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**ON THE DIRICHLET PROBLEM FOR CERTAIN HIGHER  
ORDER PARABOLIC EQUATIONS**

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The Dirichlet problem for the particular equation

$$D_x^n u + D_t u = 0$$

( $D_t \equiv \partial/\partial t$ ) on the space-time cylinder  $(0 < x < 1) \otimes (0 < t \leq T)$  is treated in this paper. However the procedure is directly applicable to the equation  $D_x^{2n} u + (-1)^n D_t u = 0$  without technical difficulty and, hence, to any equation simply reducible to this type. It can be applied as well to problems other than the Dirichlet problem. Recently P. G. Kirmser [2] made use of it in solving other interesting problems posed for the equation  $D_x^n u + D_t u = 0$ . There is also an 'uniqueness theorem' contained in his paper.

Using the methods of potential theory, as in Gevrey [1] and Tykhonov [6] for the heat equation and Zeragiya [7] for general second order equations, the problem is reduced to solving a system of integral equations. The integral equations and the integration of them are of interest in themselves.

The procedure affords information on the behavior of the solution along  $x = 0$  and  $x = 1$ . In addition, the solution obtained allows an analysis of its behavior as  $(x, t)$  approaches  $(0, 0)$  or  $(0, 1)$  as in the case of the heat equation.

**1. Statement of the problem.** The problem we pose is to find a function  $u(x, t)$  such that

$$(1.1) \quad \begin{aligned} & \text{(i) } D_x u + D_t u = 0, \quad 0 < x < 1, \quad 0 < t \leq T; \\ & \text{(ii) } u(x, 0) = 0, \quad 0 < x < 1; \\ & \text{(iii) } u(0, t) = a(t), \quad u(1, t) = b(t), \quad 0 < t \leq T; \\ & \text{(iv) } D_x u(0, t) = c(t), \quad D_x u(1, t) = d(t), \quad 0 < t \leq T \end{aligned}$$

where,  $a, b, c,$  and  $d$  are arbitrary functions from classes that we shall presently define. Certain integral operators arise which make it natural to make the following definitions:

**DEFINITION 1.** Let  $S_1$  denote the class of functions defined on  $(0, T]$  such that to each function,  $f(t)$ , there corresponds a pair of positive numbers  $(\varepsilon, \lambda)$  so that

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$$(1.2) \quad |f|_1 \equiv \sup_{t, \tau} \left\{ \sigma^\lambda \frac{|f(t) - f(\tau)|}{|t - \tau|^\varepsilon} \right\} < +\infty$$

where  $\sigma = \min(t, \tau)$ ,  $\varepsilon + 1/4 \leq \lambda < 1$ .

**DEFINITION 2.** Let  $S_2$  denote the class of all functions,  $g(t)$ , defined on  $(0, T]$  and satisfying the conditions:

(i)  $g$  uniformly  $(\varepsilon + 1/4)$  - Hölder continuous on any closed sub-interval of  $(0, T]$ , i.e., to each  $t_0 \in (0, T]$  there corresponds a constant  $c(t_0)$ , depending only on  $t_0$ , such that

$$|g(t_1) - g(t_2)| \leq c(t_0) |t_1 - t_2|^{\varepsilon+1/4}$$

for all  $t_1, t_2 \in [t_0, T]$ ;

(ii)

$$(1.3) \quad |g|_2 \equiv \sup_{t, \tau} \left\{ \sigma^\lambda \frac{\left| 4t^{-1/4}g(t) + \int_0^t [g(t) - g(s)](t-s)^{-5/4} ds \right|}{|t - \tau|^\varepsilon} \right. \\ \left. \left| \frac{-4\tau^{-1/4}g(\tau) - \int_0^\tau [g(\tau) - g(s)](\tau-s)^{-5/4} ds}{|t - \tau|^\varepsilon} \right| \right\} \\ + \sup_t t^{\lambda-1/4} |g(t)| < +\infty$$

where  $\sigma$ ,  $\lambda$  and  $\varepsilon$  are as in Definition 1.

We shall establish existence of solutions to (1.1) for  $a, b \in S_2$  and  $c, d \in S_1$ .

**2. Derivation of the integral equations.** By the standard Fourier transform techniques we find the fundamental solution:

$$(2.1) \quad k(x - y, t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi(x-y)} e^{-\xi^4(t-\tau)} d\xi, \quad 0 \leq \tau < t$$

which satisfies

$$D_x^4 k + D_t k = 0$$

and

$$D_y^4 k - D_\tau k = 0.$$

In the sequel we will frequently use the following basic estimates of the fundamental solution and of derivatives of same due to O. Ladyzhenskaya [3] (see also P. C. Rosenbloom [5]):

$$(2.2)^1 \quad |D_x^\nu k(x, t)| \leq c_1(\nu) \cdot t^{-(1+\nu)/4} \cdot \exp[-c_2(x^4/t)^{1/3}]$$

<sup>1</sup> See appendix.

where  $D_x^\nu \equiv (\partial/\partial x)^\nu$ ,  $c_1$  depends on  $\nu$ , and  $c_2$  is an absolute constant.

LEMMA 1.

$$(2.3) \quad p(x) - \int_0^1 k(x - y, t) dy = \int_0^t [D_y^3 k(x - 1, t - \sigma) - D_y^3 k(x, t - \sigma)] d\sigma$$

where

$$p(x) = \begin{cases} 1, & 0 < x < 1 \\ \frac{1}{2}, & x = 0, x = 1 \\ 0, & x < 0, x > 1. \end{cases}$$

*Proof.* Since  $D_\sigma k = D_y k$ ,

$$\begin{aligned} \int_0^1 D_\sigma k(x - y, t - \sigma) dy &= \int_0^1 D_y^4 k(x - y, t - \sigma) dy = D_y^3 k(x - y, t - \sigma) \Big|_{y=0}^{y=1} \\ &= D_y^3 k(x - 1, t - \sigma) - D_y^3 k(x, t - \sigma). \end{aligned}$$

Integrating with respect to  $\sigma$  from 0 to  $t - \varepsilon$  gives

$$\begin{aligned} \int_0^{t-\varepsilon} \left( \int_0^1 D_\sigma k(x - y, t - \sigma) dy \right) d\sigma &= \int_0^{t-\varepsilon} D_\sigma \left( \int_0^1 k(x - y, t - \sigma) dy \right) d\sigma \\ &= \int_0^1 k(x - y, \varepsilon) dy - \int_0^1 k(x - y, t) dy \\ &= \int_0^{t-\varepsilon} [D_y^3 k(x - 1, t - \sigma) - D_y^3 k(x, t - \sigma)] d\sigma. \end{aligned}$$

That is,

$$\begin{aligned} \int_0^1 k(x - y, \varepsilon) dy - \int_0^1 k(x - y, t) dy \\ = \int_0^{t-\varepsilon} [D_y^3 k(x - 1, t - \sigma) - D_y^3 k(x, t - \sigma)] d\sigma. \end{aligned}$$

