Pacific Journal of Mathematics

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Vol. 10, No. 2 October 1960

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Introduction. In this paper, the prime number theorem in the form $\psi(x) \equiv \sum_{p^m \leqq x} \log p = x + o(x \cdot \log^{-1/6+\varepsilon} x)$, for every $\varepsilon > 0$, is established via a proof that in the well-known formula

$$(1) \qquad \rho(x) \equiv \sum_{p^m \leqq x} \frac{\log p}{p^m} = \log x + O(1) \equiv \log x + a_x ,$$

$a_x = -A_0 + o(\log^{-1/6+\varepsilon} x)$. ($A_0$ is Euler's constant.)

Throughout the paper, p and q stand for prime numbers, k, m, n, t, and others are positive integers, and x, y, and z are positive real numbers.

Some well-known formulas, used in the proof, are

$$(2) \qquad \sum_{n \leqq x} \frac{\log^k n}{n} = \frac{1}{k+1} \cdot \log^{k+1} x + A_k + O\left(\frac{\log^k x}{x}\right), \quad \text{for } k = 0, 1, \cdots$$

$$(2') \qquad \sum_{y < n \leqq z} \frac{1}{n} \cdot \log^k (n/y) = \frac{1}{k+1} \cdot \log^{k+1}(z/y) + O\left(\frac{1}{y} \cdot \log^k (z/y)\right), \quad \text{for } k = 0, 1, \cdots$$

$$(3) \qquad \sum_{n \leqq x} \log^k (x/n) = O(x) , \quad \text{for } k = 1, 2, \cdots$$

$$(4) \qquad \sum_{p^m \leqq x} \log p \cdot \log^k (x/p^m) = O(x) , \quad \text{for } k = 0, 1, \cdots$$

$$(5) \qquad \sum_{n \leqq x} \mu(n)/n = O(1) \ (\mu(n) \text{ is Moebius' function.})$$

Two other formulas, used prominently, are

$$(6) \qquad \sigma(x) \equiv \sum_{p^m \leqq x} \frac{\log p}{p^m} \cdot \log (x/p^m) = \frac{1}{2} \cdot \log^2 x - A_0 \cdot \log x + g_x \ (g_x = O(1))$$

$$(7) \qquad \tau(x) \equiv \sum_{p^m \leqq x} \frac{\log p}{p^m} \cdot \log^2 (x/p^m) = \frac{1}{3} \cdot \log^3 x - A_0 \cdot \log^2 x + (2 \cdot A_0^2 + 4 \cdot A_1) \log x + O(1) .$$

With the help of (1), (2), and (4), (6) can be proved easily:

$$\sigma(x) = \sum_{p^m \leqq x} \frac{\log p}{p^m} \cdot \left(\sum_{n \leqq x/p^m} 1/n - A_0 + O(p^m/x) \right), \quad \text{or, with } k = n \cdot p^m ,$$

$$\sigma(x) = \sum_{k \leqq x} \frac{1}{k} \cdot \sum_{p^m / k} \log p - A_0 \cdot \log x + O(1)$$

$$= \sum_{k \leqq x} \frac{\log k}{k} - A_0 \log x + O(1) = \frac{1}{2} \cdot \log^2 x - A_0 \log x + O(1) .$$

Received December 31, 1958.

Also, again with $k = n \cdot p^m$,

$$\sum_{k \leqq x} \frac{\log k}{k} \cdot \log \frac{x}{k}$$

$$= \sum_{k \leqq x} \frac{1}{k} \cdot \log \frac{x}{k} \cdot \sum_{p^m / k} \log p = \sum_{p^m \leqq x} \log p \cdot \sum_{n \leqq x/p^m} \frac{1}{n \cdot p^m} \cdot \log\left(\frac{x}{n \cdot p^m}\right)$$

$$= \sum_{p^m \leqq x} \frac{\log p}{p^m} \cdot \left\{ \log\left(\frac{x}{p^m}\right) \cdot \sum_{n \leqq x/p^m} \frac{1}{n} - \sum_{n \leqq x/p^m} \frac{\log n}{n} \right\}$$

$$= \sum_{p^m \geqq x} \frac{\log p}{p^m} \cdot \left\{ \log^2(x/p^m) + A_0 \log(x/p^m) - \frac{1}{2} \log^2(x/p^m) - A_1 \right\} + O(1)$$

(by (2) and (4))

$$= \frac{1}{2} \cdot \tau(x) + A_0 \cdot \sigma(x) - A_1 \cdot \rho(x) + O(1) .$$

(7) follows now by (1), (2), and (6).

The proof now proceeds in the following steps: in part I, certain asymptotic formulas for a_n (see (1)) and g_n (see (6)) are derived; they suggest that "on the average," a_n is $-A_0$, and g_n is $A_0^2 + 2A_1$. In part II, formulas for a_n and g_n are derived which are of the type of Selberg's asymptotic formula for $\psi(x)$; part III contains the final proof.

