Pacific Journal of Mathematics

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Vol. 10, No. 3 November 1960

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It is well known (Perron [12], Frobenius [6, 7]) that if A is an $n \times n$ matrix over the real field with elements $\geqq 0$, the spectral radius[1] of A, $r(A)$, is a characteristic number, with at least one characteristic vector whose coordinates are $\geqq 0$. If A has positive elements throughout, then r is > 0, of algebraic and geometric multiplicity one, and exceeds all other elements of the spectrum in absolute value.[2] Generalizations of this theorem to integral equations were obtained by Jentzsch [9] and E. Hopf [8]. In an operator-theoretic setting, the result did not appear until 1948 when Krein and Rutman published their most comprehensive work [11]. Further results were obtained by Bonsall [2]-[4] and, in the framework of a general locally convex space, by the author [15, 17] For compact positive operators in an order-complete Banach lattice, see Ando [1].

While the key to many results generalizing the Perron-Frobenius theorem is compactness in one form or another, a good many spectral properties of positive linear operators are independent of it. Such properties were established by Bonsall (e.c., cf. Prop. 1 below), the author [17], and recently Putnam [13] who considers, however, only the rather special case of a bounded matrix with non-negative elements in l_2. The present paper establishes new and more general results on the (spectral) character of the spectral radius r of a positive operator T, valid in arbitrary ordered Banach spaces.[3] Section 2 collects some theorems for which no hypothesis or r is made; leaning heavily on topological properties of the positive cone K, they apply to any positive operator. Throughout § 3, r is assumed to be a pole of the resolvent of T. The stress is here on the notion of quasi-interior map; together with the assumption on r, this concept yields strong results earlier obtained by Krein and Rutman [11] for strongly positive operators[17] which are compact and defined on a space whose positive cone K has interior points. This is interesting since in many concrete examples of partially ordered (B)-spaces, K has empty interior [16, p. 130]. The paper concludes with two problems.

Received October 20, 1959. Based on research sponsored by the Office of Ordnance Research, U.S. Army.

[1] For the terminology adopted, see § 1.

[2] A short proof in [14]. Cf. also [5].

[3] With only minor modifications, the results of the present paper carry over to bounded positive endomorphisms of a partially ordered, quasi-complete locally convex space.

1. Auxiliary material. A (real or complex) Banach space E is *partially ordered* if an order relation[4], denoted $x \leqq y$ and invariant under addition and multiplication by positive scalars, is defined on E. It is well known that such an order structure is completely determined by the set $\{x: x \geqq 0\}$ of positive elements which will be called the *positive cone* K. Unless otherwise stated, we shall always suppose that K is closed in E and *proper*, i.e., such that $K \cap -K = \{0\}$[5]. K is *generating* if $E = K - K$, *normal* if $\| x + y \| \geqq \| y \|$ for all $x, y \in K$ and some real norm $x \to \| x \|$ generating the topology of E. K is a *B-cone* (BZ-Kegel in [**16**]) if for some fundamental system of bounded sets $\boldsymbol{B}$, the closed convex symmetric hulls of the sets $B \cap K, B \in \boldsymbol{B}$, form again a fundamental system of bounded subsets of E.[6] We say K is spanned by a set C if $K = \bigcup_{\lambda \geqq 0} \lambda C$. If E' is the topological dual of $E, K' \subset E'$ is the set of those linear forms which are $\geqq 0$ on K (resp. if E is complex, whose real parts are $\geqq 0$ on K). K' is called the cone *conjugate* to K. An $f \in E'$ is *positive* (resp. *strictly positive*) with respect to a given partial ordering of E if $\operatorname{Re} f(x) \geqq 0$ for $x \in K$ (resp. if $\operatorname{Re} f(x) > 0$ for $0 \neq x \in K$). If E is a real Banach space, F its complexification in the usual sense, and K is a normal cone (resp. a B-cone) in E, then $K + iK$ is a normal cone (resp. a B-cone) in F [**17**, p. 264].

Let E denote a real or complex Banach space, partially ordered by a proper closed cone K.

LEMMA 1. *If K is normal, then $E' = K' - K'$. If K is a normal B-cone, then so is K' for the strong topology on E'.*

The first part is proved (for real spaces) in [**10**]. For the second part, see [**3**, p. 146], and [**17**, p. 262/3] in the complex case. (It follows from a simple category argument that in a Banach space, every generating cone is a B-cone.)