Since  $k(x - y, \varepsilon) = \varepsilon^{-1/4} k((x - y)/\varepsilon^{1/4}, 1)$  and  $k(-z, 1) = k(z, 1)$ ,

$$\int_0^1 k(x - y, \varepsilon) dy = \int_0^1 k((x - y)/\varepsilon^{1/4}, 1) \frac{dy}{\varepsilon^{1/4}} = \int_{-x/\varepsilon^{1/4}}^{(1-x)/\varepsilon^{1/4}} k(z, 1) dz.$$

Hence

$$\lim_{\varepsilon \downarrow 0} \int_0^1 k(x - y, \varepsilon) dy = \begin{cases} 0, & x < 0, x > 1 \\ \int_{-\infty}^0 k(z, 1) dz, & x = 1 \\ \int_0^\infty k(z, 1) dz, & x = 0 \\ \int_{-\infty}^\infty k(z, 1) dz, & 0 < x < 1. \end{cases}$$

Now

$$\begin{aligned} \int_{-\infty}^{\infty} k(z, 1) dz &= \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi \cdot z} \cdot e^{-\xi^4} d\xi \right) dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iz \cdot w} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi \cdot z} \cdot e^{-\xi^4} d\xi \right) dz \Big|_{w=0} = e^{-w^4} \Big|_{w=0} = 1 \end{aligned}$$

and

$$\int_0^{\infty} k(z, 1) dz = \int_{-\infty}^0 k(z, 1) dz = \frac{1}{2} \int_{-\infty}^{\infty} k(z, 1) dz = \frac{1}{2} .$$

Thus,

$$\begin{aligned} &\int_0^t [D_y^3 k(x - 1, t - \sigma) - D_y^3 k(x, t - \sigma)] d\sigma \\ &= \lim_{\varepsilon \downarrow 0} \int_0^{t-\varepsilon} [D_y^3 k(x - 1, t - \sigma) - D_y^3 k(x, t - \sigma)] d\sigma \\ &= p(x) - \int_0^1 k(x - y, t) dy . \end{aligned}$$

Q.E.D.

In particular, since  $D_y^3 k(0, t - \sigma) \equiv 0$

$$\frac{1}{2} - \int_0^1 k(1 - y, t) dy = - \int_0^t D_y^3 k(1, t - \sigma) d\sigma$$

and

$$\frac{1}{2} - \int_0^1 k(-y, t) dy = \int_0^t D_y^3 k(-1, t - \sigma) d\sigma .$$

However, since  $k(-y, t) = k(y, t)$ ,

$$D_y^3 k(-1, t - \sigma) = -D_y^3 k(1, t - \sigma) ,$$

and

$$\int_0^1 k(1 - y, t) dy = \int_0^1 k(y, t) dy ,$$

$$(2.4) \quad \frac{1}{2} - \int_0^1 k(y, t) dy = - \int_0^t D_y^3 k(1, t - \sigma) d\sigma .$$

In deriving the integral equations we will need the following limit relations.

LEMMA 2.

(a)  $f \in S_i , \quad i = 1, 2$

$$(2.5) \quad \lim_{\substack{x \downarrow 0 \\ 0 < x < 1}} \int_0^t f(\sigma) D_y^3 k(x, t - \sigma) d\sigma = -\frac{1}{2} f(t)$$

$$(2.6) \quad \lim_{\substack{x \uparrow 1 \\ 0 < x < 1}} \int_0^t f(\sigma) D_y^3 k(x - 1, t - \sigma) d\sigma = \frac{1}{2} f(t)$$

$$(b) \quad g \in S_2$$

$$(2.7) \quad \begin{aligned} \lim_{\substack{x \downarrow 0 \\ 0 < x < 1}} \int_0^t g(\sigma) D_y^4 k(x, t - \sigma) d\sigma \\ = -g(t)k(0, t) - \int_0^t [g(t) - g(\sigma)] D_\sigma k(0, t - \sigma) d\sigma \end{aligned}$$

$$(2.8) \quad \begin{aligned} \lim_{\substack{x \uparrow 1 \\ 0 < x < 1}} \int_0^t g(\sigma) D_y^4 k(x - 1, t - \sigma) d\sigma \\ = -g(t)k(0, t) - \int_0^t [g(t) - g(\sigma)] D_\sigma k(0, t - \sigma) d\sigma . \end{aligned}$$

*Proof. Part (a).* We shall prove (2.5) for  $f \in S_1$ . The proofs for the remaining cases are essentially the same.

We write

$$\begin{aligned} \int_0^t f(\sigma) D_y^3 k(x, t - \sigma) d\sigma \\ = f(t) \int_0^t D_y^3 k(x, t - \sigma) d\sigma - \int_0^t [f(t) - f(\sigma)] D_y^3 k(x, t - \sigma) d\sigma . \end{aligned}$$

From (2.2) and the hypothesis on  $f$

$$\begin{aligned} |[f(t) - f(\sigma)] \cdot | D_y^3 k(x, t - \sigma) | &\leq |f|_1 \cdot \sigma^{-\lambda} (t - \sigma)^2 \cdot c_1 (t - \sigma)^{-1} \cdot e^{-c_2 (x^4 / (t - \sigma))^{1/3}} \\ &\leq (\text{constant}) \cdot \sigma^{-\lambda} (t - \sigma)^{2-1} . \end{aligned}$$

Hence, by the dominated convergence theorem:

$$\begin{aligned} \lim_{\substack{x \downarrow 0 \\ 0 < x < 1}} \int_0^t [f(t) - f(\sigma)] \cdot D_y^3 k(x, t - \sigma) d\sigma \\ = \int_0^t \lim_{x \downarrow 0} [f(t) - f(\sigma)] \cdot D_y^3 k(x, t - \sigma) d\sigma = 0 . \end{aligned}$$

Thus

$$\lim_{x \downarrow 0} \int_0^t f(\sigma) \cdot D_y^3 k(x, t - \sigma) d\sigma = f(t) \lim_{x \downarrow 0} \int_0^t D_y^3 k(x, t - \sigma) d\sigma$$

which by (2.3) equals

$$f(t) \lim_{x \downarrow 0} \left\{ \int_0^t D_y^3 k(x - 1, t - \sigma) d\sigma + \int_0^1 k(x - y, t) dy - 1 \right\}$$

$$= f(t) \left\{ \int_0^t D_y^3 k(-1, t - \sigma) d\sigma + \int_0^1 k(-y, t) dy - 1 \right\}$$

and by (2.4) this equals

$$f(t) \left\{ \frac{1}{2} - 1 \right\} = -\frac{1}{2} f(t).$$

Part (b). We shall give the proof of (2.7). As above write

$$\begin{aligned} & \int_0^t g(\sigma) \cdot D_y^4 k(x, t - \sigma) d\sigma \\ &= g(t) \int_0^t D_y^4 k(x, t - \sigma) d\sigma - \int_0^t [g(t) - g(\sigma)] D_y^4 k(x, t - \sigma) d\sigma \\ &= g(t) \int_0^t D_\sigma k(x, t - \sigma) d\sigma - \int_0^t [g(t) - g(\sigma)] D_\sigma k(x, t - \sigma) d\sigma \\ &= g(t) \cdot \left\{ k(x, t - \sigma) \Big|_{\sigma=0}^{\sigma \uparrow t} \right\} - \int_0^t [g(t) - g(\sigma)] \cdot D_\sigma k(x, t - \sigma) d\sigma \\ &= -g(t) \cdot k(x, t) - \int_0^t [g(t) - g(\sigma)] \cdot D_\sigma k(x, t - \sigma) d\sigma \\ &= -g(t) \cdot k(x, t) - \int_0^{t/2} [g(t) - g(\sigma)] \cdot D_\sigma k(x, t - \sigma) d\sigma \\ &\quad - \int_{t/2}^t [g(t) - g(\sigma)] \cdot D_\sigma k(x, t - \sigma) d\sigma. \end{aligned}$$

For given  $t \in (0, T]$ , the first two terms are continuous in  $x$ , for all  $x$ , and the interchange of limit and integration in the latter is justified as above, using the Hölder continuity of  $g$ . From these remarks, the proof follows.

