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**ON GROUPS OF EXPONENT FOUR WITH GENERATORS OF  
ORDER TWO**

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# ON GROUPS OF EXPONENT FOUR WITH GENERATORS OF ORDER TWO

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1. If  $x, y, \dots$  are elements of a group  $G$ , we define the *commutator*  $(x, y)$  of  $x$  and  $y$  by  $(x, y) = x^{-1}y^{-1}xy$ . More generally, we define *extended commutators* inductively by  $(x, \dots, y, z) = ((x, \dots, y), z)$ . In this paper we shall also be concerned with higher commutators of type  $((a_1, \dots, a_s), (b_1, \dots, b_t), \dots, (c_1, \dots, c_r))$  which we denote by  $(a_1, \dots, a_s; b_1, \dots, b_t; \dots; c_1, \dots, c_r)$ . If we let  $G_i$  be the subgroup of  $G$  which is generated by all extended commutators of length  $i$ , (i.e., with  $i$  entries), then  $G_i$  is a characteristic subgroup of  $G$ , and the series  $G = G_1 \supset G_2 \supset \dots$  is called the *lower central series* of  $G$ .<sup>1</sup>

Let  $G(n)$  ( $n = 1, 2, \dots$ ) be the freest group of exponent 4 on  $n$  generators of order 2. That is,  $G(n)$  is a group in which the fourth power of every element is the identity, 1,  $G(n)$  is generated by  $n$  elements of order 2, and if  $H$  is any other group with these properties, then  $H$  is a homomorphic image of  $G(n)$ .

We prove  $G(n)_{n+2} = 1$ . For this purpose it may be assumed, since  $G(n)$  is finite<sup>2</sup> and hence nilpotent, that  $G(n)_{n+3} = 1$ . Moreover, it will be enough to show  $(x_1, \dots, x_{n+2}) = 1$  for all choices of  $x_1, \dots, x_{n+2}$  from among the generators of  $G(n)$ .

2. LEMMA 2.1. *If  $x, y, \dots, z$  are elements of order 2 in a group of exponent 4, then  $(x, y)^2 = 1$ ,  $(x, y, \dots, z)^2 = 1$ , and  $(x, y, x) = 1$ .*

*Proof.* Since  $(x, y) = xyxy = (xy)^2$ ,  $(x, y)^2 = 1$ . By induction,  $(x, y, \dots, z)^2 = 1$ , while  $(y, x) = yxyx = x(x, y)x = (x, y)(x, y, x)$ , so that  $(x, y, x) = (y, x)^2 = 1$ .

The relation  $(x, y, \dots, z)^2 = 1$  will be the justification for future substitutions and will be used without specific mention.

THEOREM 2.1.  $G(2)_3 = 1$ .

*Proof.* By Lemma 2.1, if the generators of  $G(2)$  are  $a$  and  $b$ , then  $(a, b, a) = (b, a, a) = (a, b, b) = (b, a, b) = 1$ .

3. LEMMA 3.1. *If  $a, b$  and  $c$  are elements of order 2 in a group  $G$  of exponent 4, then*

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<sup>1</sup> For properties of commutators and the lower central series see Hall, [1], Ch. 10.

<sup>2</sup> See Sanov, [2], or Hall, [1], pp. 324-325.

$$(1) \quad (a, b, c) \equiv (b, c, a)(c, a, b) \pmod{G_5}$$

$$(2) \quad (a, b; c, a) = (a, c; b, a) \equiv (a, c, b, a) \pmod{G_5}$$

$$(3) \quad (a, b, c, a) \equiv (b, c, a, b)(c, a, b, c) \pmod{G_5}.$$

*Proof.* We may assume that  $a, b$  and  $c$  generate  $G$ . Now

$$abcabc = aba(a, c)b(b, c) = (a, b)(a, c)(a, c, b)(b, c).$$

Thus, modulo  $G_5$ ,  $(abc)^2 = (a, b)(a, c)(b, c)(a, c, b)$ . Hence

$$1 \equiv [(a, b)(a, c)(b, c)]^2 \pmod{G_5}, \text{ so that, modulo } G_5,$$

$$1 = (a, b)(a, c)(b, c)(a, b)(a, c)(b, c) = (a, b)(a, c)(a, b)(a, b; b, c)(a, c)(a, c; b, c),$$