PART I

First, the following five formulas will be derived; $K_1, K_2, \cdots$, are constants, independent of x.

$$(8) \qquad \sum_{n \leqq x} \frac{1}{n} \cdot a_n = -A_0 \log x + g_x + K_2 + O\left(\frac{\log x}{x}\right)$$

$$(9) \qquad \sum_{n \leqq x} \frac{1}{n} \cdot a_{x/n} = -A_0 \log x + K_3 + O\left(\frac{\log x}{x}\right)$$

$$(10) \qquad \sum_{p \leqq x} \frac{\log p}{p^m} \cdot a_{p^m} = -A_0 \log x + g_x + \frac{1}{2} a_x^2 + K_4 + O\left(\frac{\log x}{x}\right)$$

$$(11) \qquad \sum_{n \leqq x} \frac{1}{n} \cdot g_n = (A_0^2 + 2 \cdot A_1) \cdot \log x + O(1)$$

$$(12) \qquad \sum_{n \leqq x} \frac{1}{n} \cdot g_{x/n} = (A_0^2 + 2 \cdot A_1) \cdot \log x + K_5 + O\left(\frac{\log^2 x}{x}\right) .$$

Proofs.

$$\sigma(x) = \sum_{n \leqq x} \log \frac{x}{n} (\rho(n) - \rho(n-1)) = \sum_{n \leqq x} \rho(n) \cdot \log \frac{n+1}{n} + O\left(\frac{\log x}{x}\right)$$

$$= \sum_{n \leqq x} \frac{\rho(n)}{n} + K_1 + O\left(\frac{\log x}{x}\right)$$

$$= \sum_{n \leqq x} \frac{\log n}{x} + \sum_{n \leqq x} \frac{1}{n} a_n + K_1 + O\left(\frac{\log x}{x}\right) .$$

(8) follows now from (6) and (2).

Also

$$\sum_{n\leqq x}\frac{1}{n}\cdot a_{x/n}=\sum_{n\leqq x}\frac{1}{n}\cdot\Big(\sum_{p^m\leqq x/n}\frac{\log p}{p^m}-\log\frac{x}{n}\Big)$$

$$=\sum_{k\leqq x}\frac{1}{k}\sum_{p^m/k}\log p-\sum_{n\leqq x}\frac{1}{n}\log\frac{x}{n}\quad(k=n\cdot p^m)$$

$$=\sum_{k\leqq x}\frac{\log k}{k}-\sum_{n\leqq x}\frac{1}{n}\log\frac{x}{n}\,.\quad\text{which proves (9) by (2).}$$

And

$$\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot a_{p^m}=\sum_{p^m\leqq x}\frac{\log p}{p^m}\Big(\sum_{q^t\leqq p^m}\frac{\log q}{q^t}-\log(p^m)\Big)$$

$$=\frac{1}{2}\Big(\sum_{p^m\leqq x}\frac{\log p}{p^m}\Big)^2+\frac{1}{2}\sum_{p^m\leqq x}\frac{\log^2 p}{p^{2m}}-\log x\cdot\sum_{p^m\leqq x}\frac{\log p}{p^m}+\sum_{p^m\leqq x}\frac{\log p}{p^m}\log\frac{x}{p^m}\,.$$

Thus, by (1), (2) and (6),

$$\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot a_{p^m}=\frac{1}{2}(\log x+a_x)^2+K_4+O\Big(\frac{\log x}{x}\Big)-\log x\cdot(\log x+a_x)$$

$$+\frac{1}{2}\log^2 x-A_0\log x+g_x\,,\quad\text{which proves (10).}$$

In the next proof, use is made of the easily established fact that

$$\rho(n)\cdot\log\frac{n+1}{n}=\sigma(n+1)-\sigma(n)\,.$$

$$\tau(x)=\sum_{n\leqq x}\log^2\Big(\frac{x}{n}\Big)(\rho(n)-\rho(n-1))$$

$$=\sum_{n\leqq x}\rho(n)\Big(\log^2\Big(\frac{x}{n}\Big)-\log^2\Big(\frac{x}{n+1}\Big)\Big)+O(1)$$

$$=\sum_{n\leqq x}\rho(n)\log\frac{n+1}{n}\cdot\log\frac{x^2}{n(n+1)}+O(1)$$

$$=\sum_{n\leqq x}(\sigma(n+1)-\sigma(n))\cdot\log\frac{x^2}{n(n+1)}+O(1)$$

$$=\sum_{n\leqq x}\sigma(n)\cdot\log\frac{n+1}{n-1}+O(1)=\sum_{n\leqq x}\sigma(n)\cdot\frac{2}{n}+O(1)$$

$$=\sum_{n\leqq x}\frac{\log^2 n}{n}-2\cdot A_0\cdot\sum_{n\leqq x}\frac{\log n}{n}+2\cdot\sum_{n\leqq x}\frac{1}{n}\cdot g_n+O(1)\quad\text{(by (6)).}$$

This proves (11), with the help of (2) and (7).

