

Pacific Journal of Mathematics

**CONCERNING CERTAIN LOCALLY PERIPHERALLY
SEPARABLE SPACES**

L. BRUCE TREYBIG

CONCERNING CERTAIN LOCALLY PERIPHERALLY SEPARABLE SPACES

L. B. TREYBIG

In 1954, F. Burton Jones raised the question [2] "Is every connected, locally peripherally separable [3], metric space separable?" In this paper it will be shown that there exists a connected, semi-locally-connected, space Σ satisfying R. L. Moore's axioms 0 and C^1 , in which every region has a separable boundary, every pair of points is a subset of some separable continuum², and the set of all points at which Σ is not locally separable is separable. It will also be shown that every compactly connected, locally peripherally separable, metric space is completely separable.

PART 1

Let S' denote the set of all points of the Euclidean plane E . A square disk in E will be said to be horizontal if it has two horizontal sides. A point set in E will be called an H -disk only if that set is a horizontal square disk. By the *width* of a square disk will be meant the length of one of its sides.

Let K denote a definite H -disk of width d . Let $R_0(K)$ denote the H -disk of width $d/4$ whose center is on the vertical line that contains the center of K , and whose upper side lies at a distance of $d/16$ below the upper side of K . Let $R_{00}(K)$ and $R_{01}(K)$ denote the H -disks of width $d/8$ whose upper sides are at a distance of $d/32$ above the lower side of $R_0(K)$ and whose centers are on the vertical lines containing the left and right sides, respectively, of $R_0(K)$.

In general, for each positive integer n let $U_n(K)$ denote a collection of 2^n mutually exclusive congruent H -disks such that

- (1) $R_{01}(K)$ and $R_{00}(K)$ are the elements of $U_1(K)$,
- (2) if n is a positive integer and y is an element of $U_n(K)$, and x and z are H -disks of width $d/4(2)^{n+1}$ whose centers lie on the same vertical lines as the left and right sides of y , respectively, and whose upper sides lie at a distance of $d/32(2)^n$ above the lower side of y , then x and z are elements of $U_{n+1}(K)$.

If n is a positive integer and $R_{x_1x_2\cdots x_n}(K)$ is an elements of $U_n(K)$, then let the elements x and y of $U_{n+1}(K)$ whose centers lie on the same

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¹ The proof that every space which satisfies axioms 0 and C is metric is due to R. L. Moore.

² A continuum is a connected, closed set.

vertical lines as the left and right sides of $R_{x_1x_2\cdots x_n}(K)$, respectively, be denoted by $R_{x_1x_2\cdots x_n0}(K)$ and $R_{x_1x_2\cdots x_n1}(K)$, respectively. Let $C(K)$ be a collection to which x belongs if and only if x is $R_0(K)$ or in one of the collections $U_1(K)$, $U_2(K)$, \dots .

Let $L(K)$ denote the H -disk of width $d/8$ whose center is on the same vertical line as the center of K , and whose lower side is at a distance of $3d/16$ above the lower side of K . Let $P_l(K)$ and $P_r(K)$ denote the left and right-hand end points, respectively, of the lower side of $L(K)$. Let $M(K)$ denote the point set such that a point P belongs to it if and only if P is a point of the interval $P_l(K)P_r(K)$ such that there is no nonnegative integer p and positive integer q such that $PP_l(K)/P_l(K)P_r(K) = p/2^q$. Let $I(K)$ denote the collection to which x belongs if and only if x is a vertical interval containing a point of $M(K)$, and with both end points on the boundary of $L(K)$. Let an interval i of $I(K)$ be denoted by $i_x(I(K))$ if and only if it is true that if P is the lowest point of i , then $P_l(K)P/P_r(K)P_r(K) = x$.

Let R denote some definite H -disk. Let $R_0(R)$ be denoted by Q_0 ; let $R_{00}(R)$ and $R_{01}(R)$ be denoted by Q_{00} and Q_{01} , respectively. Let $R_{000}(R)$, $R_{001}(R)$, $R_{010}(R)$, and $R_{011}(R)$ be denoted by Q_{000} , Q_{001} , Q_{010} , and Q_{011} , respectively, and so forth. Let $C(R)$ be denoted by C_1 and let $I(R)$ be denoted by I_0 .

Let C_2 denote the collection to which x belongs if and only if x is an element of $C(y)$, for some element y of C_1 distinct from Q_0 . Let $R_0(Q_0)$ be denoted by $Q_{00,0}$; let $R_{01}(Q_0)$ be denoted by $Q_{00,01}$. In general, let $R_x(Q_y)$ be denoted by $Q_{y,x}$. Also, if Q_x is in C_1 and $x \neq 0$, let $I(Q_x)$ be denoted by I_x .