$$(4) \quad 1 \equiv (a, b; a, c)(a, b; b, c)(a, c; b, c) \pmod{G_5}.$$

But also

$$\begin{aligned} abc &= ca(a, c)b(b, c) \\ &= bc(c, b)a(a, b)(a, c)(a, c, b)(b, c) \\ &= ab(b, a)c(c, a)(c, b)(c, b, a)(a, b)(a, c)(a, c, b)(b, c), \end{aligned}$$

so that  $1 = (b, a)(b, a, c)(c, a)(c, b)(c, b, a)(a, b)(a, c)(a, c, b)(b, c)$ , and hence, modulo  $G_5$ ,

$$\begin{aligned} 1 &= (b, a)(c, a)(c, b)(a, b)(a, c)(b, c)(b, a, c)(c, b, a)(a, c, b) \\ &= [(a, b)(a, c)(b, c)]^2(a, b, c)(b, c, a)(c, a, b). \end{aligned}$$

Thus (1) is proved. Replacing  $b$  by  $(a, b)$  in (1) gives  $(a, b, c, a)(c, a; a, b) \equiv 1 \pmod{G_5}$  or (2). And (2) and (4) together give (3).

**LEMMA 3.2.** *If  $x_1, \dots, x_k$  and  $a$  are elements of order 2 in a group  $G$  of exponent 4, then  $(x_1, \dots, x_k, a) \equiv X \pmod{G_{k+2}}$ , where  $X$  is a product of commutators of form  $(a, y_1, \dots, y_k)$  with  $y_1, \dots, y_k$  from among  $x_1, \dots, x_k$ .*

**COROLLARY.** *If  $x_1, \dots, x_k, z_1, \dots, z_s$  and  $a$  are elements of order 2 in a group  $G$  of exponent 4, then*

$$(x_1, \dots, x_k, a, z_1, \dots, z_s) \equiv X \pmod{G_{k+s+2}}$$

*where  $X$  is a product of commutators of form  $(a, y_1, \dots, y_k, z_1, \dots, z_s)$  with  $y_1, \dots, y_k$  from among  $x_1, \dots, x_k$ .*

*Proof of Lemma 3.2.* Certainly the lemma and corollary are true if  $k = 1$ . Assume for induction that both are true for  $k = n - 1 \geq 1$ .

Now by (1), modulo  $G_{n+2}$ ,  $(x_1, \dots, x_{n-1}, x_n, a) = (x_1, \dots, x_{n-1}, a, x_n)(x_1, \dots, x_{n-1}; a, x_n)$ . But by the inductive assumption  $(x_1, \dots, x_{n-1}, a, x_n)$  is a product of terms  $(a, y_1, \dots, y_{n-1}, x_n)$ , and  $(x_1, \dots, x_{n-1}; a, x_n)$  is a product of terms  $(a, x_n, y_1, \dots, y_{n-1})$ . The lemma and its immediate corollary follow by induction.

**THEOREM 3.1.**  $G(3)_5 = 1$ .

*Proof.* Let  $a, b$  and  $c$  be the generators of  $G(3)$ . Consider any commutator  $C = (x_1, x_2, x_3, x_4, x_5)$  in arguments  $a, b$  and  $c$ . We show  $C = 1$ . There is no loss of generality in taking  $x_5 = a$ . If  $a$  does not appear again in  $C$ , then by Theorem 2.1,  $C = (1, x_5) = 1$ . If  $a$  appears again, then by Lemma 3.2 and the assumption that  $G(3)_6 = 1$ , we may suppose  $C = (a, x_2, x_3, x_4, a)$ . By Lemma 2.1, if  $a$  appears a third time, then  $C = 1$ . Thus we may take  $C = (a, b, c, b, a)$ . Now  $(a, b, c, b, a) = (b, c, a, b, a)(c, a, b, b, a) = (b, c, a, b, a)$  by (1). Replacing  $c$  by  $(b, c)$  in (3) gives  $(a, b; b, c; a) = (b; b, c; a; b) = 1$ , while replacing  $c$  by  $(b, c)$  in (2) gives  $(a, b; b, c; a) = (b, c, a, b, a)$ . Hence,  $C = (a, b, c, b, a) = (b, c, a, b, a) = (a, b; b, c; a) = 1$ , and the theorem is proved.

