

# Pacific Journal of Mathematics

**TERMINATING PROLONGATION PROCEDURES**

HAROLD H. JOHNSON

# ON TERMINATING PROLONGATION PROCEDURES\*

H. H. JOHNSON

In the classical treatments [3] of systems of differential equations there are two outstanding techniques—the Cauchy-Kowalewski theorem and completely integrable systems (the latter is really a special case of the former [1, p. 77]). In terms of systems of differential forms the Cauchy-Kowalewski theorem becomes the Cartan-Kahler theorem, and systems with independent variables which satisfy its conditions are called involutive.

Many systems are not involutive, and the central problem of prolongation theory is to construct a procedure by which one can reduce every system to an equivalent involutive system. For total prolongations Kuranishi's theorem [4, p. 44] gives a precise answer to the question of when total prolongations will lead to involutive systems. If  $S$  is the initial system in euclidean space  $E^n$ ,  $P^g(S)$  the  $g^{\text{th}}$  total prolongation in the space  $R_g$ , then for all points  $x \in E^n$ , except possibly on a proper subvariety, there is a number  $g_0$  such that if  $g \geq g_0$  and  $y \in R_g$  is a point over  $x$ , then  $P^g(S)$  is involutive at  $y$  if and only if  $y$  is an ordinary integral point [4, p. 7] and the 1-forms of  $P^g(S)$  do not imply any dependencies among the independent variables at integral points in a neighborhood of  $y$ . Then  $y$  is called a normal point.

The first part of this paper deals with an application of this theorem to certain types of differential systems. We show that under certain conditions the total prolongation process must result in normal points if there are to be *any* solutions. An application of this leads to a theorem often used in differential geometry [2, p. 14].

The second section is concerned with what can be done if normal points are not obtained for  $P^g(S)$  as is the case with an example of Kuranishi. Here we must distinguish two cases. If  $P^g(S)$  does not contain ordinary integral points, so that its 0-forms are not a regular system of equations [4, p. 7] the Cartan-Kahler theory does not apply. Let us call such systems *singular*. We shall not consider this aspect of the problem in this paper.

If, however, the problem lies in a dependency among the independent variables implied by 1-forms of  $P^g(S)$ , at generic integral points, one would naturally think of restricting the system to those points where dependencies do not occur, since solutions must lie only in these points. Thus one obtains a sort of partial prolongation which could in turn be prolonged. Such a procedure was certainly what Cartan and Kuranishi had in mind. However, it is not clear that the process will ever result

---

Received March 2, 1959, and in revised form May 25, 1959. This work was supported in part by the U. S. Army Office of Ordnance Research, Contract DA 04-200-ORD-456.

in an involutive system. One might conceivably go on obtaining non-normal systems indefinitely.

Kuranishi has recently proved a generalization of his prolongation theorem which is used to show that the above procedure does in fact ultimately stop, barring the occurrence of singular systems somewhere along the line.

The first section of this paper is part of the author's doctoral thesis at the University of California at Berkeley, written under the direction of Professor Harley Flanders to whom the author would like to record here his appreciation.

All functions, forms, and manifolds are assumed to be real analytic.

1. Kuranishi's fundamental theorem [4, p. 44] concerns a certain general type of differential system (called *normal*) which is generated by 1-forms  $\theta^\alpha$ ,  $\alpha = 1, \dots, \alpha_1$ . If  $\omega^1, \dots, \omega^p$  is a basis of a system of independent variables and  $\pi^1, \dots, \pi^m$  any other 1-forms to fill out a basis, then the  $\theta^\alpha$  are normal if  $d\theta^\alpha$  can be expressed as

$$d\theta^\alpha \equiv \sum_{i=1}^p \sum_{\lambda=1}^m A_{\varphi, i\lambda} \omega^i \wedge \pi^\lambda + \sum_{i=1}^p \sum_{j=1}^p \frac{1}{2} B_{\varphi, ij} \omega^i \wedge \omega^j$$

modulo  $(\theta^\alpha)$ . Suppose that these are defined on  $E^n$  where  $n = \alpha_1 + p + m$  of variables  $x^1, \dots, x^n$ . Then  $R_g$  is the euclidean space of variables

$$x^j, u_{i_1}^\lambda, u_{i_1 i_2}^\lambda, \dots, u_{i_1 \dots i_g}^\lambda,$$

where  $j = 1, \dots, n$ ;  $i_1, \dots, i_g = 1, \dots, p$ ;  $\lambda = 1, \dots, m$ , and the  $u_{i_1 \dots i_g}^\lambda$  are symmetric in the lower indicies.