An *order interval* in E is a set $[x, y] = \{z: x \leqq z \leqq y\}$. We note that if K is normal, every order interval is bounded.

DEFINITION. *A point x is quasi-interior to K if the order interval $[0, x]$ is a total subset of E.*

It is clear that every interior point of K is quasi-interior, and that every quasi-interior point of K is a non-support point of K in the sense of V. L. Klee. If K has non-empty interior, the three notions coincide; this is the case, in particular, if E is finite dimensional and K is total (hence K, resp. $K + iK$ if E is complex, is generating) in E.

[4] i.e., a binary relation which is reflexive and transitive. We assume always that $E \neq \{0\}$.

[5] K is proper if and only if the order relation is anti-symmetric.

[6] $S \subset E$ is symmetric if $x \in S$ implies $-x \in S$. In the (present) case of a normed space, K is a B-cone if and only if there exists an $m > 0$ such that every x in the unit ball U of E is of the form $x = \lim_{n \to \infty} (u_n - v_n)$ with $u_n, v_n \in K \cap mU$.

LEMMA 2. *Let P be a continuous projection in E such that $PK \subset K$. If $x \in PK$ is quasi-interior to K, it is quasi-interior to PK in PE.*

It is readily observed that $[0, x] \cap PE = P[0, x]$ under the conditions stated; since the linear hull of $[0, x]$ is dense in E, it follows that the linear hull of $P[0, x]$ is dense in PE.

A bounded endomorphism T of E is a *positive operator* if the positive cone K is invariant under T, i.e., if $TK \subset K$. The *spectral radius* r of T is the maximum modulus of the points in its spectrum[7] $\sigma(T)$. The complement of $\sigma(T)$ in the complex plane is denoted by $\rho(T)$, and the resolvent $(\lambda - T)^{-1}$, locally holomorphic in $\rho(T)$, by R_λ. The point spectrum of T is the set of all its *characteristic numbers*, i.e., the set of those λ for which $\lambda - T$ fails to be (1,1). For a characteristic number λ, $d(\lambda)$ denotes the (linear) dimension of the kernel of $\lambda - T$ (the *characteristic space*); an $x \neq 0$ in this kernel is called a *characteristic vector* (of T for λ). It is well known that every pole of the resolvent is a characteristic number of T.

If T is a positive operator, then so is its adjoint T' with respect to the conjugate cone K', which is a proper cone in E' if and only if K is total in E.

DEFINITION. *A positive operator T is quasi-interior if there exists $\lambda > r$* (r the spectral radius of T) *such that $TR_\lambda x$ is quasi-interior to K for every x, $0 \neq x \in K$.*[8]

This condition on T is not stronger than requiring that for each $x, 0 \neq x \in K$, the union of order intervals $\bigcup_{n=1}^{\infty} [0, T^n x]$ be total in E. (It is clear that K is a total cone in E if the set of quasi-interior positive operators on E is not empty.)

LEMMA 3. *If K is a normal B-cone or, more generally, if K and K'* (K' in the strong dual E') *are normal cones, then the set $\mathfrak{K}$ of all positive operators is a normal cone in the Banach space $\mathfrak{L}(E)$ of bounded endomorphisms of E.*

It is known [**17**, p. 269] that the assertion holds if K is a normal B-cone in E. If K and K' are both normal, then K' is a normal B-cone for the strong topology on E' (this follows from Lemma 1 and the subsequent remark); therefore by Lemma 1, the cone K'' conjugate to K' in the Banach space E'', bidual of E, is a normal B-cone. Thus the cone $\mathfrak{K}''$ of positive operators on E'' (with respect to K'') is normal

[7] If E is a real space, the terms spectrum, resolvent etc. will be understood with respect to the extension of T to the complexification of E, which may be considered as ordered with positive cone K or $K + iK$.

[8] E.g., if $E = l_2$, K the cone of all vectors with non-negative coordinates, a bounded matrix $A = (a_{i,k})$ with non-negative elements is quasi-interior if and only if for each pair (i, k) of indices, there exists $n = n(i, k)$ such that $(A^n)_{i,k} > 0$. Cf. [**13**].

in $\mathfrak{L}(E'')$ and this implies that $\mathfrak{K}$ is normal in $\mathfrak{L}(E)$ because the norm-preserving natural imbedding of $\mathfrak{L}(E)$ into $\mathfrak{L}(E'')$ maps $\mathfrak{K}$ into $\mathfrak{K}''$.