**COROLLARY 1.** *If  $a, b$  and  $c$  are elements of order 2 in a group of exponent 4, then*

- (1')  $(a, b, c) = (b, c, a)(c, a, b)$
- (2')  $(a, b; c, a) = (a, b, c, a)$
- (3')  $(a, b, c, a) = (b, c, a, b)(c, a, b, c)$

*Proof.* These follow from Lemma 3.1.

**COROLLARY 2.** *If  $x_1, \dots, x_k, y_1, \dots, y_s, z_1, \dots, z_t$  ( $s \geq 2$ ) are elements of order 2 in a group  $G$  of exponent 4, then*

$$(x_1, \dots, x_k; y_1, \dots, y_s; z_1; \dots; z_t) \equiv AB \pmod{G_{k+s+t+1}}$$

where

$$\begin{aligned} A &= (x_1, \dots, x_k; y_1, \dots, y_{s-1}; y_s; z_1; \dots; z_t) \\ B &= (x_1, \dots, x_k; y_s; y_1, \dots, y_{s-1}; z_1; \dots; z_t) . \end{aligned}$$

*Proof.* This follows from (1').

The following corollary lists some relations for future use.

**COROLLARY 3.** *If  $a, b, c, d$  and  $f$  are elements of order 2 in a group  $G$  of exponent 4, then*

$$(5) \quad (a, b, c, d, e) \equiv (a, b, d, c, d) \pmod{G_6}$$

$$(6) \quad (b, c, a; d, f, a) \equiv 1 \pmod{G_7}$$

$$(7) \quad (a, f; b, d, c) \equiv (a, f, c; b, d)(a, f; b, d; c)$$

$$(8) \quad (b, f, d; a, c)(d, f, b; a, c) \equiv (b, d, f; a, c) \pmod{G_8}.$$

*Proof.* By (3'), with  $a$  replaced by  $(a, b)$  and  $b$  replaced by  $d$ ,  $(a, b, d, c; a, b) = (d, c; a, b; d)(c; a, b; d; c) = (a, b; d, c; d)(a, b, c, d, c)$ , so that, since  $(a, b; d, c; d) = (a, b, d, c, d)$ , (5) is true. By (2') and (3') with  $b$  replaced by  $(b, c)$  and  $c$  replaced by  $(d, f)$ ,  $(b, c, a; d, f, a) = (a; b, c; d, f; a) = (b, c; d, f; a; b, c)(d, f; b, c; a; d, f)$ , so that (6) is true. Finally, (7) and (8) are obvious from (1').

4. LEMMA 4.1. *If  $a, b, c$  and  $d$  are elements of order 2 in a group  $G$  of exponent 4, then*

$$(9) \quad (a, b; c, d) \equiv (a, c; b, d)(a, d; b, c) \pmod{G_5}.$$

*Proof.* First, working modulo  $G_5$  and collecting as we did in the proof of Lemma 3.1 we obtain  $(abcd)^2 = T_2 T_3 T_4$  where

$$T_2 = (a, b)(a, c)(b, c)(a, d)(b, d)(c, d)$$

$$T_3 = (a, c, b)(a, d, c)(a, d, b)(b, d, c)$$

$$T_4 = (a, d, b, c).$$

Note that modulo  $G_5$ ,  $T_2, T_3$  and  $T_4$  commute, and  $T_3^2 = T_4^2 = 1$ . Hence, modulo  $G_5$ ,  $1 = (abcd)^4 = T_2^2$ . Collecting the  $(a, d)$ 's in  $T_2^2$  we obtain  $1 \equiv XABCY \pmod{G_5}$ , where