Then  $P^g(S)$  can be taken to be the system on  $R_g$  generated by the 1-forms

$$\pi_g \left\{ \begin{array}{l} \theta^\alpha, \\ d\pi^\lambda - \sum_{j=1}^n u_j^\lambda \omega^j, \\ du_{j_1}^\lambda - \sum_{j=1}^n u_{j_1 j}^\lambda \omega^j, \\ \vdots \\ du_{j_1 j_2 \dots j_{g-1}}^\lambda - \sum_{j=1}^n u_{j_1 j_2 \dots j_{g-1} j}^\lambda \omega^j, \end{array} \right.$$

and certain functions

$$\theta_{\varphi; i j; k_1 \dots k_t}, \quad t \leq g - 1.$$

It turns out that for  $t \leq g - 2$ ,

$$d\theta_{\varphi; i j; k_1 \dots k_t} \equiv 0 \quad \text{modulo } \pi_g,$$

while

$$d\theta_{\varphi; i j; k_1 \dots k_{g-1}} \equiv \sum_{\lambda=1}^m (A_{\varphi; i \lambda} du_{j k_1 \dots k_{g-1}}^\lambda - A_{\varphi; j \lambda} du_{i k_1 \dots k_{g-1}}^\lambda) + \sum_{k=1}^p B_{\varphi; i j; k_1 \dots k_{g-1} k} \omega^k,$$

modulo  $\pi_g$ .

These  $B$ 's are defined inductively by

$$B_{\varphi; i j; k_1 \dots k_t} = \sum_{\lambda=1}^m [(D_{k_t} A_{\varphi; i \lambda}) du_{j k_1 \dots k_{t-1}}^\lambda - (D_{k_t} A_{\varphi; j \lambda}) du_{i k_1 \dots k_{t-1}}^\lambda] + D_{k_t} B_{\varphi; i j; k_1 \dots k_{t-1}},$$

where  $D_k F$  is defined as follows.

If  $F$  is any function on  $R_{t-1}$  it can be considered to be a function on  $R_s$  for all  $s \geq t - 1$ . If we form  $dF$ , then modulo  $\pi_s$ , when  $s \geq t$ ,  $dF$  involves only  $\omega^1, \dots, \omega^p$ :

$$dF \equiv \sum_{k=1}^p F_k \omega^k \quad \text{modulo } \pi_s,$$

and the  $F_k$  are independent of  $s$  so long as  $s \geq t$ . Then one defines  $D_k F$  to be  $F_k$ .  $D_k F$  is a function on  $R_t$ .

**THEOREM 1.** *Let  $S$  be a normal system where*

- (1) *the  $A_{\varphi; i \lambda}$  are constants,*
- (2)  *$dB_{\varphi; i j} \equiv 0$  modulo  $(\omega^1, \theta^\alpha)$ .*

*Then if  $P^g(S)$  is non-singular for all  $g$ , there is a  $g_0$  such that  $P^g(S)$  is involutive for all  $g \geq g_0$  at ordinary integral points, or else there exist no solutions.*

*Proof.* If an ordinary integral point  $y \in R_g$  is not normal, then there must be a dependency among  $\omega^1, \dots, \omega^p$  implied by the 1-forms of  $P^g(S)$  at integral points  $y_1$  arbitrarily near  $y$ . This can happen only if there is a relation of the type

$$\Sigma I^{\varphi; i j; k_1 \dots k_{g-1}}(y_1) (d\theta_{\varphi; i j; k_1 \dots k_{g-1}})_{y_1} \equiv 0 \text{ modulo } (\omega^t),$$

where the left side does not vanish identically. This can only happen if

$$0 = \Sigma I^{\varphi; i j; k_1 \dots k_{g-1}}(y_1) [A_{\varphi; i \lambda}(y_1) (du_{j k_1 \dots k_{g-1}}^\lambda)_{y_1} - A_{\varphi; j \lambda}(y_1) (du_{i k_1 \dots k_{g-1}}^\lambda)_{y_1}],$$

while for some  $k$ ,

$$\Sigma I^{\varphi; i j; k_1 \dots k_{g-1}}(y_1) B_{\varphi; i j; k_1 \dots k_{g-1} k}(y_1) \neq 0.$$

Since the  $A$  depend only on  $x^1, \dots, x^n$ , we can choose the  $I$  to be functions of  $x^1, \dots, x^n$ .