2. Some properties of the spectral radius. Throughout this section, E denotes a (real or complex) partially ordered Banach space with positive cone K; E' is the (topological) dual of E, equipped with the strong topology unless otherwise stated. T is a positive operator on E with spectral radius r.

The first part of the following proposition is due to Bonsall [3, p. 148] but the proof given here, which also yields the second assertion, is entirely different from that in [3].

PROPOSITION 1. *Let K and K' be normal cones in E* resp. *E'. For each positive operator T, r is in the spectrum of T. If r is a pole of the resolvent R_λ of order k, every other pole of R_λ on $|\lambda| = r$ is of an order $\leqq k$.*

Proof. It follows from Lemma 3 that the cone $\mathfrak{K}$ of positive operators is normal in $\mathfrak{L}(E)$ with respect to the uniform topology. It is shown in [18] that if $z \to f(z)$ is an analytic function with values in a Banach space, holomorphic at 0, such that its expansion at 0, $\sum_{n=0}^{\infty} a_n z^n$, has radius of convergence 1 and the set of coefficients $\{a_n\}$ is contained in a normal cone, then $z = 1$ is singular for f and if it is a pole of order k, there is no pole of f on $|z| = 1$ of order $> k$. The proposition follows immediately by letting $f(z) = R(r/z)$ if $r > 0$ ($R_\lambda = R(\lambda)$ the resolvent of T). If $r = 0$, the result is trivial.

PROPOSITION 2. *R_λ is a positive operator for each (real) $\lambda > r$; if R_λ is positive for some $\lambda \in \rho(T)$, then λ is real and > 0.*[9] *If K, K' are normal* (hence, if K is a normal B-cone), *then $\lambda > r$ is a necessary and sufficient condition in order that R_λ be positive.*

Proof. From the expansion of R_λ at ∞, it is easily seen that the condition $\lambda > r$ is sufficient. Now assume that for some $\lambda \in \rho(T)$, R_λ is a positive operator. Select an $x_0 \in K$, $x_0 \neq 0$, and define recursively $x_n = R_\lambda x_{n-1} (n \in N)$.[10] Each x_n satisfies the equation

$$(*) \qquad \lambda x_n = T x_n + x_{n-1} \, .$$

We have $x_n \in K (n \in N)$ and since $x_n = 0$ for some n would imply $x_0 = 0$, $x_n \neq 0$ for all n. From (*) it follows that $\lambda x_1 \in K$, and by induction it is established that $\lambda^n x_n \in K$, $\lambda^{n-1} x_n \in K$ for all $n \in N$. Also,

[9] For this statement, we have to assume that $K \neq \{0\}$.

[10] N stands for the set of positive integers.

$$\lambda^n x_n \geqq \lambda^{n-1} x_{n-1} \geqq x_0 \qquad (n \in N) .$$

Thus $\lambda \neq 0$ and without loss of generality, we may assume that $|\lambda| = 1$. (For if R_λ is positive at $\lambda \neq 0$, then the resolvent of $|\lambda^{-1}| T$ is positive at $\lambda |\lambda^{-1}|$.) Let $\lambda = e^{i\varphi}$, $0 \leqq \varphi < 2\pi$, and suppose that $\varphi > 0$. It is clear that $n\varphi \neq \pi (n \in N)$ or K would not be a proper cone. Hence there is an $n_0 \in N$ such that the triangle in the complex plane with vertices 1, $e^{i(n_0-1)\varphi}$, $e^{in_0\varphi}$ contains 0 in its interior. Consider the 2-dimensional real subspace L of E (resp. of $E + iE$)[7] containing x_{n_0} and ix_{n_0}. $K \cap L$ (resp. $(K + iK) \cap L$) is a proper convex cone of vertex 0 in L containing the points x_{n_0}, $\lambda^{n_0-1}x_{n_0}$, $\lambda^{n_0}x_{n_0}$. Hence this cone contains 0 as an interior point in L which is contradictory. Thus $\varphi = 0$, and $\lambda > 0$.