Finally

$$\sum_{n\leqq x}\frac{1}{n}\cdot g_{x/n}=\sum_{n\leqq x}\frac{1}{n}\cdot\Big(\sum_{p^m\leqq x/n}\frac{\log p}{p^m}\cdot\log\frac{x}{n\cdot p^m}-\frac{1}{2}\log^2\Big(\frac{x}{n}\Big)+A_0\log\frac{x}{n}\Big)\,,$$

or, with $k=n\cdot p^m$,

$$\sum_{n\leqq x}\frac{1}{n}\cdot g_{x/n}$$

$$=\sum_{k\leqq x}\frac{1}{k}\cdot\log\frac{x}{k}\cdot\sum_{p^m/k}\log p-\frac{1}{2}\cdot\sum_{n\leqq x}\frac{1}{n}\cdot\log^2\Big(\frac{x}{n}\Big)+A_0\sum_{n\leqq x}\frac{1}{n}\cdot\log\frac{x}{n}$$

$$=\sum_{k\leqq x}\frac{1}{k}\cdot\log\frac{x}{k}\cdot\log k-\frac{1}{2}\cdot\sum_{n\leqq x}\frac{1}{n}\cdot\log^2\Big(\frac{x}{n}\Big)+A_0\sum_{n\leqq x}\frac{1}{n}\cdot\log\frac{x}{n}\,.$$

(12) now follows by (2).

Formulas (8) through (12) suggest setting

$$b_x\equiv a_x+A_0\,,\qquad h_x\equiv g_x-(A_0^2+2A_1)\,. \tag{13}$$

In terms of b_x and h_x, the five formulas read

$$\sum_{n\leqq x}\frac{1}{n}\cdot b_n=h_x+K_6+O\Big(\frac{\log x}{x}\Big) \tag{8'}$$

$$\sum_{n\leqq x}\frac{1}{n}\cdot b_{x/n}=K_7+O\Big(\frac{\log x}{x}\Big) \tag{9'}$$

$$\begin{aligned}\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot b_{p^m}&=A_0\cdot(-A_0+b_x)+A_0^2+2\cdot A_1+h_x+K_4\\&\quad+\frac{1}{2}\cdot(-A_0+b_x)^2+O\Big(\frac{\log x}{x}\Big)\\&=h_x+\frac{1}{2}\cdot b_x^2+K_8+O\Big(\frac{\log x}{x}\Big)\end{aligned} \tag{10'}$$

$$\sum_{n\leqq x}\frac{1}{n}\cdot h_n=O(1) \tag{11'}$$

$$\sum_{n\leqq x}\frac{1}{n}\cdot h_{x/n}=K_9+O\Big(\frac{\log^2 x}{x}\Big)\,. \tag{12'}$$

Next, it will be shown that

$$\sum_{n\leqq x}\frac{1}{n}\cdot b_n^2=\sum_{n\leqq x}\frac{1}{n}\cdot b_{x/n}^2+O(1)\,, \tag{14}$$

and

$$\sum_{n\leqq x}\frac{1}{n}\cdot b_n\cdot b_{x/n}=\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot h_{x/p^m}+O(1)\,. \tag{15}$$

For a proof of (14), we know, by (10'), that

$$\frac{1}{n}\cdot b_n^2=\frac{2}{n}\cdot\sum_{p^m\leqq n}\frac{\log p}{p^m}\cdot b_{p^m}-\frac{2}{n}\cdot h_n-\frac{2}{n}\cdot K_8+O\Big(\frac{\log n}{n^2}\Big)\,,$$

and

$$\frac{1}{n}\cdot b_{x/n}^2=\frac{2}{n}\cdot\sum_{p^m\leqq x/n}\frac{\log p}{p^m}\cdot b_{p^m}-\frac{2}{n}\cdot h_{x/n}-\frac{2}{n}\cdot K_8+O\Big(\frac{1}{x}\cdot\log\frac{x}{n}\Big)\,.$$

Thus, by (3), (11′) and (12′),

$$\sum_{n\leqq x}\frac{1}{n}\cdot(b_n^2-b_{x/n}^2)=2\cdot\sum_{n\leqq x}\frac{1}{n}\Big(\sum_{p^m\leqq n}\frac{\log p}{p^m}\cdot b_{p^m}-\sum_{p^m\leqq x/n}\frac{\log p}{p^m}\cdot b_{p^m}\Big)+O(1)$$

$$=2\cdot\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot b_{p^m}\Big(\sum_{p^m\leqq n\leqq x}\frac{1}{n}-\sum_{n\leqq x/p^m}\frac{1}{n}\Big)+O(1)$$

$$=2\cdot\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot b_{p^m}\cdot\Big(\log(x/p^m)+O(1/p^m)-\log(x/p^m)$$

$$-A_0-O(p^m/x)\Big)+O(1)$$

$=O(1)$, by (10′) and (4). This proves (14).