In general, let C_{n+1} denote the collection to which x belongs if and only if x is an element of $C(y)$, for some elements y of C_n , which, in case x_n is 0, is distinct from Q_{x_1,x_2,\dots,x_n} . Let the element $R_{x_{n+1}}[R_{x_n}[R_{x_{n-1}}[\dots[R_{x_1}(R)]\dots]]$ of C_{n+1} be denoted by $Q_{x_1,x_2,\dots,x_{n+1}}$. Also if w is the element Q_{x_1,x_2,\dots,x_n} of C_n and $x_n \neq 0$, then let $I(w)$ be denoted by I_{x_1,x_2,\dots,x_n} . For each n let I_n be the collection to which x belongs if and only if there is an element Q_{x_1,x_2,\dots,x_n} of C_n such that $x_n \neq 0$ and x is in $I(Q_{x_1,x_2,\dots,x_n})$.

Let W denote the point set to which a point P belongs if and only if P belongs to C_n^* for each positive integer n . For each positive integer n let B_n denote the collection of all the boundaries of the elements of C_n . The boundary of Q_{x_1,x_2,\dots,x_n} will be denoted by J_{x_1,x_2,\dots,x_n} .

Let S denote $[I_0^* + I_1^* + \dots] + [B_1^* + B_2^* + \dots] + W$.

Let C' be a collection to which w belongs if and only if w is R or $I(w)$ is a subset of S and there is a positive integer n such that w is in C_n .

³ C_n^* Means the sum of all the point sets of the collection C_n .

For each positive integer n let H_n denote a collection to which x belongs if and only if x is the common part of S and the interior of some square of $[B_n + B_{n+1} + \dots]$. For each element Q_{x_1, x_2, \dots, x_n} of C_n , let the set of all points of S in the interior of J_{x_1, x_2, \dots, x_n} be denoted by r_{x_1, x_2, \dots, x_n} .

For each positive integer n let K_n denote a collection to which x belongs if and only if, either (1) x is a segment of an arc lying on some square J of $(B_1 + B_2 + \dots)$, having length less than $1/4^n$ times the perimeter of J , and intersecting no square of the collection $(B_1 + B_2 + \dots)$ except J , or (2) x is the sum of two straight line segments p and q intersecting at their midpoints and lying on different squares J_p and J_q of $(B_1 + B_2 + \dots)$, such that p and q each have length less than $1/4^n$ times the perimeters of J_p and J_q , respectively, and such that neither p nor q intersects three squares of $(B_1 + B_2 + \dots)$.

Suppose x is a positive number such that $i_x[I_{j_1, j_2, \dots, j_n}]$ is an interval of I_{j_1, j_2, \dots, j_n} . For each positive integer n there exists a unique pair (k_n, x_n) such that k_n is a non-negative integer, x_n is a positive number less than one, and $x = (k_n + x_n)/2^n$. By $i_n[i_x(I_{j_1, j_2, \dots, j_n})]$ will be meant the vertical interval $i_{x_n}(I(y))$, where y is the H -disk of $U_n[Q_{j_1, j_2, \dots, j_n}]$ with only k_n disks of $U_n(Q_{j_1, j_2, \dots, j_n})$ to the left of it.

Suppose, for some y in C' , P is the highest point of the interval $i_x(I(y))$. By $R_n(P)$ will be meant the sum of all the sects z such that either

(1) for some positive integer d greater than or equal to n , z is the subset of $i_a[i_x(I(y))]$ with length $1/2^n$ times the length of $i_a[i_x(I(y))]$ that contains the lowest point of $i_a[i_x(I(y))]$, or

(2) z is the subset of $i_x[I(y)]$ with length $1/2^n$ times the length of $i_x(I(y))$ that contains the highest point of $i_x(I(y))$.

For each positive integer n let L_n denote a collection such that x belongs to it if and only if there exists a positive integer d greater than or equal to n , an element y of C' , and an interval of the collection $I(y)$ such that if P denotes the highest point of that interval, then $x = R_d(P)$.

For each positive integer n let N_n denote a collection to which x belongs if and only if either

(1) for some element y of C' there exists an interval i of the collection $I(y)$ such that x is a segment of i of length less than $1/2^n$ times the length of i , or

(2) for some element y of C' there exists an element i of $I(y)$ such that x is a sect lying in i , containing the lowest point of i and of length less than $1/2^n$ times the length of i .