Let K and K' be normal in E resp. E'; then the cone $\mathfrak{K}$ of positive operators is normal in $\mathfrak{L}(E)$ by Lemma 3. If we had $R_\lambda \in \mathfrak{K}$ for some λ, $0 < \lambda < r$, from the resolvent equation

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

it would follow that $R_\mu \leqq R_\lambda$ (with respect to the order relation on $\mathfrak{L}(E)$ whose positive cone is $\mathfrak{K}$) for all $\mu > \lambda$, for which $R_\mu \geqq 0$ therefore, in particular, for all $\mu > r$. This would imply $\| R_\mu \| \leqq \| R_\lambda \|$ for all $\mu > r$ and some real norm $A \to \| A \|$ generating the topology of bounded convergence on $\mathfrak{L}(E)$. This is impossible since $r \in \sigma(T)$ by Prop. 1 and consequently, $\| R_\mu \| \to \infty$ as $\mu \downarrow r$. The proof is finished.

PROPOSITION 3. *If there exists y, $0 \neq y \in K$, such that $T^p y \geqq \delta y$ for some $p \in N$ and $\delta > 0$, then $r \geqq \delta^{1/p}$.*

Proof. Since K is closed and $\neq E$, a routine argument shows that there exists a continuous linear form $h \in E'$ such that the real part $f(x) = \operatorname{Re} h(x)$ is $\geqq 0$ on K and $f(y) > 0$. For $\lambda > r$, we have

$$f(R_\lambda y) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} f(T^n y) \geqq \sum_{k=1}^{\infty} \frac{1}{\lambda^{kp+1}} f(T^{kp} y) \geqq f(y) \sum_{k=1}^{\infty} \frac{\delta^k}{\lambda^{kp+1}} = f(y) \cdot \frac{\delta}{\lambda(\lambda^p - \delta)}$$

because $T^p y \geqq \delta y$ implies $T^{kp} y \geqq \delta^k y (k \in N)$. It follows that $f(R_\lambda y)$ is unbounded as $\lambda^p \downarrow \delta$. Consequently $r \geqq \delta^{1/p}$.

THEOREM 1. *Let K be spanned by a convex set not containing 0 and compact for some locally convex topology (on E) for which T is continuous on K*[11]. *There exists a non-negative characteristic number*

[11] i.e., for which the restriction of T to K is continuous.

of T with (at least one) *characteristic vector in K. If in addition K is a normal cone generating E, then r is such a number.*[12]

Proof. Let C be the convex set and $\mathfrak{T}$ the locally convex topology in question. There exists a $\mathfrak{T}$-closed real hyperplane $H = \{x: f(x) = 1\}$ separating C strictly from 0. It is clear that $f(x) > 0$ for $0 \neq x \in K$. K is closed for $\mathfrak{T}$: Let $\boldsymbol{F}$ be a filter on K converging to $x_0 \in E$ for $\mathfrak{T}$; since f is continuous, there exists $F \in \boldsymbol{F}$ such that $\sup\{f(x): x \in F\} \leqq 1 + f(x_0)$, therefore $F \subset (1 + f(x_0))C_1$, where C_1 is the convex hull of $\{0\}$ and C. Since C_1 is compact[13], x_0 which is in the closure of F, is in K. Because $H \cap K$ is a closed subset of C_1, $H \cap K$ is compact; so $f(x_n) \rightarrow 0$ implies $x_n \rightarrow 0$ and thus $Tx_n \rightarrow 0$ for any sequence $\{x_n\} \subset K$, (all statements in this sentence referring to $\mathfrak{T}$).

Consider the real subspace $\hat{E} = K - K$ of E, equipped with the norm

$$z \rightarrow \| z \| = \inf\{f(x) + f(y): z = x - y;\, x, y \in K\} .$$

$\hat{E}$ is a Banach space. Given an arbitrary Cauchy sequence in $\hat{E}$, there exists a subsequence $\{z_k\}$ such that $\| z_{k+1} - z_k \| < 1/2^k$. By definition of the norm in $\hat{E}$, there exist two sequences $\{x_k\}$, $\{y_k\}$ in K with $z_{k+1} - z_k = x_k - y_k (k \in N)$ and $\| x_k \| + \| y_k \| \leqq 1/2^k$. Since C_1 is compact for $\mathfrak{T}$, the sequence

$$\left\{\sum_{\nu=1}^{n} x_\nu : n \in N\right\}\left(\text{resp.}\left\{\sum_{\nu=1}^{n} y_\nu : n \in N\right\}\right)$$