$$X = [(a, b)(a, c)(b, c)]^2$$

$$A = (b, c; b, d)(b, c; c, d)(b, d; c, d)$$

$$B = (a, c; a, d)(a, c; c, d)(a, d; c, d)$$

$$C = (a, b; a, d)(a, b; b, d)(a, d; b, d)$$

$$Y = (a, b; c, d)(a, c; b, d)(a, d; b, c).$$

Now modulo  $G_5$ ,  $X = 1$ , while  $A = B = C = 1$  by (2') and (3'). Hence,  $1 \equiv (a, b; c, d)(a, c; b, d)(a, d; b, c) \pmod{G_5}$ , which is (9).

COROLLARY 1. *If  $x_1, \dots, x_k$  and  $a$  are elements of order 2 in a group  $G$  of exponent 4, then for  $i = 2, \dots, k$ ,*

$$(x_1, a, x_2, a, \dots, x_i, \dots, x_k) \equiv (x_1, x_2, \dots, a, x_i, a, \dots, x_k) \pmod{G_{k+3}}.$$

Hence, if two of  $x_1, \dots, x_k, a$  are equal,  $(x_1, a, x_2, a, \dots, x_k) \equiv 1 \pmod{G_{k+3}}$ .

*Proof.* Let  $a, b, c$  and  $d$  be elements of order 2 in  $G$ . Then modulo  $G_6$ ,

$$\begin{aligned} (b, a, c, a, d) &= (b, a; c, a; d) \\ &= (b, a, d; c, a)(c, a, d; b, a) \\ &= (b, a, c; d, a)(c, a, b; d, a) \\ &= (b, c, a; d, a) \\ &= (b, c, a, d, a). \end{aligned}$$

The first statement follows. Now the second statement is clearly true if  $a$  appears a third time, since then  $(x_1, a, x_2, a, \dots, a, \dots, x_k) = (x_1, x_2, \dots, a, a, a, \dots, x_k) = 1$ . If some  $x_i$  appears twice, then modulo  $G_{k+3}(x_1, a, x_2, a, \dots, x_i, \dots, x_k) = (x_1, \dots, a, x_i, a, \dots, x_k) = (x_1, x_2, \dots, x_i, a, x_i, \dots, x_k) = (x_1, x_i x_2, x_i, \dots, a, \dots, x_k)$  (the second step following from (5)), and we are back to the case of three appearances of  $a$ . Thus the corollary is proved.

**COROLLARY 2.** *If  $a, b, c, d$  and  $f$  are elements of order 2 in a group  $G$  of exponent 4, then*

$$(10) \quad 1 \equiv (a, f, b; c, d)(a, f, c; b, d)(a, f, d; b, c) \pmod{G_6}$$

$$(11) \quad (a, c; d, f; b)(a, d; c, f; b) \equiv (c, d; a, f; b) \pmod{G_8}.$$

*Proof.* These follow from (9).

**THEOREM 4.1.**  $G(4)_6 = 1$ .

*Proof.* Let the generators of  $G(4)$  be  $a, b, c$  and  $d$  and consider any commutator  $C = (x_1, x_2, x_3, x_4, x_5, x_6)$  in  $a, b, c$  and  $d$ . It will be sufficient to prove  $C = 1$  under the assumption that  $G(4)_7 = 1$ . As in the proof of Theorem 3.1, we may suppose that  $C = (a, x_2, x_3, x_4, x_5, a)$ . Moreover, if  $x_2, x_3, x_4$  or  $x_5$  is  $a$ , then by Theorem 2.1 or Corollary 1 of Lemma 4.1,  $C = 1$ . It will thus be sufficient to prove  $(a, b, c, b, d, a) = 1, (a, b, c, d, b; a) = 1$ , and  $(a, c, b, d, b, a) = 1$ . Now by Corollary 1 of Lemma 4.1,  $(a, b, c, b, d, a) = (a, c, b, d, b, a) = 1$ , while by (1'),  $(a, b, c; b, d, a) = (a, c, b; b, d; a)(b, c, a; b, d; a)$ , so that by (6)  $(a, b, c; b, d; a) = 1$ . Thus  $(a, b, c, d, b, a) = (a, b, c, b, d, a)(a, b, c; b, d; a) = 1$ , and the theorem is proved.