Now, the functions in  $P^{g+1}(S)$  have the form

$$\theta_{\varphi; i j; k_1 \dots k_g} = \sum_{\lambda=1}^m (A_{\varphi; i \lambda} u_{j k_1 \dots k_g}^\lambda - A_{\varphi; j \lambda} u_{i k_1 \dots k_g}^\lambda) + B_{\varphi; i j; k_1 \dots k_g} .$$

Hence we have in  $P^{g+1}(S)$  the function which is not in  $P^g(S)$ ,

$$\Sigma \Gamma^{\varphi; i j; k_1 \dots k_{g-1}} \theta_{\varphi; i j; k_1 \dots k_{g-1} k} = \Sigma \Gamma^{\varphi; i j; k_1 \dots k_{g-1}} B_{\varphi; i j; k_1 \dots k_{g-1} k} .$$

Consider now these  $B$ . Since the  $A$  are constants,

$$B_{\varphi; i j; k_1 \dots k_t} = D_{k_t} B_{\varphi; i j; k_1 \dots k_{t-1}} ;$$

where

$$dB_{\varphi; i j; k_1 \dots k_{s-1}} \equiv \sum_{k=1}^p D_k B_{\varphi; i j; k_1 \dots k_{s-1}} \omega^k$$

modulo  $\pi_s$ .

By assumption (2),  $dB_{\varphi; i j}$  have the form

$$\begin{aligned} dB_{\varphi; i j} &= \sum_{k=1}^p C_{\varphi; i j; k} \omega^k + \sum_{\beta=1}^{\alpha_1} E_{\varphi; i j; \beta} \theta^\beta , \\ &= \sum_{k=1}^p C_{\varphi; i j; k} \omega^k \qquad \text{modulo } \pi_1 , \end{aligned}$$

hence

$$B_{\varphi; i j; k} = D_k B_{\varphi; i j} = C_{\varphi; i j; k}$$

are functions of  $x^1, \dots, x^n$  alone. Obviously  $dB_{\varphi; i j; k} \equiv 0$  modulo  $(\theta^\alpha, \omega^t)$  also, so the argument can be repeated to show that the functions

$$B_{\varphi; i j; k_1 \dots k_t}$$

depend only on  $x^1, \dots, x^n$ . But that means that

$$(I) \qquad \Sigma \Gamma^{\varphi; i j; k_1 \dots k_{g-1}} B_{\varphi; i j; k_1 \dots k_{g-1} k}$$

is a function in  $P^{g+1}(S)$ , not in  $P^g(S)$ , and dependent only on  $x^1, \dots, x^n$ .

Now, to any integral manifold  $I$  of  $S$  there corresponds a unique integral manifold  $I^{g+1}$  of  $P^{g+1}(S)$ , such that if  $\rho^{g+1}$  is the natural fibre bundle mapping on  $R_{g+1}$  to  $E^n$ , then  $\rho^{g+1}(I^{g+1}) = I$  [4, p. 15].  $I^{g+1}$  must annihilate the function (1). Since it is a function of  $x^1, \dots, x^n$  alone,  $I$  must itself annihilate it.

We conclude; if there exist ordinary integral points in  $R_g$  where  $P^g(S)$  is not normal, then the manifold of integral points of  $S$  where solutions can occur must satisfy an additional condition to any imposed by  $P^t(S)$ ,  $t < g$ . Clearly, this can happen at most  $n - p$  times if there are to be solutions

Since the  $A_{\varphi; i \lambda}$  are constants, every point of  $E^n$  is regular of order 0 [4, p. 36], so by Kuranishi's fundamental theorem there exists an in-

teger  $g_1$  such that if  $y$  is an ordinary integral point in  $R_{g_1}$  for  $g \geq g_1$ , then  $P^g(S)$  is involutive at  $y$  if and only if  $y$  is normal. Taking  $g_0 = g_1 + (n - p)$  one obtains the theorem.

Next an application of this theorem will be made to a certain type of system of differential equations.

