Pacific Journal of Mathematics

A SPHERE CHARACTERIZATION RELATED TO BLASCHKE'S CONJECTURE

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Vol. 10, No. 3 November 1960

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Let S be a C^3-surface supplied with a complete positive-definite Riemannian metric. If $R = R(s)$ is a geodesic ray with initial element (point and direction) $e = (P, \theta)$ and arc-length s, and $h = (P, \theta \pm \pi/2)$ is an orthogonal element at P, we call $F(e)$ the *focal distance* for e if $R(F(e))$ is the first focal point along R to the element h. (This definition will be given an analytic form in § 1.)

THEOREM 1. *If $F(e)$ is finite and independent of the initial element e in the unit tangent bundle of S, then S has constant positive curvature.*

The hypothesis implies that $2F(e)$ is the distance to the first point conjugate to P along R; hence this conjugate distance is constant. A long-standing conjecture of Blaschke's is: if the conjugate distance on S is finite and independent of the initial element, the surface has constant positive curvature. If one accepts as true a sentence (Blaschke [2], top of page 228) in his critique of Reidemeister's erroneous proof ([1], Anhang) of the conjecture, it is sufficient to prove the theorem as we state it for focal points. However, we have not been able to prove the assertion contained in that sentence.

Busemann ([3], Theorem 21.10) has proved an analogue of our theorem for G-spaces of dimension greater than two, so the generalization to Riemannian manifolds of higher dimension is contained in his result. For two-dimensional G-spaces even of class C^∞, Blaschke's conjecture is false, as examples of Hamel show [4].

In § 1 we give the analytic setting of the problem and summarize the pertinent known facts. § 2 contains a result in ordinary differential equations which implies the main theorem.

1. Express the line element of S in geodesic polar coordinates with center P:

$$ds^2 = dr^2 + G^2 d\theta^2 \,. \tag{L}$$

Then it is known that $G = G(r, \theta)$ satisfies the Jacobi differential equation

$$\begin{aligned} &G_{rr}(r, \theta) + K(r, \theta)G(r, \theta) = 0 \,, \\ &G(0, \theta) = 0, \qquad G_r(0, \theta) = 1 \,, \end{aligned} \tag{J}$$

Received July 17, 1959. Presented to the AMS, January 22, 1959, under the title "On a problem of Blaschke's". Research supported by NSF contract G-5256.

where $K(r, \theta)$ is the Gaussian curvature of S at a distance r from the point P along the geodesic ray with initial element (P, θ). Along any such ray the line element is positive definite for $0 < r < C(\theta)$, where $C(\theta)$ is the first positive value of r for which $G(r, \theta) = 0$. The point $(C(\theta), \theta)$ is the first point conjugate to P along the ray $R(s) = (s, \theta)$. The mapping from the (r, θ) plane to S is locally one-to-one along any geodesic ray (s, θ) of length less than $C(P, \theta)$. The first point focal to P along (s, θ) is defined to be the point $(F(\theta), \theta)$, where $F(\theta)$ is the first zero of $G_r(r, \theta)$ for increasing r, Geometrically, it is the first point along a ray at which the geodesic circle of that radius ceases to be (locally) strictly convex. Thus we may drop the transversality parts of the usual calculus of variations definition, provided we understand that "focal" in our sense has a restricted meaning.

For the remainder of this section we shall assume that $C(P, \theta)$ is finite and independent of P and θ on S. Without loss of generality we may assume that this constant is π. Much of Blaschke's discussion ([2], § 102) holds, and the following paragraph will be a summary without proof of the verifiable statements which he makes that we will need.

All geodesics of S are closed. Any two geodesics which pass through the point P intersect at a point P', the common point conjugate to P along all rays through P. S is covered by a homeomorph of the two-sphere with the same conjugate point distance. From now on we assume S simply-connected and call the distinct and unique point P' the antipode of P. The distance from P to P' along any ray is π; the length of every closed geodesic is 2π. The mapping of points into their antipodes is an isometric involution of S onto itself. (Most of these facts are also proved for G-spaces in Busemann [**3**].)

It follows that the line element (L) with any center P is definite for $0 < r < \pi$. Since the Gaussian curvature is a scalar function on the surface, we may consider (J) along a fixed closed geodesic as an ordinary differential equation with periodic coefficients. Choosing a different origin on the same geodesic and solving (J) yields a solution in the original coordinates with initial values prescribed at some point other than $r = 0$. Any solution $u(r)$ of (J) which vanishes at r_0 also vanishes at $r_0 = n\pi$. Since the antipodal map is an isometry, $K(r, \theta) = K(r \pm \pi, \theta)$ and the function G in (L) satisfies $G(r - \pi, \theta + \pi) = G(r, \theta)$ for $0 \leq r \leq \pi$. Because of the uniqueness of the solutions under the imposed boundary conditions, if we consider a solution $u(r)$ of (J) with the same initial conditions as defined for all r, $u(r) = -u(r - \pi)$. Consequently *every* solution of (J) is odd-harmonic and periodic of period 2π.