5. The main result, that  $G(n)_{n+2} = 1$ , has now been proved for  $n = 2, 3$  and 4. In this section we derive an identity analogous to (1) and (9) for five generators. This identity enables us to prove, in § 6, that  $G(n)_{n+2} = 1$  for  $n \geq 5$ .

**LEMMA 5.1.** *If  $a, b, c, d$  and  $f$  are elements of order 2 in a group  $G$  of exponent 4, then*

$$(12) \quad (a, b; c, d; f) \equiv (c, b; f, d; a)(f, b; a, d; c) \pmod{G_6} .$$

COROLLARY. *If  $(x_1, \dots, x_k), (y_1, \dots, y_j), (z_1, \dots, z_m), a$  and  $b$  ( $k, j, m \geq 1$ ) are elements of order 2 in a group  $G$  of exponent 4, then*

$$(13) \quad (x_1, \dots, x_k, a; y_1, \dots, y_j, b; z_1, \dots, z_m) \equiv C_1 C_2 \pmod{G_{k+j+m+3}} ,$$

where

$$C_1 = (y_1, \dots, y_j; z_1, \dots, z_m; x_1, \dots, x_k, b; a)$$

$$C_2 = (x_1, \dots, x_k; z_1, \dots, z_m; y_1, \dots, y_j, a; b) .$$

*Proof of Lemma 5.1.* First, working modulo  $G_5$ , we collect  $f$ 's in the expression  $(abcdf)^2$  to get  $(abcdf)^2 = (abcd)a(a, f)b(b, f)c(c, f)d(d, f)$ . Then collecting  $b, c$  and  $d$  in that order we obtain  $(abcdf)^2 = (abcd)^2 S_2 S_3 S_4$  where

$$S_2 = (a, f)(b, f)(c, f)(d, f)$$

$$S_3 = (a, f, d)(a, f, c)(a, f, b)(b, f, d)(b, f, c)(c, f, d)$$

$$S_4 = (a, f, c, d)(a, f, b, d)(a, f, b, c)(b, f, c, d) .$$

But as in the proof of Lemma 4.1,  $(abcd)^2 \equiv T_2 T_3 T_4 \pmod{G_5}$ , where

$$T_2 = (a, b)(a, c)(a, d)(b, d)(c, d)$$

$$T_3 = (a, c, b)(a, d, c)(a, d, b)(b, d, c)$$

$$T_4 = (a, d, b, c) .$$

Thus, modulo  $G_5$ ,  $(abcdf)^2 = T_2 T_3 T_4 S_2 S_3 S_4$ . But then, modulo  $G_6$ ,