Also

$$\sum_{n\leqq x}\frac{1}{n}\cdot b_n\cdot b_{x/n}=\sum_{n\leqq x}\frac{1}{n}\cdot b_n\cdot\Big(\sum_{p^m\leqq x/n}\frac{\log p}{p^m}-\log\frac{x}{n}+A_0\Big)$$

$$=\sum_{n\leqq x}\frac{1}{n}\cdot b_n\cdot\Big(\sum_{p^m\leqq x/n}\frac{\log p}{p^m}-\sum_{t\leqq x/n}\frac{1}{t}+2\cdot A_0+O\Big(\frac{n}{x}\Big)\Big)$$

$$=\sum_{p^m\leqq x}\frac{\log p}{p^m}\sum_{n\leqq x/p^m}\frac{1}{n}b_n-\sum_{t\leqq x}\frac{1}{t}\sum_{n\leqq x/t}\frac{1}{n}b_n+O(1)\,,\quad\text{by (8′)}$$

$$=\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot h_{x/p^m}+K_6\log x-\sum_{t\leqq x}\frac{1}{t}h_{x/t}-K_6\log x+O(1)$$

(by (8′), (1) and (4))

$$=\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot h_{x/p^m}+O(1)\,,\quad\text{by (12′)}\,.$$

From (14) and (15) it follows that

$$\sum_{n\leqq x}\frac{1}{n}\cdot(b_n\pm b_{x/n})^2=2\cdot\sum_{n\leqq x}\frac{1}{n}\cdot b_n^2\pm 2\cdot\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot h_{x/p^m}+O(1)\,,$$

and therefore

$$\sum_{n\leqq x}\frac{1}{n}\cdot b_n^2\geqq\Big|\sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot h_{x/p^m}\Big|+O(1)\,. \tag{16}$$

PART II

In the following, we shall employ the inversion formula

$$G(x)=\sum_{n\leqq x}g\Big(\frac{x}{n}\Big)\qquad\text{for all } x>0\Rightarrow g(x)=\sum_{n\leqq x}\mu(n)\cdot G\Big(\frac{x}{n}\Big)\,,$$

as well as

$$\sum_{n\leqq x}\frac{\mu(n)}{n}\log\frac{x}{n}=O(1)\,. \tag{17}$$

For a proof of (17), we make use of the fact that $\sum_{n\leqq x}x/n=x\cdot\log x+A_0x+O(1)$; thus, by the inversion formula,

$$x = \sum_{n \leqq x} \mu(n) \cdot \frac{x}{n} \log \frac{x}{n} + A_0 \cdot \sum_{n \leqq x} \mu(n) \cdot \frac{x}{n} + O(x) .$$

(17) follows now by (5).

If $f(x)$ is defined for $x > 0$, then

$$\sum_{n \leqq x} \left\{ \frac{x}{n} \cdot \log \frac{x}{n} \cdot f\left(\frac{x}{n}\right) + \frac{x}{n} \cdot \sum_{p^m \leqq x/n} \frac{\log p}{p^m} \cdot f\left(\frac{x}{n \cdot p^m}\right) \right\}$$

$$= \sum_{n \leqq x} \frac{x}{n} \cdot \log \frac{x}{n} \cdot f\left(\frac{x}{n}\right) + \sum_{k \leqq x} \frac{x}{k} \cdot f\left(\frac{x}{k}\right) \cdot \sum_{p^m / k} \log p \qquad (k = n \cdot p^m)$$

$$= \sum_{n \leqq x} \frac{x}{n} \cdot \log \frac{x}{n} \cdot f\left(\frac{x}{n}\right) + \sum_{k \leqq x} \frac{x}{k} \cdot f\left(\frac{x}{k}\right) \cdot \log k = x \cdot \log x \cdot \sum_{n \leqq x} \frac{1}{n} \cdot f\left(\frac{x}{n}\right) .$$

Thus, if we set

$$F(x) \equiv x \cdot \log x \cdot \sum_{n \leqq x} \frac{1}{n} \cdot f\left(\frac{x}{n}\right) ,$$

then, by the inversion formula,

$$x \cdot \log x \cdot f(x) + x \cdot \sum_{p^m \leqq x} \frac{\log p}{p^m} \cdot f(x/p^m) = \sum_{n \leqq x} \mu(n) \cdot F\left(\frac{x}{n}\right) .$$ [1]