Q.E.D.

We seek a solution to our problem in the following form:

$$(2.9) \quad \begin{aligned} u(x, t) &= \int_0^t \alpha(\sigma) D_y^3 k(x, t - \sigma) d\sigma + \int_0^t \beta(\sigma) \cdot D_y^3 k(x - 1, t - \sigma) d\sigma \\ &+ \int_0^t \gamma(\sigma) \cdot D_y^2 k(x, t - \sigma) d\sigma + \int_0^t \delta(\sigma) \cdot D_y^2 k(x - 1, t - \sigma) d\sigma, \quad 0 < x < 1 \end{aligned}$$

where  $\alpha, \beta \in S_2$  and  $\gamma, \delta \in S_1$ . The fact that  $u(x, t)$  satisfies the equation for  $0 < x < 1$  follows from (2.2), which justifies interchanging the order of differentiation and integration, and (2.1). From Lemma 2, we shall obtain a system of integral equations for the unknown functions  $\alpha, \beta, \gamma$ , and  $\delta$ .

From (2.5) and (2.6) we obtain, upon taking the limit of (2.9) first as  $x \downarrow 0$ , and then as  $x \uparrow 1$ , the equations

$$(2.10) \quad \begin{aligned} a(t) &= -\frac{1}{2} \alpha(t) + \int_0^t \beta(\sigma) \cdot D_y^3 k(-1, t - \sigma) d\sigma \\ &+ \int_0^t \gamma(\sigma) \cdot D_y^2 k(0, t - \sigma) d\sigma + \int_0^t \delta(\sigma) \cdot D_y^2 k(-1, t - \sigma) d\sigma \end{aligned}$$

and

$$(2.11) \quad \begin{aligned} b(t) &= \int_0^t \alpha(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma + \frac{1}{2} \beta(t) \\ &+ \int_0^t \gamma(\sigma) D_y^3 k(1, t - \sigma) d\sigma + \int_0^t \delta(\sigma) \cdot D_y^3 k(0, t - \sigma) d\sigma . \end{aligned}$$

The limits obtained from the various terms other than those where Lemma 2 is applied are by continuity which follows from (2.2).

Since

$$D_y^3 k(-1, t - \sigma) = -D_y^3 k(1, t - \sigma)$$

and

$$D_y^2 k(-1, t - \sigma) = D_y^2 k(1, t - \sigma) ,$$

we can write (2.10) as

$$(2.10)' \quad \begin{aligned} a(t) &= -\frac{1}{2} \alpha(t) - \int_0^t \beta(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma \\ &+ \int_0^t \gamma(\sigma) \cdot D_y^3 k(0, t - \sigma) d\sigma + \int_0^t \delta(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma . \end{aligned}$$

From (2.1)

$$\begin{aligned} D_x u(x, t) &= -\int_0^t \alpha(\sigma) \cdot D_y^4 k(x, t - \sigma) d\sigma - \int_0^t \beta(\sigma) \cdot D_y^4 k(x - 1, t - \sigma) d\sigma \\ &- \int_0^t \gamma(\sigma) \cdot D_y^3 k(x, t - \sigma) d\sigma - \int_0^t \delta(\sigma) \cdot D_y^3 k(x - 1, t - \sigma) d\sigma . \end{aligned}$$

Using (2.7) and (2.8) we obtain upon taking the limit of this relation as  $x \downarrow 0$  and then as  $x \uparrow 1$ , the equations

$$(2.12) \quad \begin{aligned} c(t) &= \alpha(t) \cdot k(0, t) + \int_0^t [\alpha(t) - \alpha(\sigma)] D_\sigma k(0, t - \sigma) d\sigma \\ &- \int_0^t \beta(\sigma) \cdot D_\sigma k(1, t - \sigma) d\sigma + \frac{1}{2} \gamma(t) + \int_0^t \delta(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma \end{aligned}$$

and

$$(2.13) \quad \begin{aligned} d(t) &= -\int_0^t \alpha(\sigma) \cdot D_\sigma k(1, t - \sigma) d\sigma + \beta(t) \cdot k(0, t) \\ &+ \int_0^t [\beta(t) - \beta(\sigma)] D_\sigma k(0, t - \sigma) d\sigma \\ &- \int_0^t \gamma(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma - \frac{1}{2} \delta(t) . \end{aligned}$$

Adding and subtracting (2.10)' and (2.11) gives



$$\begin{aligned}
a(t) \pm b(t) &= -\frac{1}{2} [\alpha(t) \mp \beta(t)] \pm \int_0^t [\alpha(\sigma) \mp \beta(\sigma)] \cdot D_y^3 k(1, t - \sigma) d\sigma \\
&+ \int_0^t [\gamma(\sigma) \pm \delta(\sigma)] D_y^3 k(0, t - \sigma) d\sigma \pm \int_0^t [\gamma(\sigma) \pm \delta(\sigma)] D_y^3 k(1, t - \sigma) d\sigma .
\end{aligned}$$

Similarly, adding and subtracting (2.12) and (2.13) gives

$$\begin{aligned}
c(t) \pm d(t) &= [\alpha(t) \pm \beta(t)] k(0, t) \\
&+ \int_0^t \{[\alpha(t) \pm \beta(t)] - [\alpha(\sigma) \pm \beta(\sigma)]\} D_\sigma k(0, t - \sigma) d\sigma \\
&\mp \int_0^t [\alpha(\sigma) \pm \beta(\sigma)] \cdot D_\sigma k(1, t - \sigma) d\sigma \mp \int_0^t [\gamma(\sigma) \mp \delta(\sigma)] D_y^3 k(1, t - \sigma) d\sigma \\
&+ \frac{1}{2} [\gamma(t) \mp \delta(t)] .
\end{aligned}$$