has a limit point x (resp. y) in K, and it is now easy to see that $\{z_k\}$ (and hence the given sequence) converges to $x - y$, in $\hat{E}$. It is readily verified that the restriction $\hat{T}$ of T to $\hat{E}$ is a continuous endomorphism. Moreover, K is a normal closed cone in $\hat{E}$, and it is a B-cone since it is generating (cf. the remark following Lemma 1). If $\hat{r}$ is the spectral radius of $\hat{T}$, we have $\hat{r} \in \sigma(\hat{T})$ by Prop. 1. Thus, since $\hat{R}_\lambda x$ is non-decreasing for each $x \in K$ if $\lambda \downarrow \hat{r}$, we have $\| \hat{R}_\lambda y \| \rightarrow \infty$ for some $y \in K$ as $\lambda \downarrow \hat{r}$. Let $\lambda_n \downarrow \hat{r}$ and set $x_n = \hat{R}(\lambda_n)y / \| \hat{R}(\lambda_n)y \|$. Then $\lambda_n x_n - \hat{T}x_n \rightarrow 0$ in $\hat{E}$ and also $(\hat{r} - \hat{T})x_n \rightarrow 0$ because of $\| x_n \| = 1$. By Proposition 2, $x_n \in K$; and, since $1 = \| x_n \| = f(x_n)$, it follows that $x_n \in H \cap K (n \in N)$. Now $H \cap K$ is compact for $\mathfrak{T}$ and as $\hat{r} - \hat{T}$ is continuous for $\mathfrak{T}$ on K, it follows that $(\hat{r} - \hat{T})x = 0$ for some $x \in H \cap K$. The proof of the first part is finished.

[12] The assumption that K be closed in E is not needed in Th. 1 and the corollary; the first assertion of Th. 1 is also independent of E being a Banach space and of T being bounded.

[13] In any linear topological space, the convex hull of a finite number of convex compact sets is compact. A locally convex topology is assumed to be Hausdorff by definition.

If K is a normal generating cone in E, then $r \in \sigma(T)$ by Prop. 1. It is clear that $\hat{r} \leqq r$. On the other hand, $\hat{r} < r$ would imply that $r - T$ is an algebraical automorphism of E, which is impossible.

REMARK. Using the notation of the preceding proof, the number $\hat{r}$ (which was shown to be in the point spectrum of T) may be characterized as follows:

(a) *$\hat{r}$ is the greatest real number α such that $\alpha - T$ is not an algebraical automorphism of the real subspace $K - K$ of E.*

(b) *$\hat{r}$ is the smallest real number α such that R_λ is positive for $\lambda > \alpha$, $\lambda \in \rho(T)$.*

(c) *If g is a real $\mathfrak{T}$-continuous linear form on E with $0 \notin g(C)$, then*

$$\hat{r} = \lim_{n\to\infty} \{\sup | g(T^n x) | : x \in C\}^{1/n} .$$

As an application of Th. 1, we list a proposition which is equivalent to the combination of [2, Th. 1] and [4, Th. C].

COROLLARY. *If K has non-empty interior, there exists a non-negative number in $\sigma(T)$ which is a characteristic number of T' with* (at least one) *characteristic vector in K'. If in addition K is normal, then r is such a number.*

Proof. If x_0 is interior to K, the real hyperplane $H = \{x' \in E' : \operatorname{Re}\langle x', x_0\rangle = 1\}$ intersects K' is a set compact for the weak* topology on E'. For the linear forms in this intersection are uniformly bounded on the order interval $[0, x_0]$ (which has interior points), hence equicontinuous. Obviously $H \cap K'$ spans K', and T' is continuous for the weak* topology. The assertion concerning T follows from $\sigma(T) = \sigma(T')$. Finally, if in addition K is normal, K' is a normal (B)-cone in E' spanning E' by Lemma 1 which completes the proof.

REMARK. If K is normal with non-empty interior $\mathring{K}$, then for each $x_0 \in \mathring{K}$, the norm $A \to \| A \|_{x_0} = \sup \{\| Ax \| : x \in [0, x_0]\}$ generates the topology of bounded convergence on $\mathfrak{L}(E)$. For a positive operator and a norm on E which is monotone on K, $\| T \|_{x_0} = \| Tx_0 \|$. Thus:

If K is normal with $\mathring{K} \neq \phi$ (and T positive), then

$$r = \lim_{n\to\infty} \| T^n x_0 \|^{1/n}$$

for every $x_0 \in \mathring{K}$.