In particular, if

$$\sum_{n \leqq x} \frac{1}{n} \cdot f\left(\frac{x}{n}\right) = K + O\left(\frac{\log^k x}{x}\right) ,$$

then

$$\sum_{n \leqq x} \mu(n) \cdot F\left(\frac{x}{n}\right) = K \cdot \sum_{n \leqq x} \mu(n) \cdot \frac{x}{n} \cdot \log\left(\frac{x}{n}\right) + O\left(\sum_{n \leqq x} \log^{k+1}\left(\frac{x}{n}\right)\right) = O(x) ,$$

by (17) and (3), and thus

(18) $$f(x) \cdot \log x + \sum_{p^m \leqq x} \frac{\log p}{p^m} \cdot f(x/p^m) = O(1) ,$$

$$\text{if} \quad \sum_{n \leqq x} \frac{1}{n} \cdot f\left(\frac{x}{n}\right) = K + O\left(\frac{\log^k x}{x}\right) .$$

(Selberg's asymptotic formula for $\psi(x)$ corresponds to $f(x) \equiv \psi(x)/x - 1$.) By (9′) and (12′), $f(x) \equiv b_x$ and $f(x) \equiv h_x$ both satisfy the condition of (18), and thus

(19) $$b_x \cdot \log x + \sum_{p^m \leqq x} \frac{\log p}{p^m} \cdot b_{x/p^m} = O(1)$$

(20) $$h_x \cdot \log x + \sum_{p^m \leqq x} \frac{\log p}{p^m} \cdot h_{x/p^m} = O(1) .$$

[1] Compare K. Iseki and T. Tatuzawa, "On Selberg's elementary proof of the prime number theorem." Proc. Jap. Acad. 27, 340–342 (1951).

From (16) and (20) it follows that

$$\sum_{n\leqq x}\frac{1}{n}\cdot b_n^2 \geqq |h_x|\cdot\log x + O(1)\,. \tag{21}$$

If we add to (19)

$$(\log x - A_0)\cdot\log x + \sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot(\log(x/p^m) - A_0)\,,$$

which by (1) and (6) is equal to $3/2\cdot\log^2 x - 3\cdot A_0\cdot\log x + O(1)$, we obtain

$$\rho(x)\cdot\log x + \sum_{p^m\leqq x}\frac{\log p}{p^m}\cdot\rho(x/p^m) = \frac{3}{2}\cdot\log^2 x - 3\cdot A_0\cdot\log x + O(1)\,.$$

If $0 < c < 1$, and $c\cdot x < y < x$, then it follows from the last equation that

$$\begin{aligned}\rho(x)\cdot\log x - \rho(y)\cdot\log y &\leqq \frac{3}{2}\cdot(\log^2 x - \log^2 y) + O(1)\\ &= \frac{3}{2}\cdot\log\frac{x}{y}\cdot(\log x + \log y) + O(1)\,,\end{aligned}$$

$$\log x\cdot(\rho(x) - \rho(y)) + \log\frac{x}{y}\cdot\rho(y) \leqq \frac{3}{2}\cdot\log\frac{x}{y}\cdot(\log x + \log y) + O(1)\,,$$

or, since $\rho(y) = \log y + O(1)$,

$$\begin{aligned}\log x\cdot(\rho(x) - \rho(y)) &\leqq \log\frac{x}{y}\cdot\left(\frac{3}{2}\cdot\log x + \frac{1}{2}\cdot\log y\right) + O(1)\\ &< 2\cdot\log\frac{x}{y}\cdot\log x + O(1)\,.\end{aligned}$$

Thus

$$\rho(x) - \rho(y) < 2\cdot\log\frac{x}{y} + O\left(\frac{1}{\log x}\right),$$

and, since $\rho(x) = \log x - A_0 + b_x$, it follows that $b_x - b_y < \log x/y + O(1/\log x)$. Also obviously $b_x - b_y \geqq -\log x/y$, because $\rho(x)$ is non-decreasing. Thus we obtain

$$|b_x - b_y| \leqq \log\frac{x}{y} + O\left(\frac{1}{\log x}\right) \quad \text{if } c\cdot x < y < x,\ 0 < c < 1\,. \tag{22}$$

PART III

Let $B \geqq 1$ be an upper bound of $|b_n|$.

Since $b_n - b_{n-1}$ is either $-\log[n/(n-1)]$, or $\log p/n - \log[n/(n-1)]$, it cannot happen that $b_n = b_{n-1} = 0$.