Setting

$$\begin{aligned}
\phi(t) &= \gamma(t) + \delta(t) & A(t) &= c(t) - d(t) \\
\psi(t) &= \alpha(t) - \beta(t) & B(t) &= a(t) + b(t) \\
f(t) &= \gamma(t) - \delta(t) & C(t) &= c(t) + d(t) \\
g(t) &= \alpha(t) + \beta(t) & D(t) &= a(t) - b(t)
\end{aligned}$$

we obtain the following pairs of equations

$$\begin{aligned}
(2.14) \quad &\frac{1}{2} \phi(t) + \int_0^t \phi(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma + \psi(t) k(0, t) \\
&+ \int_0^t [\psi(t) - \psi(\sigma)] D_\sigma k(0, t - \sigma) d\sigma \\
&+ \int_0^t \psi(\sigma) \cdot D_\sigma k(1, t - \sigma) d\sigma = A(t) \\
&\int_0^t \phi(\sigma) \cdot D_y^3 k(0, t - \sigma) d\sigma + \int_0^t \phi(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma - \frac{1}{2} \psi(t) \\
&+ \int_0^t \psi(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma = B(t) ,
\end{aligned}$$

and

$$\begin{aligned}
(2.15) \quad &\frac{1}{2} f(t) - \int_0^t f(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma + g(t) \cdot k(0, t) \\
&+ \int_0^t [g(t) - g(\sigma)] \cdot D_\sigma k(0, t - \sigma) d\sigma \\
&- \int_0^t g(\sigma) \cdot D_\sigma k(1, t - \sigma) d\sigma = C(t) \\
&\int_0^t f(\sigma) \cdot D_y^3 k(0, t - \sigma) d\sigma - \int_0^t f(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma - \frac{1}{2} g(t)
\end{aligned}$$

$$- \int_0^t g(\sigma) \cdot D_3^3 k(1, t - \sigma) d\sigma = D(t).$$

**3. Solution of the integral equations.** To facilitate the solution of the integral equations, we define for suitable functions  $f$  and  $g$

$$(3.1) \quad (T_1 f)(t) = \frac{1}{\Gamma(1/4)} \int_0^t f(\sigma) (t - \sigma)^{-3/4} d\sigma$$

and

$$(3.2) \quad (T_2 g)(t) = -\frac{1}{\Gamma(-1/4)} \left[ 4t^{-1/4} \cdot g(t) + \int_0^t [g(t) - g(\sigma)] (t - \sigma)^{-5/4} d\sigma \right].$$

$T_1$  is the operator which is commonly called  $I^{1/4}$  (see M. Riesz [4]). However, it is not immediately clear that  $T_2$  is  $I^{-1/4}$  because of the singularities allowed at the origin in the classes of functions under consideration. The following example will illustrate the effect of the singularity at the origin. Let

$$h(t) = t^{-1+\delta}, \quad 0 < \delta < \frac{1}{4} \quad (h \notin S_2).$$

Then

$$(T_2 h)(t) = \frac{\Gamma(\delta)}{\Gamma(\delta - 1/4)} \cdot t^{-5/4+\delta},$$

a function to which  $T_1$  (or  $I^{1/4}$ ) cannot be applied. Using the methods employed by M. Riesz in the theory of Riemann-Liouville integrals, we shall show that on the classes under consideration  $T_2$  is actually  $I^{-1/4}$ .

**THEOREM 1.** *If  $f \in S_1$ , then  $T_1 f$  is uniformly  $(\varepsilon + 1/4)$ -Hölder continuous on any closed subinterval of  $(0, T]$  where the  $\varepsilon$  is that associated with  $f \in S_1$ .*

*Proof.* Let  $t, \tau \in [\delta, T], \delta > 0$ . Assume without loss of generality  $\tau < t$ . Form the difference

$$\begin{aligned} \Delta &\equiv \Gamma\left(\frac{1}{4}\right) [(T_1 f)(t) - (T_1 f)(\tau)] \\ &= \int_0^t f(\sigma) \cdot (t - \sigma)^{-3/4} d\sigma - \int_0^\tau f(\sigma) \cdot (\tau - \sigma)^{-3/4} d\sigma. \end{aligned}$$

Adding and subtracting  $f(t)$  in the integrands gives

<sup>2</sup> This Theorem is essentially contained in Hardy, G. H. and Littlewood, J. E., "Some properties of fractional integrals", Math. Zeit., Vol 27, (1928), pp. 565-606.

$$\begin{aligned}
\Delta &= f(t) \cdot \int_0^t (t - \sigma)^{-3/4} d\sigma \\
&+ \int_0^t [f(\sigma) - f(t)] \cdot (t - \sigma)^{-3/4} d\sigma - f(t) \cdot \int_0^\tau (\tau - \sigma)^{-3/4} d\sigma \\
&- \int_0^\tau [f(\sigma) - g(t)] \cdot (\tau - \sigma)^{-3/4} d\sigma = 4f(t)(t^{1/4} - \tau^{1/4}) \\
&+ \int_0^\tau [f(\sigma) - f(t)] \cdot [(t - \sigma)^{-3/4} - (\tau - \sigma)^{-3/4}] d\sigma \\
&+ \int_\tau^t [f(\sigma) - f(t)] \cdot (t - \sigma)^{-3/4} d\sigma \equiv I_1 + I_2 + I_3,
\end{aligned}$$

say.

Regarding  $I_2$ , write it as

$$\begin{aligned}
I_2 &= \int_0^{\delta/2} [f(\sigma) - f(t)] [(t - \sigma)^{-3/4} - (\tau - \sigma)^{-3/4}] d\sigma \\
&+ \int_{\delta/2}^t [f(\sigma) - f(t)] \cdot [(t - \sigma)^{-3/4} - (\tau - \sigma)^{-3/4}] d\sigma \equiv J_{21} + J_{22}.
\end{aligned}$$

Using the mean-value theorem

$$\begin{aligned}
|f(\sigma) - f(t)| \cdot [(\tau - \sigma)^{-3/4} - (t - \sigma)^{-3/4}] \\
= |f(\sigma) - f(t)| \cdot 3/4 [(\tau - \sigma) + \theta(t - \tau)]^{-7/4} \cdot (t - \tau), \quad 0 \leq \theta \leq 1.
\end{aligned}$$

Since  $t - \sigma, \tau - \sigma \geq \delta/2$  for  $J_{21}$ ,

$$\begin{aligned}
|f(\sigma) - f(t)| \cdot [(\tau - \sigma)^{-3/4} - (t - \sigma)^{-3/4}] &\leq \sigma^{-\lambda} (t - \sigma)^\varepsilon \cdot |f|_1 \cdot 3/4 \cdot (\delta/2)^{-7/4} \cdot (t - \tau) \\
&\leq 3/4 \cdot T^\varepsilon \cdot (2/\delta)^{7/4} \cdot |f|_1 \cdot \sigma^{-\lambda} (t - \tau) = (\text{constant}) \cdot \sigma^{-\lambda} \cdot (t - \tau).
\end{aligned}$$

Thus,

$$\begin{aligned}
|J_{21}| &= \left| \int_0^{\delta/2} [f(\sigma) - f(t)] \cdot [(t - \sigma)^{-3/4} - (\tau - \sigma)^{-3/4}] d\sigma \right| \\
&\leq \int_0^{\delta/2} |f(\sigma) - f(t)| \cdot [(\tau - \sigma)^{-3/4} - (t - \sigma)^{-3/4}] d\sigma \\
&\leq (\text{constant}) \cdot (t - \tau) \int_0^{\delta/2} \sigma^{-\lambda} d\sigma = (\text{constant}) \cdot (t - \tau).
\end{aligned}$$