$$1 = (abcdf)^4 = T_2 T_3 T_4 S_2 S_3 T_2 T_3 T_4 S_2 S_3$$

$$= T_2 T_3 T_4 T_2 S_2 (S_2, T_2) S_3 (S_3, T_2) T_3 T_4 S_2 S_3$$

$$= (T_2 T_3 T_4)^2 S_2^2 (S_2, T_3) (S_2, T_2) S_3 (S_3, T_2) S_2 S_3$$

$$= S_2 (S_2, T_3) (S_2, T_2) S_3 (S_3, T_2) S_2 S_3$$

$$= S_2^2 (S_2, T_3) (S_2, T_2) S_3 (S_3, S_2) (S_3, T_2) S_3$$

$$= S_2^2 (S_2, T_3) (S_2, T_2) S_3^2 (S_3, S_2) (S_3, T_2) .$$

But modulo  $G_6$ ,  $S_3^2 = 1$ , while  $S_2^2$  is a product of commutators of weight 4. Thus the last relation may be rewritten as  $1 \equiv A \pmod{G_6}$  where  $A$  is a product of commutators in  $a, b, c, d$  and  $f$  of weight 4 or 5; hence the factors of  $A$  commute modulo  $G_6$ . Let  $A'_a$  be the product of all factors of  $A$  which do *not* contain  $a$  as argument, and let  $A_a$  be the product of the remaining factors of  $A$ . Then  $1 \equiv A'_a A_a \pmod{G_6}$ , so that, setting  $a = 1$ ,  $1 \equiv A'_a \pmod{G_6}$ , and hence  $1 \equiv A_a \pmod{G_6}$ . Continuing this argument we finally arrive at  $1 \equiv A_{abcdf} \pmod{G_6}$ , where  $A_{abcdf}$  is the product of all factors of  $A$  which contain each of  $a, b, c, d$  and  $f$ . But what are

these factors? Clearly  $S_2^2$  and  $(S_2, T_2)$  do not contain any such factors; and since each factor of  $S_2$  and  $S_3$  contains  $f$ ,  $(S_3, S_2)$  cannot contain any such factors. We are left with  $(S_2, T_3)$  and  $(S_3, T_2)$ . The product of the desired factors of  $(S_2, T_3)$  is clearly

$$(a, f; b, d, c)(b, f; a, d, c)(c, f; a, d, b)(d, f; a, c, b) ,$$

while the product of the desired factors of  $(S_3, T_2)$  is

$$(a, f, d; b, c)(a, f, c; b, d)(a, f, b; c, d)(b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) .$$

Hence, modulo  $G_6$ ,

$$\begin{aligned} 1 &= (a, f; b, d, c)(b, f; a, d, c)(c, f; a, d, b)(d, f; a, c, b) \\ &\quad \cdot (a, f, d; b, c)(a, f, c; b, d)(a, f, b; c, d) \\ &\quad \cdot (b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) . \end{aligned}$$

so that by (10)

$$\begin{aligned} 1 &= (a, f; b, d, c)(b, f; a, d, c)(c, f; a, d, b)(d, f; a, c, b) \\ &\quad \cdot (b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) . \end{aligned}$$

Using (7) on the first four factors gives, modulo  $G_6$ ,

$$\begin{aligned} 1 &= (a, f, c; b, d)(a, f; b, d; c)(b, f, c; a, d)(b, f; a, d; c) \\ &\quad \cdot (c, f, b; a, d)(c, f; a, d; b)(d, f, b; a, c)(d, f; a, c; b) \\ &\quad \cdot (b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) \\ &= (a, f, c; b, d)(a, f; b, d; c)(b, f; a, d; c)(c, f, b; a, d)(c, f; a, d; b) \\ &\quad \cdot (d, f, b; a, c)(d, f; a, c; b)(b, f, d; a, c)(c, f, d; a, b) \\ &= (a, f, c; b, d)(a, f; b, d; c)(b, f; a, d; c)(c, f, b; a, d) \\ &\quad \cdot (c, f; a, d; b)(d, f; a, c; b)(b, d, f; a, c)(c, f, d; a, b) , \end{aligned}$$

where the last step follows from (8). Now applying (11) twice gives

$$\begin{aligned} 1 &= (a, f, c; b, d)(a, b; d, f; c)(c, f, b; a, d)(a, f; c, d; b) \\ &\quad \cdot (b, d, f; a, c)(c, f, d; a, b) , \end{aligned}$$

so that by (10)

$$1 = (a, f, c; b, d)(a, b; d, f; c)(a, f; c, d; b)(b, d, f; a, c)(c, f, a; b, d)$$

and hence by (8)

$$1 = (a, b; d, f; c)(a, f; c, d; b)(b, d, f; a, c)(a, c, f; b, d) .$$

Thus, by (7)

$$1 \equiv (a, b; d, f; c)(a, f; c, d; b)(a, c; b, d; f) \pmod{G_6} ,$$



so that interchanging  $a$  with  $b$  and  $c$  with  $f$  we get

$$1 \equiv (a, b; c, d; f)(c, b; f, d; a)(f, b; a, d; c) \pmod{G_6}$$

which is (12). Thus the lemma is proved.