Let  $E^n$  be the euclidean space of variables  $x^1, \dots, x^p, z^1, \dots, z^m$ . Consider the problem of finding  $m$  functions  $f^\lambda(x^1, \dots, x^p) = z^\lambda$  which will satisfy a given set of first order partial differential equations

$$\frac{\partial z^\alpha}{\partial x^i} = \psi_i^\alpha(x, z), \quad \alpha = 1, \dots, m; i = 1, \dots, p.$$

In terms of differential forms this is the problem of finding integral manifolds of the system  $S$  generated by the 1-forms

$$\theta^\alpha = dz^\alpha - \sum_{i=1}^p \psi_i^\alpha(x, z) dx^i$$

with independent 1-forms  $dx^1, \dots, dx^p$ . Here there are no  $\pi^\lambda$ . Then

$$\begin{aligned} d\theta^\alpha \equiv & \frac{1}{2} \sum_{i=1}^p \sum_{j=1}^p \left( \sum_{\beta=1}^m \frac{\partial \psi_i^\alpha}{\partial z^\beta} \psi_j^\beta - \sum_{\beta=1}^m \frac{\partial \psi_j^\alpha}{\partial z^\beta} \psi_i^\beta \right. \\ & \left. + \frac{\partial \psi_i^\alpha}{\partial x^j} - \frac{\partial \psi_j^\alpha}{\partial x^i} \right) dx^j \wedge dx^i \quad \text{modulo } (\theta^\alpha). \end{aligned}$$

If then

$$B_{\alpha; i, j} = \sum_{\beta=1}^m \frac{\partial \psi_i^\alpha}{\partial z^\beta} \psi_j^\beta - \sum_{\beta=1}^m \frac{\partial \psi_j^\alpha}{\partial z^\beta} \psi_i^\beta + \frac{\partial \psi_i^\alpha}{\partial x^j} - \frac{\partial \psi_j^\alpha}{\partial x^i}$$

one can deduce the following theorem from the nature of the forms

$$d\theta_{\phi; i, j; k_1 \dots k_t} \equiv B_{\phi; i, j; k_1 \dots k_t} \omega^k.$$

**THEOREM 2.** *In order that the system of differential equations*

$$\frac{\partial z^\alpha}{\partial x^i} = \psi_i^\alpha(x, z)$$

*have a solution, given that the equations*

$$B_{\phi; i, j; k_1 \dots k_t} = 0, \quad t \leq g$$

*are non-singular for all  $g$ , it is necessary and sufficient that for all  $\phi, i, j, k_1, \dots, k_g$ ,*

$$B_{\phi; i, j, k_1 \dots k_g} \equiv 0 \text{ modulo } (B_{\theta; rs; h_1 \dots h_t} | t \leq g - 1)$$

for some  $g \leq m - 1$ . [See 2, p. 14].

2. In Theorem 1, the prolongation process had to yield an involutive system because whenever a non-normal prolongation occurred, this implied additional restrictions on the original system. In general this need not happen. Kuranishi gives an example of a system in which  $P^g(S)$  is not normal for any  $g \geq 1$  [4, p. 45].

Normality at integral points  $y$  of  $P^g(S)$  involves two conditions; the set of 0-forms of  $P^g(S)$  which define  $y$  must define a regular system of equations at  $y$ , and the 1-forms of  $P^g(S)$  must imply no relations among the independent variables at integral points near  $y$ . This paper will ignore the first problem. It would seem to call for a more delicate approach to the Cartan-Kahler theorem. Let  $y$  be a non-normal integral point of  $P^g(S)$  such that for all integral points  $y_1$  near  $y$  there is a dependency of the type

$$\sum_{i=1}^p A_i(y_1)(\omega^i)_{y_1}$$

in  $P^g(S)$ . Then obviously solutions can occur only at points  $y_1$  where  $A_1(y_1) = A_2(y_1) = \dots = A_p(y_1) = 0$ . Hence a natural step to solving the system would be to add  $A_1, \dots, A_p$  as 0-forms to the system  $P^g(S)$ . One would obtain a system having the same solutions as  $P^g(S)$ .

Observe also that if  $P^g(S)$  contains a 0-form which is a function on  $R_{g-1}$ , obviously any solution of  $P^{g-1}(S)$  must annihilate that function; hence, adding it to  $P^{g-1}(S)$  would generate a system having the same solutions as  $P^{g-1}(S)$ .

