Pacific Journal of Mathematics

ON GROUPS OF EXPONENT FOUR WITH GENERATORS OF ORDER TWO

CHARLES R. B. WRIGHT

Vol. 10, No. 3 November 1960

ON GROUPS OF EXPONENT FOUR WITH GENERATORS OF ORDER TWO

C. R. B. WRIGHT

1. If $x, y, \cdots$ are elements of a group G, we define the *commutator* (x, y) of x and y by $(x, y) = x^{-1}y^{-1}xy$. More generally, we define *extended commutators* inductively by $(x, \cdots, y, z) = ((x, \cdots, y), z)$. In this paper we shall also be concerned with higher commutators of type $((a_1, \cdots, a_s), (b_1, \cdots, b_t), \cdots, (c_1, \cdots, c_r))$ which we denote by $(a_1, \cdots, a_s; b_1, \cdots, b_t; \cdots; c_1, \cdots, c_r)$. If we let G_i be the subgroup of G which is generated by all extended commutators of length i, (i.e., with i entries), then G_i is a characteristic subgroup of G, and the series $G = G_1 \supset G_2 \supset \cdots$ is called the *lower central series* of G.[1]

Let $G(n)$ $(n = 1, 2, \cdots)$ be the freest group of exponent 4 on n generators of order 2. That is, $G(n)$ is a group in which the fourth power of every element is the identity, 1, $G(n)$ is generated by n elements of order 2, and if H is any other group with these properties, then H is a homomorphic image of $G(n)$.

We prove $G(n)_{n+2} = 1$. For this purpose it may be assumed, since $G(n)$ is finite[2] and hence nilpotent, that $G(n)_{n+3} = 1$. Moreover, it will be enough to show $(x_1, \cdots, x_{n+2}) = 1$ for all choices of $x_1, \cdots, x_{n+2}$ from among the generators of $G(n)$.

2. LEMMA 2.1. *If $x, y, \cdots, z$ are elements of order 2 in a group of exponent 4, then $(x, y)^2 = 1, (x, y, \cdots, z)^2 = 1$, and $(x, y, x) = 1$.*

Proof. Since $(x, y) = xyxy = (xy)^2$, $(x, y)^2 = 1$. By induction, $(x, y, \cdots, z)^2 = 1$, while $(y, x) = yxyx = x(x, y)x = (x, y)(x, y, x)$, so that $(x, y, x) = (y, x)^2 = 1$.

The relation $(x, y, \cdots, z)^2 = 1$ will be the justification for future substitutions and will be used without specific mention.

THEOREM 2.1. $G(2)_3 = 1$.

Proof. By Lemma 2.1, if the generators of $G(2)$ are a and b, then $(a, b, a) = (b, a, a) = (a, b, b) = (b, a, b) = 1$.

3. LEMMA 3.1. *If a, b and c are elements of order 2 in a group G of exponent 4, then*

Received October 26, 1959. Presented to the Society on September 3, 1959. This work was supported by a National Science Foundation predoctoral fellowship.

[1] For properties of commutators and the lower central series see Hall, [**1**], Ch. 10.

[2] See Sanov, [**2**], or Hall, [**1**], pp. 324–325.

$$(1) \qquad (a, b, c) \equiv (b, c, a)(c, a, b) \bmod G_5$$

$$(2) \qquad (a, b; c, a) = (a, c; b, a) \equiv (a, c, b, a) \bmod G_5$$

$$(3) \qquad (a, b, c, a) \equiv (b, c, a, b)(c, a, b, c) \bmod G_5 .$$

Proof. We may assume that a, b and c generate G. Now

$$abcabc = aba\,(a, c)b(b, c) = (a, b)(a, c)(a, c, b)(b, c) .$$

Thus, modulo G_5, $(abc)^2 = (a, b)(a, c)(b, c)(a, c, b)$. Hence

$1 \equiv [(a, b)(a, c)(b, c)]^2 \bmod G_5$, so that, modulo G_5,
$1 = (a, b)(a, c)(b, c)(a, b)(a, c)(b, c) = (a, b)(a, c)(a, b)(a, b; b, c)(a, c)(a, c; b, c)$,

$$(4) \qquad 1 \equiv (a, b; a, c)(a, b; b, c)(a, c; b, c) \bmod G_5 .$$

But also

$$\begin{aligned} abc &= ca(a, c)b(b, c) \\ &= bc(c, b)a(a, b)(a, c)(a, c, b)(b, c) \\ &= ab(b, a)c(c, a)(c, b)(c, b, a)(a, b)(a, c)(a, c, b)(b, c) , \end{aligned}$$

so that $1 = (b, a)(b, a, c)(c, a)(c, b)(c, b, a)(a, b)(a, c)(a, c, b)(b, c)$, and hence, modulo G_5,

$$\begin{aligned} 1 &= (b, a)(c, a)(c, b)(a, b)(a, c)(b, c)(b, a, c)(c, b, a)(a, c, b) \\ &= [(a, b)(a, c)(b, c)]^2(a, b, c)(b, c, a)(c, a, b) . \end{aligned}$$

Thus (1) is proved. Replacing b by (a, b) in (1) gives $(a, b, c, a)(c, a; a, b) \equiv 1 \bmod G_5$ or (2). And (2) and (4) together give (3).

LEMMA 3.2. *If $x_1, \cdots, x_k$ and a are elements of order 2 in a group G of exponent 4, then $(x_1, \cdots, x_k, a) \equiv X \bmod G_{k+2}$, where X is a product of commutators of form $(a, y_1, \cdots, y_k)$ with $y_1, \cdots, y_k$ from among $x_1, \cdots, x_k$.*

COROLLARY. *If $x_1, \cdots, x_k, z_1, \cdots, z_s$ and a are elements of order 2 in a group G of exponent 4, then*

$$(x_1, \cdots, x_k, a, z_1, \cdots, z_s) \equiv X \bmod G_{k+s+2}$$

where X is a product of commutators of form $(a, y_1, \cdots, y_k, z_1, \cdots, z_s)$ with $y_1, \cdots, y_k$ from among $x_1, \cdots, x_k$.