Now

$$\begin{aligned}
|J_{22}| &\leq |f|_1 \cdot \int_{\delta/2}^\tau \sigma^{-\lambda} (t - \sigma)^\varepsilon \cdot [(\tau - \sigma)^{-3/4} - (t - \sigma)^{-3/4}] d\sigma \\
&\leq (\delta/2)^{-\lambda} \cdot |f|_1 \cdot \int_{\delta/2}^\tau (t - \sigma)^\varepsilon \cdot [(\tau - \sigma)^{-3/4} - (t - \sigma)^{-3/4}] d\sigma
\end{aligned}$$

$$\leq (2/\delta)^\lambda \cdot |f|_1 \cdot \int_0^\tau (t - \sigma)^\varepsilon \cdot [(\tau - \sigma)^{-3/4} - (t - \sigma)^{-3/4}] d\sigma .$$

Set  $\tau - \sigma = (t - \sigma)s$ . Then

$$t - \sigma = \frac{t - \tau}{1 - s} ,$$

$$\tau - \sigma = \frac{s(t - \tau)}{1 - s} ,$$

and

$$d\sigma = - \frac{t - \tau}{(1 - s)^2} ds .$$

Hence

$$|J_{22}| \leq (\text{constant}) (t - \tau)^{1/4+\varepsilon} \cdot \int_0^{\tau/t} (1 - s)^{-5/4-\varepsilon} \cdot (s^{-3/4} - 1) ds$$

$$\leq (\text{constant}) (t - \tau)^{1/4+\varepsilon} \cdot \int_0^1 (1 - s)^{-5/4-\varepsilon} \cdot (1 - s^{3/4}) \cdot s^{-3/4} ds$$

$$\leq (\text{constant}) \cdot (t - \tau)^{1/4+\varepsilon} ;$$

the latter integral existing for  $\varepsilon < 3/4$  since

$$(1 - s)^{-5/4-\varepsilon} \cdot (1 - s^{3/4}) \cdot s^{-3/4}$$

$$= s^{-3/4} \cdot (1 - s)^{-1/4-\varepsilon} \cdot (1 + s + s^2)(1 + s^{3/2})^{-1}(1 + s^{3/4})^{-1} .$$

Now

$$|I_3| \leq \int_\tau^t |f(\sigma) - f(t)| \cdot (t - \sigma)^{-3/4} d\sigma \leq |f|_1 \cdot \int_\tau^t \sigma^{-\lambda} (t - \sigma)^{\varepsilon-3/4} d\sigma$$

$$\leq (2/\delta)^\lambda \cdot |f|_1 \cdot \int_\tau^t (t - \sigma)^{\varepsilon-3/4} d\sigma = (\text{constant}) (t - \tau)^{\varepsilon+1/4} .$$

This completes the proof.

Q.E.D.

**THEOREM 2.**  $f \in S_1$

- (i)  $T_2 T_1 = I_1$ , where  $I_1$  is the identity transformation on  $S_1$ .
- (ii)  $T_1 f \in S_2$ .

*Proof.*

$$(i) \quad [T_2(T_1 f)](t) = \frac{-1}{\Gamma(1/4) \cdot \Gamma(-1/4)} \left\{ 4t^{-1/4} \int_0^t f(\sigma)(t - \sigma)^{-3/4} d\sigma \right.$$

$$\left. + \int_0^t \left[ \int_0^t f(\sigma)(t - \sigma)^{-3/4} d\sigma - \int_0^\tau f(\sigma)(\tau - \sigma)^{-3/4} d\sigma \right] (t - \tau)^{-5/4} d\tau \right\}$$

We proceed as in the theory of Riemann-Liouville integrals. Define

$$F(\mu) = \frac{1}{\Gamma(\mu)} \int_0^t \left[ \int_0^t f(\sigma)(t - \sigma)^{-3/4} d\sigma - \int_0^\tau f(\sigma)(\tau - \sigma)^{-3/4} d\sigma \right] (t - \tau)^{\mu-1} d\tau$$

which exists and is analytic for  $\Re\mu > -1/4 - \varepsilon$  by Theorem 1. Now restrict  $\mu$  so that  $\Re\mu > 0$ . Then

$$\begin{aligned} F(\mu) &= \frac{1}{\Gamma(\mu)} \int_0^t f(\sigma)(t - \sigma)^{-3/4} d\sigma \cdot \int_0^t (t - \tau)^{\mu-1} d\tau \\ &\quad - \frac{1}{\Gamma(\mu)} \int_0^t \left( \int_0^\tau f(\sigma)(\tau - \sigma)^{-3/4} d\sigma \right) (t - \tau)^{\mu-1} d\tau \\ &= \frac{t^\mu}{\mu\Gamma(\mu)} \cdot \int_0^t f(\sigma) \cdot (t - \sigma)^{-3/4} d\sigma - \frac{1}{\Gamma(\mu)} \int_0^t \left( \int_0^\tau f(\sigma)(\tau - \sigma)^{-3/4} d\sigma \right) \\ &\quad \times (t - \tau)^{\mu-1} d\tau . \end{aligned}$$

Interchanging the order of integration in the second term and setting  $\tau - \sigma = (t - \sigma) \cdot s$  in the inner integral gives

$$\begin{aligned} &\frac{1}{\Gamma(\mu)} \int_0^t f(\sigma) \cdot (t - \sigma)^{\mu-3/4} d\sigma \cdot \left( \int_0^1 s^{-3/4}(1 - s)^{\mu-1} ds \right) \\ &= \frac{\Gamma(1/4)}{\Gamma(\mu + 1/4)} \cdot \int_0^t f(\sigma) \cdot (t - \sigma)^{\mu-3/4} d\sigma . \end{aligned}$$

Adding and subtracting  $f(t)$  in the integrand of this latter integral gives

$$\begin{aligned} &\frac{\Gamma(1/4)}{\Gamma(\mu + 1/4)} \int_0^t f(\sigma) \cdot (t - \sigma)^{\mu-3/4} d\sigma = \frac{\Gamma(1/4) \cdot f(t)}{\Gamma(\mu + 1/4)} \cdot \int_0^t (t - \sigma)^{\mu-3/4} d\sigma \\ &\quad + \frac{\Gamma(1/4)}{\Gamma(\mu + 1/4)} \int_0^t [f(\sigma) - f(t)](t - \sigma)^{\mu-3/4} d\sigma \\ &= \frac{\Gamma(1/4)}{\Gamma(\mu + 1/4)} f(t) \cdot t^{\mu+1/4} + \frac{\Gamma(1/4)}{\Gamma(\mu + 1/4)} \int_0^t [f(\sigma) - f(t)] \cdot (t - \sigma)^{\mu-3/4} d\sigma . \end{aligned}$$

This latter term has a zero at  $\mu = -1/4$  since the integral defines a function analytic for  $\Re\mu > -1/4 - \varepsilon$  and  $(\Gamma(\mu + 1/4))^{-1}$  is an entire function with a zero at  $\mu = -1/4$ .