We introduce the following definition: let the system  $T$  in independent variables  $x^1, \dots, x^p$ , and dependent variables  $y^1, \dots, y^r, z^1, \dots, z^m$  be called *complete* if the 1-forms of  $T$  contain no forms of the type  $\Sigma A_i \omega^i$ , where  $\omega^1, \dots, \omega^p$  is a basis of independent variables,  $A_i$  not in  $T$ .

LEMMA. *Let  $S$  be any system with independent variables  $x^1, \dots, x^p$ , and dependent variables  $z^1, \dots, z^m$ . Then there exists a sequence  $\{S^g\}$  of differential systems  $S^g$ , closed, on  $R_g$  such that*

- (1)  $S^g$  has the same solutions as  $P^g(S)$ ,
- (2)  $S^g$  is complete,
- (3)  $P(S^{g-1}) \subseteq S^g$ , and
- (4) the 0-forms of  $S^g$  contain no functions on  $R_{g-1}$  except those in  $S^{g-1}$ ,
- (5)  $S^g$  is generated by 0-forms,  $\pi_g$ , and their derivatives.

*Proof.* Let  $X$  be the set of all sequences  $\{T^g | g = 1, 2, \dots\}$ , where  $T^g$  is a closed differential system on  $R_g$  generated by 0-forms,  $\pi_g$  and their derivatives and having the same solutions as  $P^g(S)$  and  $P(T^{g-1}) \subseteq T^g$ . The elements of  $X$  can be partially ordered by inclusion:  $\{U^g\} \supseteq \{T^g\}$  if  $U^g \supseteq T^g$  for all  $g = 1, 2, \dots$ . If  $A = \{\{T_a^g\} | a \in A\}$  is a nest in  $X$ ,

then  $\{T^g\}$ , where  $T^g$  is the closed differential system generated by  $U\{T_a^g | a \in A\}$ , is in  $X$  and is  $\geq$  every element of  $A$ . Hence,  $X$  contains a maximal element,  $\{S^g\}$ . By definition,  $\{S^g\}$  satisfies (1) and (3). If  $S^h$  were not complete, one could add to  $S^h$  the coefficients of forms of the type  $\Sigma A_i \omega^i$  to obtain a still larger system  $\bar{S}^h$ , and  $\{\bar{S}^g\}$ , where  $\bar{S}^g = S^g$  for  $g < h$ , and  $\bar{S}^g = P^{g-h}(\bar{S}^h)$  for  $g \geq h$ , would be properly greater than  $\{S^g\}$ . Similarly, if condition (4) did not hold for some  $S^h$ , we could enlarge  $S^{h-1}$ . Hence the lemma.

The construction of such a sequence, given  $S$ , could proceed as follows. Form  $P(S)$  and complete it in the obvious way to form a system  $T^1$ . If the resulting system involves any functions on  $R_0$  i.e., depending only on the coordinates of  $R_0$ , add these to the system  $S$  and begin again. Otherwise, form  $P(T^1)$  and complete to form  $T^2$ . If  $T^2$  contains functions on  $R_1$ , add these to  $T_1$  and begin again at that step. Observe that the addition of new functions to any one system on, say,  $R_g$ , is limited by the dimension of  $R_g$ , since each such addition reduces the dimension of the variety of integral points, which must have at least dimension  $p$  if there are to be any solutions at all.

Granted that such a sequence  $\{S^g\}$  as given in the lemma exists, it is still not clear whether any  $S^g$  is involutive. Of course, the 0-forms might not define a regular system of equations for the integral points. But barring this one can prove that for  $g$  sufficiently large,  $S^g$  is involutive. This follows from a recent extension of Kuranishi's prolongation theorem [5, Theorem III. 1], where the required conditions are precisely those of the lemma.

**THEOREM 3.** *Given a differential system  $S$  with independent variable  $dx^1, \dots, dx^p$ , there exists a sequence  $\{S^g\}$  of closed differential systems, where  $S^g$  is on  $R_g$ ,  $g = 1, 2, \dots$ , which have the same solutions as  $P^g(S)$ . Moreover, if for all  $g \geq g_0$ ,  $S^g$  is non-singular, then there exists a  $g_1$  such that for  $g \geq g_1$ ,  $P(S^{g-1}) = S^g$  and  $S^g$  is involutive.*

#### REFERENCES

1. E. Cartan, *Les systemes differentielles exterieurs et leurs applications geometriques*, Paris, 1945.
2. L. P. Eisenhart, *Non-Riemannian geometry*, New York, 1927.
3. E. Goursat, *Lecons sur l'integration des equations aux derivees partielles du premier order*, Paris, 1921.
4. M. Kuranishi, *On E. Cartan's prolongation theorem of exterior differential systems*, Amer. J. Math., vol. **79** (1957), pp. 1-47.
5. ———, *On the abstract approach to the local theory of continuous infinite pseudo groups*, Project Report, University of Chicago, 1957.





# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DAVID GILBARG  
Stanford University  
Stanford, California

F. H. BROWNELL  
University of Washington  
Seattle 5, Washington

A. L. WHITEMAN  
University of Southern California  
Los Angeles 7, California

L. J. PAIGE  
University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH  
T. M. CHERRY  
D. DERRY

E. HEWITT  
A. HORN  
L. NACHBIN

M. OHTSUKA  
H. L. ROYDEN  
M. M. SCHIFFER

E. SPANIER  
E. G. STRAUS  
F. WOLF

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE COLLEGE  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE COLLEGE  
UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
HUGHES AIRCRAFT COMPANY  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

# Pacific Journal of Mathematics

Vol. 10, No. 2

October, 1960

Maynard G. Arsove, <i>The Paley-Wiener theorem in metric linear spaces</i> . . . . .	365
Robert (Yisrael) John Aumann, <i>Acceptable points in games of perfect information</i> . . . . .	381
A. V. Balakrishnan, <i>Fractional powers of closed operators and the semigroups generated by them</i> . . . . .	419
Dallas O. Banks, <i>Bounds for the eigenvalues of some vibrating systems</i> . . . . .	439
Billy Joe Boyer, <i>On the summability of derived Fourier series</i> . . . . .	475
Robert Breusch, <i>An elementary proof of the prime number theorem with remainder term</i> . . . . .	487
Edward David Callender, Jr., <i>Hölder continuity of <math>n</math>-dimensional quasi-conformal mappings</i> . . . . .	499
L. Carlitz, <i>Note on Alder's polynomials</i> . . . . .	517
P. H. Doyle, III, <i>Unions of cell pairs in <math>E^3</math></i> . . . . .	521
James Eells, Jr., <i>A class of smooth bundles over a manifold</i> . . . . .	525
Shaul Foguel, <i>Computations of the multiplicity function</i> . . . . .	539
James G. Glimm and Richard Vincent Kadison, <i>Unitary operators in <math>C^*</math>-algebras</i> . . . . .	547
Hugh Gordon, <i>Measure defined by abstract <math>L_p</math> spaces</i> . . . . .	557
Robert Clarke James, <i>Separable conjugate spaces</i> . . . . .	563
William Elliott Jenner, <i>On non-associative algebras associated with bilinear forms</i> . . . . .	573
Harold H. Johnson, <i>Terminating prolongation procedures</i> . . . . .	577
John W. Milnor and Edwin Spanier, <i>Two remarks on fiber homotopy type</i> . . . . .	585
Donald Alan Norton, <i>A note on associativity</i> . . . . .	591
Ronald John Nunke, <i>On the extensions of a torsion module</i> . . . . .	597
Joseph J. Rotman, <i>Mixed modules over valuations rings</i> . . . . .	607
A. Sade, <i>Théorie des systèmes démosiens de groupoïdes</i> . . . . .	625
Wolfgang M. Schmidt, <i>On normal numbers</i> . . . . .	661
Berthold Schweizer, Abe Sklar and Edward Oakley Thorp, <i>The metrization of statistical metric spaces</i> . . . . .	673
John P. Shanahan, <i>On uniqueness questions for hyperbolic differential equations</i> . . . . .	677
A. H. Stone, <i>Sequences of coverings</i> . . . . .	689
Edward Oakley Thorp, <i>Projections onto the subspace of compact operators</i> . . . . .	693
L. Bruce Treybig, <i>Concerning certain locally peripherally separable spaces</i> . . . . .	697
Milo Wesley Weaver, <i>On the commutativity of a correspondence and a permutation</i> . . . . .	705
David Van Vranken Wend, <i>On the zeros of solutions of some linear complex differential equations</i> . . . . .	713
Fred Boyer Wright, Jr., <i>Polarity and duality</i> . . . . .	723