Theorem 13. But the reasoning of Remark 1, No. 33, shows that this property of $f(\Gamma)$ contradicts the fact that $R(\Gamma)=\infty$ and, therefore, $qe^{i\pi/4}\in H$ exists. Take a subsequence $\{k_j\}\subset\{k\}$ such that

$$\lim_{j\to\infty} a^0_{k_j}=\lim_{j\to\infty} a^{\pi/2}_{-k_j}=qe^{i\pi/4}\,.$$

Then

$$L_{k_j}+2\rho_{k_j}\geqq\frac{p-q}{2}>0$$

for sufficiently great j, contrary to (17) and (18). Consequently, w_0 does not exist and $H=\{0\}$.

Finally, if $f(\Gamma)^*\supsetneqq H$, then $f(\Gamma)$ would again have a "free" subset, contrary to the fact that $R(\Gamma)=\infty$. We conclude that $f(\Gamma)^*=\{0\}$.

41. Transform the region D by $\zeta=1/z$ and, for simplicity, denote the image again by D. For the sequence $G_n\uparrow D$ of No. 37, we take

$$\begin{aligned}G_n=(|z|<n!+c_{n+1})\cap D\\ -\bigcup_{h=1}^{3}\Big\{z;\ 1-c_{n+1}\leqq|z|,\ \frac{h\pi}{2}-\frac{c_n+c_{n+1}}{2}\leqq\arg z\\ \leqq\frac{h\pi}{2}+\frac{c_n+c_{n+1}}{2}\Big\}\,,\end{aligned}$$

$n=1,2,\cdots$, and consider the extremal function $f_n(z)$. We shall show:

If $c_k=-c_{-k}$ *decreases sufficiently fast* (e.g., $c_k=e^{-k!}$), *then* $\lim_{n\to\infty}f_n(z)=z$ *uniformly on every compact set in* D.

In order to prove this, we estimate the Dirichlet integral of $\log|f_n(z)/z|$ over $\Delta=(|z|\leqq1/2)$:

$$\begin{aligned}&D_\Delta(\log|f_n(z)|-\log|z|)\leqq D_{G_n}(\log|f_n(z)|-\log|z|)\\ &=\int_{\partial G_n}(\log|f_n|\cdot d\arg f_n-\log|z|\cdot d\arg f_n\\ &\quad-\log|f_n|\cdot d\arg z+\log|z|\cdot d\arg z)\\ &=\int_{\partial G_n}(\log|f_n|\cdot d\arg f_n-2\log|f_n|\cdot d\arg z\\ &\quad+\log|z|\cdot d\arg z)\\ &=2\pi\log R(\Gamma_n;\ G_n)-2\log R(\Gamma_n;\ G_n)\int_{\Gamma_n}d\arg z\\ &\quad+\int_{\Gamma_n}\log|z|\,d\arg z\leqq2\pi\{\log n!-\log R(\Gamma_n;\ G_n)\}\,.\end{aligned}$$

To estimate the last term, we shall use the relation $\log R(\Gamma_n; G_n) = \lim_{\varepsilon\to 0}(\log \varepsilon + 2\pi\lambda\{\gamma\}_\varepsilon^{(n)})$, where the sequence is increasing (No. 22). Here $\{\gamma\}_\varepsilon^{(n)}$ is the family of curves in $G_n - (|z| \leqq \varepsilon)$ connecting Γ_n with $|z| = \varepsilon$. We take the closed disks

$$\varDelta_n^h : |z - e^{i\pi h/2}| \leqq c_n ,$$

$$\varDelta_n'^h : |z - n!\, e^{i\pi h/2}| \leqq n!\, c_n ,$$

$h = 0, 1, 2, 3$; $n = 1, 2, \cdots$. Let $\{\gamma_1\}_\varepsilon^{(n)} \subset \{\gamma\}_\varepsilon^{(n)}$ be the family of curves connecting $|z| = \varepsilon$ with $\bigcup_{h,n} \varDelta_n^h \cup \varDelta_n'^h$ and put $\{\gamma_2\}_\varepsilon^{(n)} = \{\gamma\}_\varepsilon^{(n)} - \{\gamma_1\}_\varepsilon^{(n)}$. By (VI), No. 9,

$$\frac{1}{\lambda\{\gamma\}_\varepsilon^{(n)}} \leqq \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \qquad (\lambda_\nu = \lambda\{\gamma_\nu\}_\varepsilon^{(n)},\ \nu = 1, 2) ,$$

or

$$\lambda\{\gamma\}_\varepsilon^{(n)} \geqq \lambda_2 - \frac{\lambda_2^2}{\lambda_1} .$$

It is evident that

$$\frac{1}{2\pi - 8c_n} \log \frac{n! + c_n}{\varepsilon} \geqq \lambda_2 \geqq \frac{1}{2\pi} \log \frac{n!}{\varepsilon} .$$

Therefore,

$$\log R(\Gamma_n;\ G_n) \geqq \log \varepsilon + 2\pi\lambda\{\gamma\}_\varepsilon^{(n)} \geqq \log n! - 2\frac{\lambda_2^2}{\lambda_1} ,$$

whence

$$D_\varDelta(\log |f_n(z)| - \log |z|) \leqq 4\pi^2 \frac{\lambda_2^2}{\lambda_1} .$$

If c_n is taken sufficiently small, then $\lim_{n\to\infty} \lambda_2^2/\lambda_1 = 0$. For instance, if $c_n = e^{-n!}$, we have $\lambda_1 \sim (8\cdot n!)/\pi$ $(n \to \infty)$ by Lemma 1, No. 13, and $\lambda_2^2/\lambda_1 \to 0$. In such a case, $\lim_{n\to\infty} D_\varDelta (\log |f_n(z)| - \log |z|) = 0$ and we conclude that $\lim_{n\to\infty} f_n(z) = z$ uniformly on each compact set in D.

Consequently $R(\Gamma) = \infty$ for our region, but $\lim_{n\to\infty} f_n(z)$ does not transform Γ to a point.

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