For each positive integer n let G_n denote a collection to which x belongs if and only if it lies in $H_n + K_n + L_n + N_n$. S is the set of

all points of Σ . A subset r of S is a *region* in Σ if and only if r belongs to G_1^4 .

R. L. Moore's axioms 0 and C are as follows:

Axiom 0. Every region is a point set.

Axiom C. There exists a sequence G_1, G_2, \dots such that

(1) for each positive integer n , G_n is a collection such that each element of G_n is of region and G_n covers S ,

(2) for each n , G_{n+1} is a subcollection of G_n ,

(3) if A is a point, B is a point and R is a region containing A , then there exists a positive integer n such that if x is a region of G_n containing A and y is a region of G_n intersecting x , then

(a) y is a subset of R and

(b) if B is not A , y does not contain B ,

(4) if M_1, M_2, \dots is a sequence of closed point sets such that for each n there exists a region g_n of G_n such that M_n is a subset of \bar{g}_n and for each n M_n contains M_{n+1} , then there is a point common to all the point sets of this sequence.

It is obvious that in the space Σ each region has a countable, and therefore separable, boundary, and that the sequence G_1, G_2, \dots defined for the space Σ satisfies conditions (1) and (2) of axiom C. It will be shown that it also satisfies conditions (3) and (4) of this axiom.

Suppose that P is a point of W , that $r = r_{x_1, x_2, \dots, x_n}$ is a region of H_n containing P , and that Q is a point of r distinct from P . If q is a region containing a point of W , then q must belong to H_1 . Since each element of C_{n+1} which contains P has a side of length less than or equal $1/4$ times the length of a side of Q_{x_1, x_2, \dots, x_n} , and each element of C_{n+2} which contains P has a side of length less than or equal $1/4^2$ times the length of side of Q_{x_1, x_2, \dots, x_n} , and so forth; it is obvious that there is a $d > n$ such that if q is a region of H_d which contains P , then \bar{q} does not intersect Q and is a subset of r . Suppose that x and y are two intersecting regions of G_{n+1} such that x contains P . x belongs to H_{n+1} and is therefore a subset of r . Every region of G_{n+1} which intersects x is a subset of r , so clearly, y is a subset of r .

Now suppose that P is a point of J_{x_1, x_2, \dots, x_n} of B_n and r is a region containing P , and Q is a point of r distinct from P . There exists a circle J in E with center at P such that every point of S in the interior of J belongs to r , but Q is not in the interior of J . There exists a positive integer d such that $1/4^d$ times the perimeter of any square of $(B_1 + B_2 + \dots)$ to which P belongs is less than the radius of J , and such that no region of H_d contains P . If R^1 is a region of G_{d+1} containing P , then \bar{R}^1 does not contain Q and is a subset of r . If $n > d + 2$

⁴ The collection G_1 of regions is a basis for the space Σ .

and x and y are two intersecting regions of G_n such that x contains P , then $x + y$ is a subset of r .

Now suppose that P is a point of $i_x(I(y))$, for y in C' , and that r is a region containing P and that Q is a point of r distinct from P .

Case 1. Suppose P is not the highest point of $i_x(I(y))$. There exists a segment t containing P , or a sect in case P is the lowest point of $i_x(I(y))$, such that t is a subset of r and does not contain Q nor the highest point of $i_x(I(y))$. There exists a positive number ε such that every point of $i_x(I(y))$ which is at a distance from P of less than ε lies in t . There exists a positive integer d such that

(1) no region of L_a intersects t and no region of H_a intersects $i_x(I(y))$, and

(2) $1/2^a$ times the length of $i_x(I(y))$ is less than ε . Therefore, if k is a region of G_{a+1} containing P , then \bar{k} is a subset of r and does not contain Q . Also, if x and y are two intersecting regions of G_{a+2} such that x contains P , then $x + y$ is a subset of r .

Case 2. Suppose P is the highest point of $i_x(I(y))$. Whether Q belongs to $i_x(I(y))$ or there is a positive integer p such that Q belongs to $i_p[i_x(I(y))]$ or r is in H_1 and Q does not belong to $i_x(I(y)) + i_1[i_x(I)] + i_2[i_x(I(y))] + \dots$, there is a positive integer d such that

(1) $R_d(P)$ does not contain Q and is a subset of r , and

(2) no region of H_a contains P . If k is a region of G_{a+1} containing P , then \bar{k} is a subset of r and does not contain Q . Also, if x and y are two intersecting regions of G_{a+3} such that x contains P , then $x + y$ is a subset of r .

Therefore G_1, G_2, \dots satisfies the third part of axiom C .