The corollary follows immediately.

6. Having proved the crucial relation (12), we are now in a position to prove the main theorem.

**THEOREM 6.1.** *Let  $G(n)$ , ( $n = 1, 2, \dots$ ) be the freest group of exponent 4 generated by  $n$  elements of order 2. Then  $G(n)_{n+2} = 1$ .*

*Proof.* The proof is by induction on  $n$ . We have the result for  $n = 1, 2, 3$  and 4. Assuming the result true for  $n$  we now prove it for  $n + 1$ . As before, we may assume  $G(n + 1)_{n+4} = 1$ . Consider a commutator  $C = (y_1, y_2, \dots, y_{n+3})$  in the generators  $x_1, \dots, x_n, a$  and  $b$  of  $G(n + 1)$ . As before, we may restrict attention to the case  $C = (a, y_2, \dots, y_{n+2}, a)$ . There are two possibilities to consider—*Case 1:*  $a$  appears again; *Case 2:*  $b$  appears twice. In either case we may assume that every  $x_i$  appears once, since otherwise, by the inductive assumption,  $C = 1$ .

*Case 1.* The proof in this case is by induction on the position of the middle  $a$ . Clearly  $(a, y_2, a, \dots, a) = 1$ . Assume that for some  $i \geq 3$ ,  $(a, y_2, \dots, y_{i-1}, a, \dots, a) = 1$ . Then

$$\begin{aligned} (a, y_2, \dots, y_i, a, y_{i+1}, \dots, y_{n+2}, a) & \\ &= (a, y_2, \dots, y_{i-1}; y_i, a; y_{i+1}; \dots; y_{n+2}; a) \\ &= (a, y_2, \dots, y_{i-1}; y_i, a; a, y_{n+2}, \dots, y_{i+1}) , \end{aligned}$$

where the last step follows from  $G(n)_{n+2} = 1$ . But by (13),

$$(a, y_2, \dots, y_{i-1}; y_i, a; a, y_{n+2}, \dots, y_{i+1}) = C_1 C_2$$

where

$$\begin{aligned} C_1 &= (a, y_2, \dots, y_{i-2}, y_i; a, y_{n+2}, \dots, y_{i+1}, a; y_{i-1}) \\ C_2 &= (a, y_2, \dots, y_{i-2}; a, y_{n+2}, \dots, y_{i+1}; y_{i-1}, a; y_i) . \end{aligned}$$

Since  $y_i$  and  $y_{i-1}$  appear only once, by the assumption that  $G(n)_{n+2} = 1$  we have  $C_1 = C_2 = 1$ . Hence, by induction,  $C = 1$  if  $a$  appears three times.

*Case 2.* In this case also the proof is by induction, this time on the distance between the  $b$ 's. Let

$$C = (a, z_1, \dots, z_i, b, z_{i+1}, \dots, z_j, b, z_{j+1}, \dots, z_{n-1}, a) ,$$

where  $0 \leq i < j \leq n - 1$  (that is, there might be no entries between

the  $a$ 's and the  $b$ 's). If  $j - i = 1$ , then clearly  $C = 1$ . Assume that  $C = 1$  for  $j - i = k \geq 1$ . Then as in Case 1,

$$\begin{aligned} &(a, z_1, \dots, z_i, b, z_{i+1}, \dots, z_{i+k+1}, b, z_{i+k+2}, \dots, z_{n-1}, a) \\ &= (a, z_1, \dots, z_i, b, z_{i+1}, \dots, z_{i+k}; z_{i+k+1}, b; a, z_{n-1}, \dots, z_{i+k+2}) \\ &= C_1 C_2 \end{aligned}$$

where

$$\begin{aligned} C_1 &= (a, \dots, b, \dots, z_{i+k-1}, z_{i+k+1}; a, z_{n-1}, \dots, z_{i+k+2}, b; z_{i+k}) = 1 \\ C_2 &= (a, \dots, b, \dots, z_{i+k-1}; a, z_{n-1}, \dots, z_{i+k+2}; z_{i+k}, b; z_{i+k+1}) = 1 \end{aligned}$$

Thus  $C = 1$  for  $j - i = k + 1$ , so that by induction  $C = 1$  if  $b$  appears twice.