Let the integers $r_1, r_2, \cdots, r_t, \cdots$ be the indices n for which the b_n change signs. Precisely:

$$(23)\qquad \begin{cases} r_1 = 1\,;\ n = r_t \ \text{ if } \ b_n \cdot b_{n+1} \leqq 0, \text{ and } b_{n+1} \neq 0\,; \\ \text{if } \ r_t < v \leqq w < r_{t+1} \text{ then } b_v \cdot b_w > 0\,; \text{ and} \\ |b_{r_t}| < (\log r_t)/r_t \text{ for } t > 1. \end{cases}$$

Let $\{s_k\}$ be a sequence of integers, determined as follows: every r_t is an s_k; if $\log(r_{t+1}/r_t) < 7 \cdot B$, and $r_t = s_k$, then $r_{t+1} = s_{k+1}$; if $\log(r_{t+1}/r_t) \geqq 7 \cdot B$, enough integers s_{k+v} are inserted between $r_t = s_k$ and $r_{t+1} = s_{k+m}$ such that $3 \cdot B \leqq \log(s_{k+v+1}/s_{k+v}) < 7 \cdot B$, for $v = 0, 1, \cdots, m-1$. If there is a last $r_{t_0} = s_{k_0}$, a sequence $\{s_{k_0+v}\}$ is formed such that $3 \cdot B \leqq \log(s_{k_0+v+1}/s_{k+v}) < 7 \cdot B$. Thus the s_k form a sequence with the following properties:

$$(24)\qquad \begin{cases} s_1 = 1\,;\ \log(s_{k+1}/s_k) < 7 \cdot B\,;\ \text{for } k > 1, \textit{ either} \\ \log(s_{k+1}/s_k) \geqq 3 \cdot B, \textit{ or } |b_{s_k}| \text{ and } |b_{s_{k+1}}| \text{ are both} \\ \text{less than } \dfrac{\log s_k}{s_k}\,;\ b_v \cdot b_w > 0 \text{ for } s_k < v \leqq w < s_{k+1}\,. \end{cases}$$

Assume now that $\alpha\,(0 < \alpha < 1/2)$ is such that

$$(25)\qquad \textit{not} \quad h_x = O(\log^{-\alpha} x).$$

Then $|h_x| \cdot \log^{\alpha} x$ is unbounded. Let x be large, and such that $|h_x| \cdot \log^{\alpha} x \geqq |h_y| \cdot \log^{\alpha} y$ for all $y \leqq x$. Let c and d be positive integers such that

$$(26)\qquad s_{c-1} < \log x \leqq s_c\,, \quad \text{and} \quad s_d \leqq x < s_{d+1}\,.$$

It will be shown that

$$\frac{1}{2} \cdot (1 - \alpha - o(1)) \cdot S(x) \leqq |h_x| \cdot \log x \leqq \frac{1}{3} \cdot (1 + o(1)) \cdot S(x)\,,$$

where

$$(27)\qquad S(x) \equiv \sum_{k=c+1}^{d} |h_{s_k} - h_{s_{k-1}}| \cdot \log(s_k/s_{k-1})\,.$$

From this it will follow that $\alpha \geqq 1/3$.

Clearly

$$|h_x| \cdot \log x = |h_x| \cdot \log^{\alpha} x \cdot \left\{ \log^{1-\alpha} x - \log^{1-\alpha} s_d + \sum_{k=2}^{d} (\log^{1-\alpha} s_k - \log^{1-\alpha} s_{k-1}) \right\}$$

$$\geqq \frac{1}{2} \cdot \sum_{k=c+1}^{d} (|h_{s_k}| \cdot \log^{\alpha} s_k + |h_{s_{k-1}}| \cdot \log^{\alpha} s_{k-1}) \cdot (\log^{1-\alpha} s_k - \log^{1-\alpha} s_{k-1})$$

$$\geqq \frac{1}{2} \cdot \sum_{k=c+1}^{d} |h_{s_k} - h_{s_{k-1}}| \cdot \log^{\alpha} s_{k-1} \cdot (\log^{1-\alpha} s_k - \log^{1-\alpha} s_{k-1})\,.$$

If $y < z$, it is easily shown by the mean value theorem that

$$y^{\alpha} \cdot (z^{1-\alpha} - y^{1-\alpha}) > (1-\alpha) \cdot \frac{y}{z} \cdot (z-y) > \left(1 - \alpha - \frac{z-y}{z}\right)(z-y) .$$

With $y = \log s_{k-1}$, $z = \log s_k$, and from the fact that $s_k > \log x$, $\log (s_k/s_{k-1}) < 7 \cdot B$, it follows by (27) that

$$|h_x| \cdot \log x > \frac{1}{2} \cdot \left(1 - \alpha - \frac{7 \cdot B}{\log \log x}\right) \cdot S(x) . \tag{28}$$

For the next estimate, we need the following lemma.

