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**AN INVERSION THEOREM FOR LAPLACE-STIELTJES
TRANSFORMS**

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E. Phragmén [2; p. 360] showed that under certain assumptions of boundedness for $F(x)$,

$$\lim_{s \rightarrow +\infty} \int_0^t F(\tau) [1 - \exp(-e^{(t-\tau)s})] d\tau = \int_0^t F(\tau) d\tau .$$

If we write $1 - \exp(-e^{s(t-\tau)}) = \sum_{n=1}^{\infty} (-1)^{n+1} e^{ns(t-\tau)} / n!$ in the above formula, and interchange sum and integral, we formally obtain

$$\lim_{s \rightarrow \infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} e^{nst} \int_0^t e^{-ns\tau} F(\tau) d\tau = \int_0^t F(\tau) d\tau .$$

G. Doetsch [1; pp. 286-288] showed that for reals s , if $f(s) = \int_0^{\infty} e^{-s\tau} F(\tau) d\tau$ converges absolutely in some half-plane, then

$$\int_0^t F(\tau) d\tau = \lim_{s \rightarrow +\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} f(ns) e^{nst} \text{ for } t > 0 .$$

This paper will generalize this result to Laplace-Stieltjes transforms

$$(I) \quad f(s) = \int_0^{\infty} e^{-st} d\alpha(t)$$

and will eliminate the assumption of absolute convergence. Unless specifically written otherwise, all integrals will be evaluated from 0 to $+\infty$ and all summations from 1 to ∞ . We shall need the following two propositions [3; pp 39,41]:

LEMMA 1. *If the integral*

$$f(s_0) = \int e^{-s_0 t} d\alpha(t)$$

converges with $Rs_0 > 0$, then

$$f(s_0) = s_0 \int e^{-s_0 t} \alpha(t) dt - \alpha(0)$$

and $\int e^{-s_0 t} \alpha(t) dt$ converges absolutely if s_0 is replaced by any number with larger real part.

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Lemma I remains valid for $Rs_0 < 0$ if we insist that $\alpha(\infty) = 0$. In this paper we shall make the following

Assumption. In (I), s is real and positive, and $\alpha(t)$ is of bounded variation in $(0, R)$, for every $R > 0$.

LEMMA 2. *If the integral*

$$\int_0^\infty e^{-s\tau} d\alpha(\tau)$$

converges for $s = s_0$ and if the real part γ of s_0 is positive, then $\alpha(\tau) = O(e^{\gamma\tau})$ as $\tau \rightarrow \infty$.

We shall now prove some useful lemmas.

LEMMA 3. *If (I) converges in some half plane Γ , then*

- (a) $\lim_{s \rightarrow \infty} \left| \int_\sigma^\infty [1 - \exp(-e^{-s\tau})] d\alpha(\tau) \right| = 0$ for fixed $\sigma > 0$,
- (b) $\lim_{\sigma \rightarrow \infty} \left| \int_\sigma^\infty [1 - \exp(-e^{-s\tau})] d\alpha(\tau) \right| = 0$ for fixed $s > 0$.

Proof. Since $1 - \exp(-e^{-s\tau}) = O(e^{-s\tau})$ for $s, \tau \geq 0$, a standard argument involving integration by parts shows that

$$\int_\sigma^\infty [1 - \exp(-e^{-s\tau})] d\alpha(\tau) = O\{e^{-s\sigma} [|\alpha(\sigma)| + s]\}$$

for $s \in \Gamma$ and $\sigma \geq 0$. The desired result now follows from Lemmas 1 and 2.