Suppose that M_1, M_2, \dots is a sequence of closed point sets such that

(1) for each n M_n contains M_{n+1} , and

(2) for each n there is a region g_n of G_n such that M_n is a subset of \bar{g}_n .

In case, for each n , g_n is in H_n , then by definition of W , there is a point common to M_1, M_2, \dots because some point of W can be easily shown to be a limit point or point of M_n for each n .

In case there is a positive integer j such that g_j belongs to K_j , then for $n > j$, g_n belongs to K_n . But M_j, M_{j+1}, \dots is a sequence of closed and compact point sets such that for $n \geq j$ M_n contains M_{n+1} . So there is a point common to M_j, M_{j+1}, \dots and thus common to M_1, M_2, \dots .

In case there is a positive integer j such that g_j belongs to N_j , then for $n > j$, g_n belongs to N_n . So, for the same reason as in the

previous case, there is a point common to M_1, M_2, \dots .

The only case not considered is the one where there is a positive integer j_1 such that, for $n \geq j_1$, g_n belongs to L_n . In this case g_{j_1} must be $R_{x_1}(P)$ for some point P and positive integer x_1 . There is a positive integer $j_2 > j_1$ such that $g_{j_2} = R_{x_2}(P)$, where $x_2 > x_1$. There is a positive integer $j_3 > j_2$ such that $g_{j_3} = R_{x_3}(P)$, for $x_3 > x_2$, and so forth. P is common to the sets $R_{x_1}(P), R_{x_2}(P), \dots$. But if P does not belong to each of the sets M_{j_1}, M_{j_2}, \dots then there is a positive integer d such that $\bar{R}_{x_d}(P)$ contains no point of M_{x_j} for any j . But $R_{x_d}(P)$ contains $M_{j_{d+1}}$. So P is common to the sets M_{j_1}, M_{j_2}, \dots and thus common to M_1, M_2, \dots .

Thus, Σ satisfies the fourth part of axiom C .

In order to show that Σ is connected, an indirect argument will be used. Suppose that S is the sum of two mutually separated sets H and K . Since $W + (B_1^* + B_2^* + \dots)$ is connected, let H' be the one of the sets H and K that contains this set and let K' be the other. There exists an element y of C' such that for some x $i_x[I(y)]$ is a subset of K' . But there exists a positive integer d_1 such that for $n \geq d_1$, $i_n[i_x(I(y))]$, belongs to K' . There exists a positive integer d_2 such that for $n \geq d_2$ $i_n[i_{d_1}(i_x(I(y)))]$ belongs to K' . So, obviously, there is a positive integer sequence, d_1, d_2, \dots such that if j is a positive integer and $n \geq d_j$, then $i_n(i_{d_{j-1}}(i_{d_{j-2}}(\dots i_{d_1}(i_x(I(y))) \dots)))$ belongs to K' . But from this fact it is easily seen that some point of W is a limit point of K' . So Σ is connected.

It has been shown that in any space satisfying axioms 0 and C (1) if M is a separable point set, M is completely separable, and (2) if M is separable, any subset of M is separable.

In order to show that any two points of S lie in a separable continuum, suppose first that P and Q are two points of S . Obviously, $(B_1^* + B_2^* + \dots)$ is separable and connected, and therefore $W + (B_1^* + B_2^* + \dots)$ is a separable continuum. In case P and Q both lie in $W + (B_1^* + B_2^* + \dots)$, this continuum has the desired properties. In case P does not belong to this set, P belongs to $i_x[I(y)]$ for some y in C' . Let M_P be the set to which point R belongs if and only if, either

(1) there is a finite positive integer sequence x_1, x_2, \dots, x_n such that R belongs to $i_{x_1}[i_{x_2}[\dots i_{x_n}[i_x(I(y)) \dots]]$, or

(2) there is a positive integer q such that R belongs to $i_q[i_x(I(y))]$, or

(3) R belongs to $i_x[I(y)]$. $M_P + (B_1^* + B_2^* + \dots) + W$ is a separable continuum. If Q does not belong to this set, let M_Q be a set related to Q like M_P was related to P . The continuum $M_P + M_Q + (B_1^* + B_2^* + \dots) + W$ is separable.

The statement that Σ is locally separable at the point P means that there is a region R containing P such that R is separable. Alexandroff [1] has shown that if β is a connected, locally completely separable,

space satisfying axioms 0 and C , then β is completely separable. It is interesting to note that Σ is locally separable, and therefore locally completely separable, at each point except those of a separable set, and yet, Σ is not separable.