From the identity theorem from ‘function theory’

$$\begin{aligned} &\frac{1}{\Gamma(\mu)} \int_0^t \left[ \int_0^t f(\sigma)(t - \sigma)^{-3/4} d\sigma - \int_0^\tau f(\sigma)(\tau - \sigma)^{-3/4} d\sigma \right] (t - \tau)^{\mu-1} d\tau = F(\mu) \\ &= \frac{t^\mu}{\Gamma(\mu + 1)} \int_0^t f(\sigma)(t - \sigma)^{-3/4} d\sigma - \frac{\Gamma(1/4)}{\Gamma(\mu + 1/4)} \int_0^t [f(\sigma) - f(t)](t - \sigma)^{\mu-3/4} d\sigma \end{aligned}$$

$$- \frac{\Gamma(1/4)}{\Gamma(\mu + 5/4)} \cdot f(t) \cdot t^{\mu+1/4}$$

for  $\Re\mu > -1/4 - \varepsilon$ . Therefore we find that

$$\begin{aligned} & \frac{1}{\Gamma(-1/4)} \int_0^t \left[ \int_0^t f(\sigma)(t - \sigma)^{-3/4} d\sigma - \int_0^\tau f(\sigma)(\tau - \sigma)^{-3/4} d\sigma \right] (t - \tau)^{-5/4} d\tau \\ &= \frac{t^{-1/4}}{\Gamma(3/4)} \int_0^t f(\sigma) \cdot (t - \sigma)^{-3/4} d\sigma - \Gamma(1/4) \cdot f(t) \\ &= - \frac{4 \cdot t^{-1/4}}{\Gamma(-1/4)} \cdot \int_0^t f(\sigma) \cdot (t - \sigma)^{-3/4} d\sigma - \Gamma(1/4) \cdot f(t) . \end{aligned}$$

Thus,

$$\begin{aligned} [T_2(T_1f)](t) &= \frac{-1}{\Gamma(1/4) \cdot \Gamma(-1/4)} \left\{ 4t^{-1/4} \cdot \int_0^t f(\sigma) \cdot (t - \sigma)^{-3/4} d\sigma \right. \\ &\quad \left. - 4t^{-1/4} \int_0^t f(\sigma) \cdot (t - \sigma)^{-3/4} d\sigma - \Gamma(1/4) \cdot \Gamma(1/4) f(t) \right\} = f(t) . \end{aligned}$$

(ii) All that remains to be shown is that

$$\sup_t t^{\lambda-1/4} |(T_1f)(t)| < +\infty .$$

Adding and subtracting  $f(t)$  in the integrand we have

$$\begin{aligned} t^{\lambda-1/4}(T_1f)(t) &= \frac{t^{\lambda-1/4}}{\Gamma(1/4)} \int_0^t f(\sigma) \cdot (t - \sigma)^{-3/4} d\sigma \\ &= \frac{t^{\lambda-1/4}}{\Gamma(1/4)} \int_0^t [f(\sigma) - f(t)] \cdot (t - \sigma)^{-3/4} d\sigma + \frac{t^{\lambda-1/4}}{\Gamma(1/4)} \cdot f(t) \cdot \int_0^t (t - \sigma)^{-3/4} d\sigma . \end{aligned}$$

Thus,

$$\begin{aligned} t^{\lambda-1/4} \cdot |(T_1f)(t)| &\leq t^{\lambda-1/4} \cdot \frac{|f|_1}{\Gamma(1/4)} \cdot \int_0^t \sigma^{-\lambda} \cdot (t - \sigma)^{\varepsilon-3/4} d\sigma + \frac{4}{\Gamma(1/4)} \cdot t^\lambda \cdot |f(t)| \\ &= t^{\lambda-1/4} \cdot \frac{|f|_1}{\Gamma(1/4)} \cdot t^{-\lambda+1/4+\varepsilon} \cdot \int_0^1 s^{-\lambda} \cdot (1 - s)^{\varepsilon-3/4} ds + \frac{4}{\Gamma(1/4)} \cdot t^\lambda \cdot |f(t)| \\ &\leq (\text{constant}) \cdot T^\varepsilon |f|_1 + \frac{4}{\Gamma(1/4)} \left[ \frac{t^\lambda |f(t) - f(T)|}{|t - T|^\varepsilon} \right] (T - t)^\varepsilon + \frac{4}{\Gamma(1/4)} t^\lambda |f(T)| \\ &\leq (\text{constant}) \cdot T^\varepsilon |f|_1 + \frac{4}{\Gamma(1/4)} \cdot T^\lambda |f(T)| < +\infty . \end{aligned}$$

Q.E.D.

**THEOREM 3.**  $g \in S_2$

(i)  $T_1T_2 = I_2$ , where  $I_2$  is the identity transformation on  $S_2$ .

(ii)  $T_2g \in S_1$ .

*Proof.* Part (i) is proven exactly as part (i) of Theorem 2 and part (ii) follows directly from the definitions of  $S_2$  and  $T_2$ .

Q.E.D.

Consider the following system of equations made up from the terms with singular kernels in (2.14).

$$(3.3) \quad \begin{aligned} \frac{1}{2} \phi(t) + \psi(t) \cdot k(0, t) + \int_0^t [\psi(t) - \psi(\sigma)] \cdot D_\sigma k(0, t - \sigma) d\sigma &= f(t) \\ \int_0^t \phi(\sigma) \cdot D_\nu^2 k(0, t - \sigma) d\sigma - \frac{1}{2} \psi(t) &= g(t)^3 \end{aligned}$$

where  $f \in S_1$  and  $g \in S_2$ . Now

$$\begin{aligned} D_\nu^2 k(0, t - \sigma) &= \frac{-1}{2\pi} \int_{-\infty}^{\infty} \xi^2 \cdot e^{-\xi^4(t-\sigma)} d\xi \\ &= \frac{-1}{2\pi} (t - \sigma)^{-3/4} \int_{-\infty}^{\infty} \eta^2 \cdot e^{-\eta^4} d\eta = -\frac{1}{\pi} (t - \sigma)^{-3/4} \int_0^{\infty} \eta^2 e^{-\eta^4} d\eta \\ &= -\frac{1}{4\pi} (t - \sigma)^{-3/4} \cdot \int_0^{\infty} \xi^{-1/4} \cdot e^{-\xi} d\xi = \frac{\Gamma(3/4)}{4\pi} \cdot (t - \sigma)^{-3/4}. \end{aligned}$$

Similarly,

$$D_\sigma k(0, t - \sigma) = \frac{\Gamma(5/4)}{4\pi} \cdot (t - \sigma)^{-5/4}$$

and

$$k(0, t) = \frac{\Gamma(1/4)}{4\pi} \cdot t^{-1/4}.$$

Then using the fact that  $\Gamma(3/4) \cdot \Gamma(1/4) = \pi \csc \pi/4 = \sqrt{2} \cdot \pi$

$$(3.4) \quad \begin{aligned} \int_0^t \phi(\sigma) \cdot D_\nu^2 k(0, t - \sigma) d\sigma &= -\frac{\Gamma(3/4)}{4\pi} \int_0^t \phi(\sigma) \cdot (t - \sigma)^{-3/4} d\sigma \\ &= -\frac{\Gamma(3/4) \cdot \Gamma(1/4)}{4\pi} (T_1 \phi)(t) = -\frac{1}{2\sqrt{2}} (T_1 \phi)(t). \end{aligned}$$

