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ON THE LINE SEGMENTS OF A CONVEX SURFACE IN E_3

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1. Introduction. For integral $n \geq 2$ let C be a bounded open convex subset of Euclidean n -space E_n , and let C' be the boundary (surface) of C . Let B_n be the closed unit ball in E_n , that is, the set of points x in E_n with $\|x\| \leq 1$, and let $S_{(n-1)}$ be the boundary of B_n , that is, the set of points x in E_n with $\|x\| = 1$. Let D be the set of directions of straight line segments lying in C' , specifically, the set of points $(a - b)/\|a - b\|$, where a and b are distinct points of a line segment lying in C' . Thus D is contained in $S_{(n-1)}$.

V. L. Klee [2] has stated that D is an F_σ set and has raised two questions: Is D of first category in $S_{(n-1)}$? Is D of $(n - 1)$ -dimensional measure zero? Both of these questions are herein answered affirmatively for the case $n = 3$. The method employed unfortunately does not generalize to $n > 3$. (For $n = 2$ the case is trivial, for then D is countable. The case is also trivial if C is of dimension less than n , for then the $(n - 2)$ -dimensional measure of D cannot be greater than the $(n - 2)$ -dimensional measure of $S_{(n-2)}$ which is finite. The restriction to bounded sets is only a matter of convenience, for any answers to the questions posed are easily made to serve the unbounded case.)

In one sense though, for the case $n = 3$, we show somewhat more, namely, that D is contained in the union of the ranges of a countable family of Lipschitz functions each on B_1 to S_2 . By virtue of the Lipschitz nature of these functions, they possess total differentials (Lebesgue measure) almost everywhere [4; straight forward generalizations of Definition 1, V. 2.2, and Lemmas 1 and 2, V. 2.3, to cover the case of a Lipschitz function on a domain contained in E_1 to E_3] and their ranges are compact and have finite one dimensional measure [1]. The affirmative answers to Klee's questions for this case immediately follow from these last two properties.

2. Preliminaries. We assume henceforth that $n = 3$.

Let a *flat side* of C' be a two dimensional intersection of C' with a plane supporting C' . It is easy to check that the set of flat sides is countable. (Check, for instance, that relative to C' , the interior of each flat side is non-vacuous and, that no two such interiors intersect.) Thus the set of directions of line segments lying in flat sides is the union of a countable family of great circles lying on S_2 and can certainly be represented as the union of the ranges of an appropriately

chosen countable family of Lipschitz functions on B_1 to S_2 .

We go on to show that the set of directions of line segments not lying in flat sides can be similarly represented.

Let \mathcal{L} be the set of closed line segments each of which is the middle third line segment contained in a maximal line segment of C' not lying in a flat side. Clearly \mathcal{L} is disjointed, for if any two members intersected they would be forced by the convexity of C to lie in a flat side determined by the plane containing the two line segments.

Now choose a point a in C and let 2δ be the distance from a to C' . Let \mathcal{H} be the family of open right circular cylinders of radius δ extending infinitely in two directions whose axis is a line radiating out from a infinitely in two directions. Thus each member of \mathcal{H} intersects C' in a set open relative to C' and having two components. Let \mathcal{M} be the set of all these components corresponding to all cylinders of \mathcal{H} .

Since \mathcal{M} forms an open covering of the compact space C' we can reduce it to a finite subcovering \mathcal{M}' .

Now let \mathcal{P} be the family of planes each of which intersects C and perpendicularly intersects a coordinate axis in a point with rational coordinates. Let \mathcal{Q} be the family of pairs of distinct parallel members of \mathcal{P} .

Clearly every member of \mathcal{L} intersects at least one member of \mathcal{M}' and every such intersection intersects both planes of at least one pair in \mathcal{Q} .

Since \mathcal{M}' is finite and \mathcal{Q} is countable, we will have achieved our aim when we have shown that corresponding to each member m of \mathcal{M}' and each pair (P_1, P_2) of planes in \mathcal{Q} both intersecting m there exist two Lipschitz functions each on B_1 to S_2 whose ranges together contain the set of directions of the members of \mathcal{L} each of which intersects both $m \cap P_1$ and $m \cap P_2$. With m, P_1 , and P_2 fixed and letting \mathcal{L}' be the set of members of \mathcal{L} each intersecting both $m \cap P_1$ and $m \cap P_2$, we proceed to secure the required functions.