Σ is obviously locally separable at all points not belonging to W . Since every region that contains a point of W contains uncountably many mutually exclusive domains, Σ is not locally separable at any point of W . Furthermore $(B_1^* + B_2^* + \dots)$ is separable, and so $(\overline{B_1^* + B_2^* + \dots})$ is separable, and thus, since W is a subset of the latter, W is separable.

Σ is said to be *semi-locally-connected* [5] at point P if and only if it is true that if R is a region containing P , R contains a region R' containing P such that $S - R$ does not intersect infinitely many components of $S - R'$. Σ is said to be semi-locally-connected if and only if Σ is semi-locally-connected at each point.

The space Σ is obviously semi-locally-connected because S minus any region has only a finite number of components.

PART 2

Suppose that Σ is a space satisfying the conditions specified on the first page of this paper.

For each positive integer j let G_j denote the collection of all open sets which have diameter less than j^{-1} .

Let P denote some definite point, and suppose n is a positive integer such that no countable subcollection of G_n covers S . Let R_n be some region of G_n which contains P , let $H_1 = \{R_n\}$, and let K_1 be the boundary of R_n .

For each point Q of S let $A(Q)$ be the least integer $j > n$ such that some region $R(Q)$ of G_n contains every region of G_j that intersects a region of G_j that contains Q .

It has been shown that in a space satisfying these axioms if L is a separable point set and G is a collection of open sets covering L , then some countable subcollection of G covers L . Therefore, there is a countable point set T_1 dense in K_1 such that the collection H_2 of all $R(Q)$'s, for Q 's in T_1 , covers K_1 . Let K_2 be the sum of the boundaries of all the sets in $H_1 + H_2$. There is a countable point set T_2 dense in K_2 such that the collection H_3 of all $R(Q)$'s for Q 's in T_2 , covers K_2 . Let K_3 be the sum of the boundaries of the sets in $H_1 + H_2 + H_3$, and so forth.

There is a point B not in the closure of $H = (H_1 + H_2 + \dots)^*$. Let M be a compact continuum containing P and B .

Case 1. Suppose some point A of $M - M \cdot H$ is a limit point of $K = K_1 + K_2 + \dots$.

Let R'_1 be a region of G_n containing A , let Q_1 be a point of $T = T_1 + T_2 + \dots$ in R'_1 , and let x_1 be the largest integer i such that R'_1 belongs to G_i . Let R'_2 be a region of C_{x_1+1} containing A such that \bar{R}'_2 lies in $R'_1 - Q_1$. Let Q_2 be a point of T in R'_2 and let x_2 be the largest integer i such that R'_2 is in G_i . Obtain R'_3, Q_3 , and x_3 similarly, and so forth. $n \leq x_1 < x_2 < x_3 < \dots$. For each i , $\Delta(Q_i) > x_i$. Otherwise, for some i , $R(Q_i)$ would contain R'_i , and thus A . However, there is a positive integer $t > n$ such that if x, y , and z are regions of G_t such that $x \cdot y$ and $y \cdot z$ exist and x contains A , then R_n contains $x + y + z$. For some $s > t$, $\Delta(Q_s) > t$. But R_n contains every region of G_t that intersects a region of G_t that contains G_s . So $\Delta(Q_s) \leq t$, which is a contradiction.

Case 2. Suppose no point of $M - M \cdot H$ is a limit point of K . For each point Q of $M - M \cdot H$ let g_Q be a region containing Q such that g_Q contains no point of $K + P$. Some finite subcollection C of the g_Q 's covers this set of limit points. Let $D = H - H \cdot \bar{C}^*$. Let C_1 be the component of $M - M \cdot \bar{D}$ which contains B . Some point z of $M \cdot \bar{D}$ is a limit point of C_1 . But z lies in a region r of H , and therefore C_1 would intersect the boundary of r , and thus contain a limit point of K . This yields a contradiction.

Since, for each n , some countable subcollection of G_n covers S , Σ is completely separable.

REFERENCES

1. Paul Alexandroff, *Über die Metrizierung der im Kleinen Kompakten topologischen Räume*, Mathematische Annalen, **92** (1924) 294-301.
2. F. B. Jones, *On a property related to separability in metric spaces*, Journal of the Elisha Mitchell Scientific Society, **70** (1954), 30-33.
3. ———, *A theorem concerning locally peripherally separable spaces*, Bull. Amer. Math. Soc. **41** (1935), 437-439.
4. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloq. Publ., vol 13 1932.
5. G. T. Whyburn, *Semi-locally-connected sets*, Amer. J. Math. **61** (1939), 733-749.

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