3. Operators for which r is a pole of R_λ. As in § 2, E denotes a (real or complex) partially ordered Banach space; but we shall assume that T is a positive operator for which the spectral radius r is a pole of the resolvent R_λ. The positive cone K is assumed proper and closed.

PROPOSITION 4. *The leading coefficient in the principal part of R_λ at $\lambda = r$ is a positive operator. Hence, if K is total in E, there exists* (at least) *one characteristic vector of T for r in K, and of T' for r in K'.*

Proof. Since the leading coefficient in the principal part of R_λ is the limit[14] (r being a pole of order k) of $(\lambda - r)^k R_\lambda$ as $\lambda \downarrow r$, the first assertion follows from the facts that R_λ is positive for $\lambda > r$ and that K is closed in E. Further, if K is a closed proper cone total in E, then K' is a closed proper cone weak* total in E'. The remainder is clear.

THEOREM 2. *Let T be quasi-interior. Then:*

1°. *$r > 0$ and r is a simple pole of R_λ.*

2°. *Every characteristic vector pertaining to r, of T in K* (resp. *of T' in K'*) *is quasi-interior to K* (resp. *a strictly positive linear form*).

3°. *Each of these conditions implies that $d(r) = 1$:*
   (a) *K has non-empty interior*
   (b) *$d(r)$ is finite*
   (c) *E is a Banach lattice.*[15]

Proof. The assumption $r = 0$ implies, by Prop. 4, that $Tx = 0$ for some x, $0 \neq x \in K$. (Since T is a quasi-interior map, K has quasi-interior points and is therefore total in E.) But then $TR_\lambda x = 0$ for every $\lambda \in \rho(T)$ which contradicts the definition of a quasi-interior map. Hence $r > 0$.

Let x_0, $0 \neq x_0 \in K$, be a characteristic vector of T for r. By definition, there exists $\lambda > r$ such that $TR_\lambda x_0$ is quasi-interior to K. From

$$TR_\lambda x_0 = \sum_1^\infty \frac{1}{\lambda^n} T^n x_0 = x_0 \sum_1^\infty \left(\frac{r}{\lambda}\right)^n$$

it follows that x_0 is quasi-interior to K. Similarly, if f is a characteristic vector of T' in K' for r, we have $r^n f(x) = f(T^n x) (n \in N)$ for $x \in E$, hence with $f_1(x) = \operatorname{Re} f(x)$

$$f_1(x) \sum_1^\infty \left(\frac{r}{\lambda}\right)^n = \sum_1^\infty \frac{1}{\lambda^n} f_1(T^n x) = f_1(TR_\lambda x) > 0$$

[14] For the topology of bounded convergence.

[15] In the sense of G. Birkhoff (Lattice Theory, New York 1948). A Banach lattice is by definition a real space; for our purposes, it is sufficient to assume that the underlying real space of E is a Banach lattice.

for every $0 \neq x \in K$, for f_1 must be > 0 at every quasi-interior point of K.

We show that r is a simple pole of R_λ. Let k be the order of r; if A is the leading coefficient in the principal part of R_λ at $\lambda = r$, we have $A = P(T - r)^{k-1}$ where

$$P = \frac{1}{2\pi i}\int_C R_\lambda d\lambda$$

(C a positively oriented circle enclosing r, and having no other elements of $\sigma(T)$ in its interior or on its boundary), is the continuous projection of E onto the subspace pertaining to the spectral set $\{r\}$. K being total in E, we have $Av \neq 0$ for some $v \in K$ and Av is quasi-interior to K by 2°. Let $f \in K'$ be a characteristic vector of T' for r (Prop. 4), then $P'f = f$ (P' the adjoint of P) and

$$f_1(Av) = f_1[(T - r)^{k-1}v] = [(T' - r)^{k-1}f]_1(v) > 0$$

which implies $k = 1$. Therefore, r is a simple pole.