LEMMA 4. *If (I) converges in some half-plane Γ , then for $s \in \Gamma'$ where Γ' is a half-plane properly contained in Γ ,*

$$\sum \frac{(-1)^{n+1}}{n!} \int e^{ns(t+\tau)} d\alpha(\tau) = \int d\alpha(\tau) \sum \frac{(-1)^{n+1}}{n!} e^{ns(t-\tau)}.$$

Proof. Upon integration by parts, application of Lemma 2, and some algebra, the desired equality takes the form

$$\sum_1^\infty \frac{(-1)^{n+1}}{(n-1)!} \int e^{ns(t-\tau)} \alpha(\tau) d\tau = \int \sum_1^\infty \frac{(-1)^{n+1}}{(n-1)!} e^{ns(t-\tau)} \alpha(\tau) d\tau.$$

To verify this latter equality, it suffices to show that

$$\sum_0^\infty \frac{1}{(n-1)!} \int e^{ns(t-\tau)} |\alpha(\tau)| d\tau < \infty,$$

but this follows from Lemma 1.

LEMMA 5. *If (I) converges in some half-plane Γ , then*

$$\alpha(t) - \alpha(0) = \lim_{s \rightarrow +\infty} \sum \frac{(-1)^{n+1}}{n!} f(ns) e^{nst}$$

for all non-negative t which are points of continuity of $\alpha(t)$.

Proof. We have

$$\begin{aligned} \sum \frac{(-1)^{n+1}}{n!} f(ns) e^{nst} &= \int d\alpha(\tau) \sum \frac{(-1)^{n+1}}{n!} e^{ns(t-\tau)} \\ &= \int [1 - \exp(-e^{s(t-\tau)})] d\alpha(\tau), \end{aligned}$$

the interchange in the order of summation and integration being justified by Lemma 4. For $t = 0$ ($t > 0$) and a point of continuity of $\alpha(t)$, write the integral on the right as

$$\int_0^\delta + \int_\delta^\infty \left(\text{or, for } t > 0, \int_0^{t-\delta} + \int_{t-\delta}^{t-\delta} + \int_{t+\delta}^\infty \right),$$

with $0 < \delta < t$ chosen that the total variation of α on $[0, \delta]$ (respectively. $[t - \delta, t + \delta]$) is less than ε , and apply Lemma 3 to \int_δ^∞ (resp., $\int_{t+\delta}^\infty$). We see that \int_0^δ (resp., $\int_{t-\delta}^{t+\delta}$) is less than ε for all $s \geq 0$. (For $t > 0$, $\int_0^{t-\delta} = \alpha(t - \delta) - \alpha(0) - \int_0^{t-\delta} \exp[-e^{s(t-\tau)}] d\alpha(\tau)$, and this clearly tends to $\alpha(t - \delta) - \alpha(0)$ as $s \rightarrow \infty$. Thus the integral \int_0^∞ is $\alpha(t) - \alpha(0) + o(1)$ as $s \rightarrow \infty$).

We can now prove our main result.

THEOREM *If $\alpha(t) = [\alpha(t^+) + \alpha(t^-)]/2$ for $t > 0$ and (I) converges for some $s > 0$, then*

$$\lim_{s \rightarrow \infty} \sum \frac{(-1)^{n+1}}{n!} f(ns) e^{nst} = \begin{cases} [\alpha(0^+) - \alpha(0)] (1 - e^{-1}), & , t = 0 \\ \alpha(t) - \alpha(0) - [\alpha(t^+) - \alpha(t^-)] (e^{-1} - 1/2), & , t > 0 \end{cases}$$

Proof. Define

$$\beta(\tau) = \begin{cases} \alpha(\tau) - [\alpha(0^+) - \alpha(0)] \text{ sign } \tau, & \tau > 0 \\ \alpha(0) & , \tau = 0 \end{cases}$$

for $t = 0$, and

$$\beta(\tau) = \begin{cases} \alpha(\tau) - [\alpha(t^+) - \alpha(t^-)] \text{ sign } (\tau - t), & \tau > 0 \\ \alpha(\tau) & , \tau = 0 \end{cases}$$

for $t > 0$, β is then continuous at t , and

$$F(s) = \int_0^{\infty} e^{-s\tau} d\beta(\tau) = \begin{cases} f(s) - \alpha(0^+) + \alpha(0) & , t = 0 \\ f(s) - [\alpha(t^+) - \alpha(t^-)]e^{-st} & , t > 0 . \end{cases}$$

Now apply Lemma 5 with β and F substituted for α and f , respectively.

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