3. The Lipschitz direction functions. Let f be the set of all pairs (x, y) such that $x \in \lambda \cap P_1$ and $y \in \lambda \cap P_2$ for some $\lambda \in \mathcal{L}'$. Let A be the domain of f . Since \mathcal{L}' is disjointed and since $\lambda \cap P_1$ and $\lambda \cap P_2$ are singletons we infer that f is a function. The key to the construction of the required functions lies in the

LEMMA. *f is Lipschitz.*

Momentarily leaving aside its proof, we first show how it is used to obtain these functions.

Drawing upon the lemma, we apply a method due to McShane [3; or 4, V. 2.4, Lemma 1] to get a Lipschitz extension f^* of f on the

closure of $P_1 \cap m$, that is, a Lipschitz function f^* on the closure of $P_1 \cap m$ to P_2 that agrees with f on A .

We next let h be a function that assigns to each member x of the closure of $P_1 \cap m$ one of the directions of the line connecting x to $f^*(x)$, specifically for x in the closure of $P_1 \cap m$ we let

$$h(x) = \frac{f^*(x) - x}{\|f^*(x) - x\|} .$$

Upon checking that the difference of two Lipschitz functions is Lipschitz and that the ratio of a Lipschitz function whose values are bounded away from the origin (in our case bounded by the distance between P_1 and P_2) with its norm is Lipschitz, we infer that h is Lipschitz. It is easy to construct a Lipschitz homeomorphism g on B_1 onto the closure of $P_1 \cap m$. So finally upon defining functions k and k' on B_1 to S_2 to be such that for x in B_1

$$k(x) = h(g(x)), \quad k'(x) = -k(x) ,$$

and noting that the composition of Lipschitz functions is Lipschitz, we conclude that k and k' are Lipschitz and furthermore that their ranges together contain the set of directions of members of \mathcal{L}' . These are the functions we seek.

We now turn our attention to the lemma and close our discussion with its proof.

4. Proof of the Lemma. We show that f is Lipschitz by showing that it can be represented as the composition of Lipschitz functions. To do this let us project m perpendicularly onto a plane perpendicular to the axis of the cylinder in \mathcal{H} associated with m . Let m' be the projected set and let p be the projecting function. Thus p is on m onto m' . From the convexity of C and the nature of the cylinder determining m we readily check that p is a Lipschitz homeomorphism on m onto m' whose inverse is also Lipschitz. For x' in $p(A)$ let $f'(x') = p(f(p^{-1}(x')))$. For x in A clearly $f(x) = p^{-1}(f'(p(x)))$. We have only to show that f' is Lipschitz.

Let λ_1 and λ_2 be two members of \mathcal{L}' . Let $x_1 \in \lambda_1 \cap P_1$ and $x_2 \in \lambda_2 \cap P_1$. Let l_1 and l_2 be maximal line segments contained in C' containing respectively λ_1 and λ_2 . Let l_1' and l_2' be the respective perpendicular projections of l_1 and l_2 onto the plane of m' . Clearly l_1 and l_2 fail to intersect or intersect only in an end point of both l_1 and l_2 . Consequently the same is true of l_1' any l_2' . If l_1' and l_2' are parallel or, when extended, intersect on the side of P_2 opposite from P_1 , then clearly

$$(1) \quad \|p(f(x_1)) - p(f(x_2))\| \leq \|p(x_1) - p(x_2)\| .$$

If, on the other hand, l_1' and l_2' , when extended, intersect in a point b , on the same side of P_2 that P_1 lies on, then either an end point of l_1' lies at b or between b and P_1 , or an end point of l_2' lies at b or between b and P_1 . We may assume the first of these two main disjunctions without loss of generality. Now since the line segment connecting $p(x_1)$ with $p(f(x_1))$ is contained in the middle third segment of l_1' , we have

$$\|p(f(x_1)) - p(x_1)\| \leq \|p(x_1) - b\| .$$

and hence

$$(2) \quad \|p(f(x_1)) - b\| = \|p(f(x_1)) - p(x_1)\| + \|p(x_1) - b\| \leq 2\|p(x_1) - b\| .$$

As P_1 and P_2 are parallel, we may use a property of similar triangles to get

$$(3) \quad \frac{\|p(f(x_1)) - p(f(x_2))\|}{\|p(x_1) - p(x_2)\|} = \frac{\|p(f(x_1)) - b\|}{\|p(x_1) - b\|} .$$

Combining (2) and (3) we get

$$(4) \quad \|p(f(x_1)) - p(f(x_2))\| \leq 2\|p(x_1) - p(x_2)\| .$$

Since equations (1) and (4) show that for any x_1' and x_2' in the domain of f'

$$\|f'(x_1') - f'(x_2')\| \leq 2\|x_1' - x_2'\| ,$$

and hence that f' is Lipschitz, our proof is complete.

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