LEMMA. *Let v and w be positive integers such that*

(1) $\log \frac{w}{v} = O(1)$;

(2) $b_n > 0$ *for* $v \leqq n \leqq w$;

(3) $b_v < \frac{\log v}{v}$

Then

$$\sum_{v \leqq n \leqq w} \frac{1}{n} \cdot b_n^2 \leqq \frac{2}{3} \cdot \log \frac{w}{v} \cdot \sum_{v \leqq n \leqq w} \frac{1}{n} b_n + O\left(\frac{\log (w/v)}{\log v}\right) .$$

Proof. If $b_n \leqq 1/3 \cdot \log w/v$ for every n in $[v, w]$, the lemma is obviously correct. Otherwise, let n_1 be such that

$$b_{n_1} \geqq \frac{1}{3} \cdot \log \frac{w}{v} , \qquad b_n < \frac{1}{3} \cdot \log \frac{w}{v} \quad \text{for} \quad v \leqq n < n_1 .$$

If $\log (n_1/v) > 1/3 \log (w/v)$, let z $(v \leqq z < n_1)$ be such that $\log (n_1/z) = 1/3 \log (w/v)$; otherwise, let $z = v$. Thus by (22), in every case, $\log (n_1/z) = 1/3 \log (w/v) + O(1/\log v)$. Clearly $b_n - 2/3 \cdot \log w/v < 0$ for $v \leqq n \leqq z$. Thus

$$\begin{aligned} T &\equiv \sum_{v \leqq n \leqq w} \frac{1}{n} \cdot b_n^2 - \frac{2}{3} \cdot \log \frac{w}{v} \cdot \sum_{v \leqq n \leqq w} \frac{1}{n} \cdot b_n \\ &\leqq \sum_{z \leqq n \leqq w} \frac{1}{n} \cdot b_n^2 - \frac{2}{3} \cdot \log \frac{w}{v} \cdot \sum_{z \leqq n \leqq w} \frac{1}{n} \cdot b_n , \end{aligned}$$

$$T \leqq \sum_{z \leqq n \leqq w} \frac{1}{n} \cdot \left(b_n - \frac{1}{3} \cdot \log \frac{w}{v}\right)^2 - \frac{1}{9} \cdot \log^2 (w/v) \cdot \log (w/z) + O\left(\frac{\log (w/v)}{v}\right) .$$

By (22),

$$\begin{aligned} \left| b_n - \frac{1}{3} \log (w/v) \right| &= |b_n - b_{n_1}| + O\left(\frac{\log v}{v}\right) \\ &\leqq |\log (n_1/n)| + O\left(\frac{1}{\log v}\right) = |\log (n_1/z) - \log (n/z)| + O\left(\frac{1}{\log v}\right) , \end{aligned}$$

and thus

$$\left|b_n - \frac{1}{3}\cdot\log\frac{w}{v}\right| \leqq \left|\log\frac{n}{z} - \frac{1}{3}\log\frac{w}{v}\right| + O\Big(\frac{1}{\log v}\Big)\,.$$

Thus

$$\begin{aligned} T &\leqq \sum_{z\leqq n\leqq w}\frac{1}{n}\cdot\Big(\log\frac{n}{z} - \frac{1}{3}\cdot\log\frac{w}{v}\Big)^2 \\ &\quad - \frac{1}{9}\cdot\log^2(w/v)\cdot\log(w/z) + O\Big(\frac{\log(w/v)}{\log v}\Big) \\ &= \sum_{z\leqq n\leqq w}\frac{1}{n}\cdot\log^2(n/z) - \frac{2}{3}\cdot\log\frac{w}{v}\cdot\sum_{z\leqq n\leqq w}\frac{1}{n}\cdot\log\frac{n}{z} + O\Big(\frac{\log(w/v)}{\log v}\Big) \\ &= \frac{1}{3}\cdot\log^3(w/z) - \frac{2}{3}\cdot\log^2(w/v)\cdot\frac{1}{2}\cdot\log^2(w/z) + O\Big(\frac{\log(w/v)}{\log v}\Big)\,, \end{aligned}$$

by (2′), and thus $T \leqq O(\log(w/v)/\log v)$. This completes the proof of the lemma.

COROLLARY 1. *If condition* (3) *is replaced by* $b_w < \log w/w$, *the conclusion still holds*; *if* $b_n < 0$ *in* $v \leqq n \leqq w$, *the conclusion holds if* b_n *is replaced by* $|b_n|$.

COROLLARY 2. *If instead of* (3) *it is known that* $b_v < \log v/v$ *and* $b_w < \log w/w$ *then*

$$\sum_{v\leqq n\leqq w}\frac{1}{n}\cdot b_n^2 \leqq \frac{1}{3}\cdot\log\frac{w}{v}\cdot\sum_{v\leqq n\leqq w}\frac{1}{n}\cdot|b_n| + O\Big(\frac{\log(w/v)}{\log v}\Big)\,.$$

For a proof, we split $[v, w]$ *into two intervals by a division point at* $(v\cdot w)^{1/2}$, *and apply the lemma separately to each subinterval.*

COROLLARY 3.

$$(29)\qquad \sum_{s_{k-1}<n\leqq s_k}\frac{1}{n}\cdot b_n^2 \leqq \frac{1}{3}\cdot\log(s_k/s_{k-1})\cdot\sum_{s_{k-1}<n\leqq s_k}\frac{1}{n}\cdot|b_n| + O\Big(\frac{\log(s_k/s_{k-1})}{\log s_k}\Big)\,.$$

Proof. If $\log(s_k/s_{k-1}) < 3\cdot B$, this follows from (24) and Corollary 2; if $\log(s_k/s_{k-1}) \geqq 3B$, it is obvious, since $|b_n| \leqq B$.