We show now that 3°. holds. Since r is a simple pole of R_λ, P is a positive operator by Prop. 4. If $x_0 \in K$ is a characteristic vector of T for r, x_0 is quasi-interior to K by 2°. Therefore, the cone PK can have no boundary points $\neq 0$ which are not quasi-interior to PK in PE by Lemma 2. If a) K has interior points, then so has PK in PE; thus we must have $d(r) = 1$. If b) $d(r)$ is finite, i.e., if P is of finite rank, then every quasi-interior point of PK is actually interior to PK in PE and the conclusion is the same.

There remains to show that 3°. c) is sufficient for $d(r) = 1$. Let x_0 be any characteristic vector of T for r. We have $rx_0 = Tx_0$ and consequently $r|x_0| \leqq T|x_0|$, $|x_0|$ denoting the absolute of x_0 in the lattice-theoretic sense. If in the latter relation equality does not hold, we obtain

$$rf_1(|x_0|) < f_1(T|x_0|) = rf_1(|x_0|)$$

for every characteristic vector $f \in K'$ of T' for r (f is then strictly positive by 2°). This is contradictory; hence, $r|x_0| = T|x_0|$ for every characteristic vector x_0, whether or not in K, of T for r. Now $x_0 = x_0^+ - x_0^-$ where the summands are disjoint. Since $|x_0| = x_0^+ + x_0^-$, x_0^+ and x_0^- are both in the characteristic space of T pertaining to r. Assume that for some x_0, both $x_0^+ \neq 0$ and $x_0^- \neq 0$. Since the order interval $[0, x_0^+]$ is disjoint from x_0^- and the lattice operations are continuous, x_0^+ cannot be quasi-interior to K which contradicts 2°.[16] Consequently, either

[16] It becomes clear from this that if E is a Banach lattice, the points quasi-interior to K are weak units of E in the sense of Birkhoff (l.c.).

$x_0^+ = 0$ or $x_0^- = 0$. This implies that for each characteristic vector of T in K (for r), either $x_0 \in K$ or $x_0 \in -K$; therefore $d(r) = 1$.

The theorem is proved.

If the assumptions that T be quasi-interior and r be a pole of R_λ are satisfied, r need not be the only element of $\sigma(T)$ on $|\lambda| = r$ even if E is finite dimensional. For let E be Euclidean 2-space in its natural order (i.e., K being the set of all vectors with non-negative coordinates). The positive operator on E represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is quasi-interior: for $\lambda = 2$, R_λ is the matrix $1/3 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The characteristic numbers of T are 1 and -1.

PROPOSITION 5. *Let T be such that for each $x, 0 \neq x \in K$, there exists a positive integer $n = n(x)$ for which $T^n x$ is an interior point of K.*[17] *Then r is the only element in the point spectrum of T on $|\lambda| = r$.*

Proof. We note first that if T has the stated property and E is a real space, the extension of T to the complexification $E + iE$ has the same property provided $E + iE$ is considered as partially ordered with positive cone $K + iK$. Hence we assume E as complex.

By Theorem 2 (since T is obviously quasi-interior) there exists x_0 interior to K with $rx_0 = Tx_0$. Because of $r > 0$, we may assume that $r = 1$. Suppose that for some $\varphi, 0 < \varphi < 2\pi, e^{i\varphi}$ is in the point spectrum of T and $e^{i\varphi}x = Tx (x \neq 0)$. Consider the 3-dimensional real subspace E_3 of E that contains x_0, x, ix; obviously E_3 is invariant under T. x_0, which is an interior point of K, is interior to $K_3 = K \cap E_3$ in E_3. Identifying E_3 (which we may for our purpose) as Euclidean 3-space with coordinate axes x_0, x, ix, the restriction of T to E_3 is a rotation through φ about x_0. Let $w \neq 0$ be a point of K_3 which has maximum angular distance from x_0; then $T^n w$ must have the same property for every $n \in N$. This implies that no $T^n w (n \in N)$ is interior to K_3 and a contradiction is established.

4. Problems. Let E be a partially ordered Banach space with positive cone K, T a positive operator on E with spectral radius r. Under what general conditions, if any, are these implications true:

a. If r is an isolated singularity of R_λ, every singularity of R_λ on $|\lambda| = r$ is isolated.

b. If r is a pole of R_λ, R_λ has no singularities on $|\lambda| = r$ other than poles.[18]

[17] Operators T with this property are called *strongly positive* in **[11]**.

[18] E.g., are a. and b. true if K is a normal B-cone in E?

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 10, No. 3 November, 1960