Since  $C = 1$  in both cases, we conclude that  $G(n + 1)_{n+3} = 1$ , so that by induction  $G(n)_{n+2} = 1$  for  $n = 1, 2, \dots$ .

7. The author conjectures that the class of  $G(n)$  is precisely  $n + 1$  for  $n > 2$ . As supporting evidence, he has constructed  $G(n)/G(n)''$  and shown that its class is exactly  $n$ . Moreover, for  $n = 3$  and  $n = 4$ ,  $G(n)''$  is fairly large, and  $G(n)_{n+1} \neq 1$ .

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Glen Earl Baxter, <i>An analytic problem whose solution follows from a simple algebraic identity</i> .....	731
Leonard D. Berkovitz and Melvin Dresher, <i>A multimove infinite game with linear payoff</i> .....	743
Earl Robert Berkson, <i>Sequel to a paper of A. E. Taylor</i> .....	767
Gerald Berman and Robert Jerome Silverman, <i>Embedding of algebraic systems</i> .....	777
Peter Crawley, <i>Lattices whose congruences form a boolean algebra</i> .....	787
Robert E. Edwards, <i>Integral bases in inductive limit spaces</i> .....	797
Daniel T. Finkbeiner, II, <i>Irreducible congruence relations on lattices</i> .....	813
William James Firey, <i>Isoperimetric ratios of Reuleaux polygons</i> .....	823
Delbert Ray Fulkerson, <i>Zero-one matrices with zero trace</i> .....	831
Leon W. Green, <i>A sphere characterization related to Blaschke's conjecture</i> .....	837
Israel (Yitzchak) Nathan Herstein and Erwin Kleinfeld, <i>Lie mappings in characteristic 2</i> .....	843
Charles Ray Hobby, <i>A characteristic subgroup of a p-group</i> .....	853
R. K. Juberg, <i>On the Dirichlet problem for certain higher order parabolic equations</i> .....	859
Melvin Katz, <i>Infinitely repeatable games</i> .....	879
Emma Lehmer, <i>On Jacobi functions</i> .....	887
D. H. Lehmer, <i>Power character matrices</i> .....	895
Henry B. Mann, <i>A refinement of the fundamental theorem on the density of the sum of two sets of integers</i> .....	909
Marvin David Marcus and Roy Westwick, <i>Linear maps on skew symmetric matrices: the invariance of elementary symmetric functions</i> .....	917
Richard Dean Mayer and Richard Scott Pierce, <i>Boolean algebras with ordered bases</i> .....	925
Trevor James McMinn, <i>On the line segments of a convex surface in <math>E_3</math></i> .....	943
Frank Albert Raymond, <i>The end point compactification of manifolds</i> .....	947
Edgar Reich and S. E. Warschawski, <i>On canonical conformal maps of regions of arbitrary connectivity</i> .....	965
Marvin Rosenblum, <i>The absolute continuity of Toeplitz's matrices</i> .....	987
Lee Albert Rubel, <i>Maximal means and Tauberian theorems</i> .....	997
Helmuth Heinrich Schaefer, <i>Some spectral properties of positive linear operators</i> .....	1009
Jeremiah Milton Stark, <i>Minimum problems in the theory of pseudo-conformal transformations and their application to estimation of the curvature of the invariant metric</i> .....	1021
Robert Steinberg, <i>The simplicity of certain groups</i> .....	1039
Hisahiro Tamano, <i>On paracompactness</i> .....	1043
Angus E. Taylor, <i>Mittag-Leffler expansions and spectral theory</i> .....	1049
Marion Franklin Tinsley, <i>Permanents of cyclic matrices</i> .....	1067
Charles J. Titus, <i>A theory of normal curves and some applications</i> .....	1083
Charles R. B. Wright, <i>On groups of exponent four with generators of order two</i> .....	1097