Similarly,

$$(3.5) \quad \psi(t) \cdot k(0, t) + \int_0^t [\psi(t) - \psi(\sigma)] \cdot D_\sigma k(0, t - \sigma) d\sigma = \frac{1}{2\sqrt{2}} (T_2 \psi)(t).$$

Thus from (3.4) and (3.5) we can write (3.3) as

<sup>3</sup> This is just the system of integrals equations one obtains for the problem on the half-space  $(0 < x < \infty) \otimes (0, T)$ .

$$(3.3)' \quad \begin{cases} \frac{1}{2}\phi + \frac{1}{2\sqrt{2}}T_2\psi = f \\ -\frac{1}{2\sqrt{2}}T_1\phi - \frac{1}{2}\psi = g. \end{cases}$$

Using Theorems 2 and 3, we can solve this system of equations by formally applying  $T_1$  and  $T_2$ . Applying  $(1/\sqrt{2})T_2$  to the second equation and adding to the first gives

$$-\frac{1}{4}T_2T_1\phi + \frac{1}{2}\phi = f + \frac{1}{\sqrt{2}}T_2g.$$

Since  $T_2T_1 = I_1$ , we find that

$$\phi = 4\left(f + \frac{1}{\sqrt{2}}T_2g\right) = 8\left(\frac{1}{2}f + \frac{1}{2\sqrt{2}}T_2g\right).$$

Similarly, we find that

$$\psi = -8\left(\frac{1}{2\sqrt{2}}T_1f + \frac{1}{2}g\right).$$

Thus the solution of (3.3)' is given by

$$(3.6) \quad \begin{cases} \phi = 8\left(\frac{1}{2}f + \frac{1}{2\sqrt{2}}T_2g\right) \\ \psi = -8\left(\frac{1}{2\sqrt{2}}T_1f + \frac{1}{2}g\right). \end{cases}$$

Defining

$$M = \begin{pmatrix} \frac{1}{2}I_1 & \frac{1}{2\sqrt{2}}T_2 \\ -\frac{1}{2\sqrt{2}}T_1 & -\frac{1}{2}I_2 \end{pmatrix}$$

where  $M$  is an operator on the product  $S_1 \otimes S_2$ , we can write (3.3)' as

$$(3.3)'' \quad M\phi = F$$

where

$$\phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad F = \begin{pmatrix} f \\ g \end{pmatrix};$$

and (3.6) as

$$(3.6)' \quad \phi = 8MF.$$



Thus,

$$(3.7) \quad M^{-1} = 8M.$$

Define for suitable functions:

$$(3.8) \quad \begin{aligned} (Sf)(t) &= \int_0^t f(\sigma) \cdot D_y^3 k(1, t - \sigma) d\sigma \\ (Uf)(t) &= \int_0^t f(\sigma) \cdot D_\sigma k(1, t - \sigma) d\sigma \\ (Vf)(t) &= \int_0^t f(\sigma) \cdot D_y^2 k(1, t - \sigma) d\sigma. \end{aligned}$$

In terms of the above defined operators and  $T_1$ , and  $T_2$ , we can write the general system (2.14) as:

$$(3.9) \quad \begin{cases} \frac{1}{2} \phi + S\phi + \frac{1}{2\sqrt{2}} T_2 \psi + U\psi = A \\ -\frac{1}{2\sqrt{2}} T_1 \phi + V\phi - \frac{1}{2} \psi + S\psi = B \end{cases}$$

or as,

$$(3.9)' \quad M\phi + N\psi = F \in S_1 \otimes S_2$$

where

$$(3.10) \quad N = \begin{pmatrix} S & U \\ V & S \end{pmatrix}, \quad F = \begin{pmatrix} A \\ B \end{pmatrix}.$$

From (2.2) it follows that all of the kernels in the operators in  $N$  are bounded (in fact, they are  $C^\infty$  functions).

Write (3.9)' as

$$(3.9)'' \quad (I + NM^{-1})M\phi = F$$

where

$$I = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}$$

is the identity transformation on  $S_1 \otimes S_2$ . This is certainly meaningful since  $NM^{-1}$  is well-defined and likewise  $(I + NM^{-1})M$ .

**LEMMA.** *All of the kernels in the operators in  $NM^{-1}$  are bounded and differentiable.*

*Proof.*

$$\begin{aligned}
 NM^{-1} &= 8 \begin{pmatrix} S & U \\ V & S \end{pmatrix} \begin{pmatrix} \frac{1}{2} I_1 & \frac{1}{2\sqrt{2}} T_2 \\ -\frac{1}{2\sqrt{2}} T_1 & -\frac{1}{2} I_2 \end{pmatrix} \\
 &= 8 \begin{pmatrix} \frac{1}{2} S - \frac{1}{2\sqrt{2}} UT_1 & \frac{1}{2\sqrt{2}} ST_2 - \frac{1}{2} U \\ \frac{1}{2} V - \frac{1}{2\sqrt{2}} ST_1 & \frac{1}{2\sqrt{2}} VT_2 - \frac{1}{2} S \end{pmatrix}.
 \end{aligned}$$

We shall carry out the proof for  $ST_2$ . The proofs of the remaining ones follows exactly the same lines.

For  $f \in S_2$ :

$$\begin{aligned}
 &[S(T_2 f)](t) \\
 &= \int_0^t \left\{ \frac{-1}{\Gamma(-1/4)} \left[ 4\tau^{-1/4} \cdot f(\tau) + \int_0^\tau [f(\tau) - f(\sigma)](\tau - \sigma)^{-5/4} d\sigma \right] \right\} D_y^3 k(1, t - \tau) d\tau \\
 &= -\frac{4}{\Gamma(-1/4)} \int_0^t \tau^{-1/4} \cdot f(\tau) \cdot D_y^3 k(1, t - \tau) d\tau \\
 &\quad - \frac{1}{\Gamma(-1/4)} \int_0^t \left( \int_0^\tau [f(\tau) - f(\sigma)] \cdot (\tau - \sigma)^{-5/4} d\sigma \right) \cdot D_y^3 k(1, t - \tau) d\tau.
 \end{aligned}$$

We proceed as in the proof of Theorem 2. Let

$$F(\mu) = \frac{1}{\Gamma(\mu)} \int_0^t \left( \int_0^\tau [f(\tau) - f(\sigma)] \cdot (\tau - \sigma)^{\mu-1} d\sigma \right) \cdot D_y^3 k(1, t - \tau) d\tau.$$

This defines an analytic function for  $\Re\mu > -1/4$ . Now restrict  $\mu$  so that  $\Re\mu > 0$ .