If we adjoin the condition that $F(e)$ is constant (and equals $\pi/2$), then every solution of (J) which vanishes at r_0 has zero first derivative at $r_0 + \pi/2$.

2. The analytic situation which we shall exploit may now be summarized as follows:

(H) Every solution of the differential equation

$$y''(r) + K(r)y(r) = 0 \tag{J'}$$

is odd-harmonic (i.e., $y(r - \pi) = -y(r)$), periodic of period 2π, and if $y(r_0) = 0$, then $y'(r_0 + \pi/2) = 0$ (focal point condition). Here $K(r)$ is continuous and of period π.

In this section $w(r, a)$ will always designate a solution of (J') satisfying $w(a, a) = 0$, $w_r(a, a) = 1$. We set $u(r) = w(r, 0)$ and let $v(r)$ be that solution of (J') with the initial conditions $v(0) = 1$, $v'(0) = 0$.

LEMMA. *Under the assumptions (H), $w(a + \pi/2, a)$ is independent of a.*

Proof. In terms of the independent basis of solutions we may write

$$w(r, a) = v(a)u(r) - u(a)v(r) . \tag{1}$$

Then the focal point condition becomes

$$v(a)u'(a + \pi/2) = u(a)v'(a + \pi/2) \tag{2}$$

for every a. Putting $b = a + \pi/2$ and using the odd-harmonicity,

$$v(b + \pi/2)u'(b) = u(b + \pi/2)v'(b) \tag{3}$$

for every b. Now differentiate (1) with respect to a:

$$w_a(r, a) = v'(a)u(r) - u'(a)v(r) .$$

Using (3), we see that $w_a(a + \pi/2, a) = 0$. Combining this with the hypothesis on the derivative with respect to the first variable, we conclude that the total derivative with respect to a of $w(a + \pi/2, a)$ is zero for all a. This proves the lemma.

THEOREM 2. *Under the hypotheses (H), $K(r)$ is identically* 1.

Proof. At any point a where $u'(a + \pi/2) \neq 0$ we may solve (2) for $v(a)$ and substitute in (1) to find the constant value of $w(a + \pi/2, a)$:

$$\begin{aligned} u(\pi/2) &= w(\pi/2, 0) = w(a + \pi/2, a) \\ &= [u'(a + \pi/2)]^{-1}[u(a)v'(a + \pi/2)u(a + \pi/2) \\ &\quad - u(a)v(a + \pi/2)u'(a + \pi/2)] \\ &= -u(a)/u'(a + \pi/2) , \end{aligned} \tag{4}$$

using the fact that the Wronskian is constant. In particular, at every such point a, $u(a) \neq 0$ also. Hence we may divide (2) further to find that, for all intervals in which $u'(a + \pi/2) \neq 0$,

$$v(a)/u(a) = v'(a + \pi/2)/u'(a + \pi/2) .$$

Differentiating and substituting from (J'):

$$\begin{aligned}
&[u(a)v'(a) - v(a)u'(a)]/[u(a)]^2 \\
&= \frac{u'(a + \pi/2)v''(a + \pi/2) - v'(a + \pi/2)u''(a + \pi/2)}{[u'(a + \pi/2)]^2} \\
&= \frac{K(a + \pi/2)[v'(a + \pi/2)u(a + \pi/2) - u'(a + \pi/2)v(a + \pi/2)]}{[u'(a + \pi/2)]^2} .
\end{aligned}$$

Another recognition of the Wronskian allows us to conclude that

$$K(a + \pi/2) = \left[\frac{u'(a + \pi/2)}{u(a)}\right]^2 = [u(\pi/2)]^{-2} ,$$

a constant independent of a. The fact that the zeroes of $u'(r)$ are isolated together with the continuity of K completes the argument that K is constant everywhere. (Or one may notice that the choice of origin was arbitrary.) Since the period of the solutions is 2π, the value of K is determined. This completes the proof of Theorem 2, and clearly implies the truth of Theorem 1.

3. Warren Loud has indicated to me how to obtain non-constant periodic functions $K(r)$ such that all solutions of (J') are periodic and odd-harmonic (cf. his monograph [6], § 3). Hence the method given here cannot be directly extended to prove Blaschke's conjecture. (Perhaps this remark should be added to Blaschke's "Warnungstafel".)

If one considers the equation (J) as a one-parameter family of ordinary differential equations and uses the hypothesis that the spectral value 0 does not change under perturbation, one obtains (in the analytic case) an infinite sequence of conditions on the derivatives of $K(r, \theta)$, of which that given by Blaschke ([2], page 226) is only the first.

By using classical Sturm-Liouville techniques, one can also prove

THEOREM 3. *Let $K(r, \theta)$ be analytic for $|\theta| < \theta_0$, and such that $K(r, \theta) \to \infty$ uniformly in some interval of r as $\theta \to \theta_0$. Then it cannot happen that every solution of (J) is periodic with period independent of θ.*

We omit the proof. (It is essentially given, e.g., in Haupt [5].) If it were not for the growth hypothesis, which doesn't correspond to the geometrical situation, this would imply the truth of Blaschke's conjecture, since any two points on S lie on a geodesic circle.

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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