By (26), $\sum_{n\leqq s_c} 1/n\cdot b_n^2 = O(\log\log x)$, and $\sum_{s_d<n\leqq x} 1/n\cdot b_n^2 = O(1)$; also

$$\sum_{k=c+1}^{d}\frac{\log(s_k/s_{k-1})}{\log s_k} \leqq \sum_{k=c+1}^{d}\log\Big(\frac{\log s_k}{\log s_{k-1}}\Big) \leqq \log\log x\,.$$

It follows from (29) that

$$\sum_{n\leqq x}\frac{1}{n}\cdot b_n^2 \leqq \frac{1}{3}\cdot\sum_{k=c+1}^{d}\log(s_k/s_{k-1})\cdot\sum_{s_{k-1}<n\leqq s_k}\frac{1}{n}\cdot|b_n| + O(\log\log x)\,.$$

By (8′) $\sum_{s_{k-1}<n\leqq s_k} 1/n\cdot|b_n| = |h_{s_k} - h_{s_{k-1}}| + O(\log s_k/s_k)$, and thus, by (21) and (27),

$$|h_x| \cdot \log x \leqq \frac{1}{3} \cdot S(x) + O(\log \log x) . \tag{30}$$

It follows from (28) and (30) that

$$\left[\frac{1}{3} - \frac{1}{2} \cdot \left(1 - \alpha - \frac{7 \cdot B}{\log \log x}\right)\right] \cdot S(x) \geqq O(\log \log x) ,$$

and since by (25) and (30) $S(x) \geqq K \cdot \log^{1/2} x$, this implies that $\alpha \geqq 1/3$. Thus $h_x = o(\log^{-1/3+\varepsilon} x)$, for every $\varepsilon > 0$, and therefore, by (8′),

$$\sum_{s_{k-1} < n \leqq s_k} \frac{1}{n} \cdot |b_n| = o(\log^{-1/3+\varepsilon} s_k) . \tag{31}$$

In order to find a bound for $|b_x|$, we consider now a particular interval $I_k = (s_{k-1}, s_k]$; let us assume that $b_n > 0$ in I_k. Let $n_2 \in I_k$ be such that $b_{n_2} \geqq b_n$ for every $n \in I_k$. Let n_1 $(s_{k-1} \leqq n_1 < n_2)$ be such that

$$b_{n_1} \leqq \frac{1}{2} \cdot b_{n_2} < b_{n_1+1} .$$

Then

$$\sum_{n \in I_k} \frac{1}{n} \cdot b_n > \sum_{n=n_1+1}^{n_2} \frac{1}{n} \cdot b_n > \frac{1}{2} \cdot b_{n_2} \cdot \log (n_2/n_1) - O(1/s_k) .$$

But by (22),

$$\log (n_2/n_1) \geqq b_{n_2} - b_{n_1} - O\left(\frac{1}{\log s_k}\right) \geqq \frac{1}{2} \cdot b_{n_2} - O\left(\frac{1}{\log s_k}\right) .$$

Thus

$$\sum_{n \in I_k} \frac{1}{n} \cdot b_n > \frac{1}{4} \cdot b_{n_2}^2 - O\left(\frac{1}{\log s_k}\right) .$$

It follows from (31) that $b_{n_2}^2 = o(\log^{-1/3+\varepsilon} n_2)$, and thus

$$b_x = o(\log^{-1/6+\varepsilon} x) . \tag{32}$$

Finally,

$$\begin{aligned}
\psi(x) &= \sum_{n \leqq x} n \cdot (\rho(n) - \rho(n-1)) = [x] \cdot \rho([x]) - \sum_{n \leqq x-1} \rho(n) \\
&= x \cdot (\log x - A_0 + b_x) - \sum_{n \leqq x} (\log n - A_0 + b_n) + O(\log x) \\
&= x \cdot \log x - A_0 \cdot x + b_x \cdot x - x \cdot \log x + x + A_0 \cdot x - \sum_{n \leqq x} b_n + O(\log x) \\
&= x + o(x \cdot \log^{-1/6+\varepsilon} x) + o\left(\sum_{n \leqq x} \log^{-1/6+\varepsilon} n\right) , \quad \text{by (32).}
\end{aligned}$$

The last sum is easily seen to be $o(x \cdot \log^{-1/6+\varepsilon} x)$, and thus

$$\psi(x) = x + o(x \cdot \log^{-1/6+\varepsilon} x) . \tag{33}$$

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Pacific Journal of Mathematics
Vol. 10, No. 2 October, 1960