Then

$$\begin{aligned}
 F(\mu) &= \frac{1}{\Gamma(\mu)} \int_0^t f(\tau) \cdot \left( \int_0^\tau (\tau - \sigma)^{\mu-1} d\sigma \right) \cdot D_y^3 k(1, t - \tau) d\tau \\
 &\quad - \frac{1}{\Gamma(\mu)} \int_0^t \left( \int_0^\tau f(\sigma) (\tau - \sigma)^{\mu-1} d\sigma \right) \cdot D_y^3 k(1, t - \tau) d\tau \\
 &= \frac{1}{\mu\Gamma(\mu)} \int_0^t f(\tau) \cdot \tau^\mu \cdot D_y^3 k(1, t - \tau) d\tau \\
 &\quad - \frac{1}{\Gamma(\mu)} \int_0^t \left( \int_0^\tau f(\sigma) \cdot (\tau - \sigma)^{\mu-1} d\sigma \right) \cdot D_y^3 k(1, t - \tau) d\tau.
 \end{aligned}$$

Interchanging the order of integration in the second term gives

$$\frac{1}{\Gamma(\mu)} \int_\sigma^t f(\sigma) \left( \int_0^t (\tau - \sigma)^{\mu-1} \cdot D_y^3 k(1, t - \tau) d\tau \right) d\sigma.$$

Integrating the inner integral by parts  $n$  gives

$$\frac{1}{\Gamma(\mu + n)} \int_0^t f(\sigma) \left( (-1)^n \int_\sigma^t (\tau - \sigma)^{\mu+n-1} D_\tau^n D_y^3 k(1, t - \tau) d\tau \right) d\sigma,$$

which is analytic for  $\Re\mu > -n$ .

Thus from the identity theorem we have that

$$\begin{aligned} & \frac{1}{\Gamma(\mu)} \int_0^t \left( \int_0^\tau [f(\tau) - f(\sigma)] \cdot (\tau - \sigma)^{\mu-1} d\sigma \right) \cdot D_y^3 k(1, t - \tau) d\tau \\ &= F(\mu) = \frac{1}{\mu\Gamma(\mu)} \int_0^t f(\tau) \cdot \tau^\mu \cdot D_y^3 k(1, t - \tau) d\tau \\ &+ \frac{(-1)^{n+1}}{\Gamma(\mu + n)} \int_\sigma^t f(\sigma) \cdot \left( \int_\sigma^t (\tau - \sigma)^{\mu+n-1} \cdot D_\tau^n D_y^3 k(1, t - \tau) d\tau \right) d\sigma, \Re\mu > -1/4. \end{aligned}$$

Taking the limit as  $\mu \downarrow -1/4$ , we get

$$\begin{aligned} & \frac{1}{\Gamma(-1/4)} \int_0^t \left( \int_0^\tau [f(\tau) - f(\sigma)] \cdot (\tau - \sigma)^{-5/4} d\sigma \right) \cdot D_y^3 k(1, t - \tau) d\tau \\ &= -\frac{4}{\Gamma(-1/4)} \int_0^t f(\tau) \cdot \tau^{-1/4} D_y^3 k(1, t - \tau) d\tau \\ &+ \frac{(-1)^{n+1}}{\Gamma(n - 1/4)} \int_0^t f(\sigma) \cdot \left( \int_\sigma^t (\tau - \sigma)^{n-5/4} \cdot D_\tau^n D_y^3 k(1, t - \tau) d\tau \right) d\sigma. \end{aligned}$$

Hence

$$[S(T_2 f)](t) = \frac{(-1)^n}{\Gamma(n - 1/4)} \int_0^t f(\sigma) \cdot \left( \int_\sigma^t (\tau - \sigma)^{n-5/4} \cdot D_\tau^n D_y^3 k(1, t - \tau) d\tau \right) d\sigma.$$

From (2.2)

$$\begin{aligned} \left| \int_\sigma^t (\tau - \sigma)^{n-5/4} \cdot D_\tau^n D_y^3 k(1, t - \tau) d\tau \right| &\leq (\text{Constant}) \cdot \int_\sigma^t (\tau - \sigma)^{n-5/4} d\tau \\ &= (\text{Constant}) \cdot (t - \sigma)^{n-1/4}. \end{aligned}$$

Clearly the kernel is continuous and differentiable for  $0 \leq \sigma \leq t$ . In fact, we could conclude that it is infinitely often differentiable.

Q.E.D.

The above Lemma shows that  $I + NM^{-1}$  is essentially a perturbation of the identity. That is, the problem is reduced to solving a system of Volterra type integral equations with bounded and differentiable kernels.

APPENDIX: *Derivation of Estimate (2.2).* This appendix is included at the suggestion of the referee in order to make this paper essentially self contained.

From (2.1)

$$D_x^n k(x, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} (-iz)^n \exp(-izx - z^4 t) dz, \quad t > 0.$$

Making the change of variable  $zt^{1/4} = y$  gives

$$(1) \quad D_x^n k(x, t) = (-i)^n (2\pi)^{-1} t^{-(1+n)/4} \int_{-\infty}^{\infty} y^n \exp(-iyxt^{-1/4} - y^4) dy.$$

The integral in (1) considered as an integral in the complex plane is easily seen to be equal to

$$(2) \quad \int_{-\infty}^{\infty} (y + ic)^n \exp[-ia^3(y + ic) - (y + ic)^4] dy$$

where  $c$  is any real number and  $a = (xt^{-1/4})^{1/3}$ .

Denoting the integral (2) by  $I$  we find upon expanding that

$$I = (\exp[a^3c - c^4]) \sum_{j=0}^n \binom{n}{j} (ic)^{n-j} \times \int_{-\infty}^{\infty} y^j \exp[-i(a^3y + 4y^3c - 4yc^3) - (y^4 - 6y^2c^2)] dy.$$

Using the inequality  $6y^2c^2 \leq 9R^{-1}y^4 + Rc^4$  with  $R > 9$  it follows that

$$|I| \leq (\exp[a^3c + (R - 1)c^4]) \sum_{j=0}^n \binom{n}{j} |c|^{n-j} \int_{-\infty}^{\infty} |y|^j \exp[-y^4(1 - 9R^{-1})] dy.$$

Setting

$$A(n) = \max_{0 \leq j \leq n} \left\{ \int_{-\infty}^{\infty} |y|^j \exp[-y^4(1 - 9R^{-1})] dy \right\}$$

we obtain the inequality

$$|I| \leq A(n)(1 + |c|)^n \exp[a^3c + (R - 1)c^4].$$

Now choose  $c = -\mu(R - 1)^{-1/3}a$ ,  $0 < \mu < 1$ . Then

$$a^3c + (R - 1)c^4 = -\mu(1 - \mu^3)(R - 1)^{-1/3}a^4 < 0$$

and

$$|I| \leq A(n)[1 + \mu(R - 1)^{-1/3}|a|]^n \exp[-\mu(1 - \mu^3)(R - 1)^{-1/3}a^4].$$

Setting

$$B(n) = \{\max_{z \geq 0}\} [1 + \mu(R - 1)^{-1/3}z]^n \exp[-2^{-1}\mu(1 - \mu^3)(R - 1)^{-1/3}z^4]$$

and replacing  $a$  by  $(xt^{-1/4})^{1/3}$  we get the inequality

$$(3) \quad |I| \leq A(n)B(n) \exp[-2^{-1}\mu(1 - \mu^3)(R - 1)^{-1/3}(x^4t^{-1})^{1/3}].$$

Estimate (2.2) is obtained from (1), (2), and (3) with

$$C_1 = (2\pi)^{-1}A(n)B(n) \text{ and } C_2 = 2^{-1}\mu(1 - \mu^3)(R - 1)^{-1/3}.$$

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