Proof of Lemma 3.2. Certainly the lemma and corollary are true if $k = 1$. Assume for induction that both are true for $k = n - 1 \geq 1$.

Now by (1), modulo G_{n+2}, $(x_1, \cdots, x_{n-1}, x_n, a) = (x_1, \cdots, x_{n-1}, a, x_n)(x_1, \cdots, x_{n-1}; a, x_n)$. But by the inductive assumption $(x_1, \cdots, x_{n-1}, a, x_n)$ is a product of terms $(a, y_1, \cdots, y_{n-1}, x_n)$, and $(x_1, \cdots, x_{n-1}; a, x_n)$ is a product of terms $(a, x_n, y_1, \cdots, y_{n-1})$. The lemma and its immediate corollary follow by induction.

THEOREM 3.1. $G(3)_5 = 1$.

Proof. Let a, b and c be the generators of $G(3)$. Consider any commutator $C = (x_1, x_2, x_3, x_4, x_5)$ in arguments a, b and c. We show $C = 1$. There is no loss of generality in taking $x_5 = a$. If a does not appear again in C, then by Theorem 2.1, $C = (1, x_5) = 1$. If a appears again, then by Lemma 3.2 and the assumption that $G(3)_6 = 1$, we may suppose $C = (a, x_2, x_3, x_4, a)$. By Lemma 2.1, if a appears a third time, then $C = 1$. Thus we may take $C = (a, b, c, b, a)$. Now $(a, b, c, b, a) = (b, c, a, b, a)(c, a, b, b, a) = (b, c, a, b, a)$ by (1). Replacing c by (b, c) in (3) gives $(a, b; b, c,; a) = (b; b, c; a; b) = 1$, while replacing c by (b, c) in (2) gives $(a, b; b, c; a) = (b, c, a, b, a)$. Hence, $C = (a, b, c, b, a) = (b, c, a, b, a) = (a, b;, b, c; a) = 1$, and the theorem is proved.

COROLLARY 1. *If a, b and c are elements of order 2 in a group of exponent 4, then*

$$(1') \qquad (a, b, c) = (b, c, a)(c, a, b)$$

$$(2') \qquad (a, b; c, a) = (a, b, c, a)$$

$$(3') \qquad (a, b, c, a) = (b, c, a, b)(c, a, b, c)$$

Proof. These follow from Lemma 3.1.

COROLLARY 2. *If $x_1, \cdots, x_k, y_1, \cdots, y_s, z_1, \cdots, z_t$ $(s \geq 2)$ are elements of order 2 in a group G of exponent 4, then*

$$(x_1, \cdots, x_k; y_1, \cdots, y_s; z_1; \cdots; z_t) \equiv AB \bmod G_{k+s+t+1}$$

where

$$A = (x_1, \cdots, x_k; y_1, \cdots, y_{s-1}; y_s; z_1; \cdots; z_t)$$
$$B = (x_1, \cdots, x_k, y_s; y_1, \cdots, y_{s-1}; z_1; \cdots; z_t) .$$

Proof. This follows from (1′).

The following corollary lists some relations for future use.

COROLLARY 3. *If a, b, c, d and f are elements of order 2 in a group G of exponent 4, then*

$$(5) \qquad (a, b, c, d, c) \equiv (a, b, d, c, d) \bmod G_6$$

$$(6) \qquad (b, c, a; d, f, a) \equiv 1 \bmod G_7$$

$$(7) \qquad (a, f; b, d, c) \equiv (a, f, c; b, d)(a, f; b, d; c)$$

$$(8) \qquad (b, f, d; a, c)(d, f, b; a, c) \equiv (b, d, f; a, c) \bmod G_8 .$$

Proof. By (3′), with a replaced by (a, b) and b replaced by d, $(a, b, d, c; a, b) = (d, c; a, b; d)(c; a, b; d; c) = (a, b; d, c; d)(a, b, c, d, c)$, so that, since $(a, b; d, c; d) = (a, b, d, c, d)$, (5) is true. By (2′) and (3′) with b replaced by (b, c) and c replaced by (d, f), $(b, c, a; d, f, a) = (a; b, c; d, f; a) = (b, c; d, f; a; b, c)(d, f; b, c; a; d, f)$, so that (6) is true. Finally, (7) and (8) are obvious from (1′).

4. LEMMA 4.1. *If a, b, c and d are elements of order 2 in a group G of exponent 4, then*

$$(9) \qquad (a, b; c, d) \equiv (a, c; b, d)(a, d; b, c) \bmod G_5 .$$

Proof. First, working modulo G_5 and collecting as we did in the proof of Lemma 3.1 we obtain $(abcd)^2 = T_2 T_3 T_4$ where

$$\begin{aligned} T_2 &= (a, b)(a, c)(b, c)(a, d)(b, d)(c, d) \\ T_3 &= (a, c, b)(a, d, c)(a, d, b)(b, d, c) \\ T_4 &= (a, d, b, c) . \end{aligned}$$

Note that modulo G_5, T_2, T_3 and T_4 commute, and $T_3^2 = T_4^2 = 1$. Hence, modulo G_5, $1 = (abcd)^4 = T_2^2$. Collecting the (a, d)'s in T_2^2 we obtain $1 \equiv XABCY \bmod G_5$, where

$$\begin{aligned} X &= [(a, b)(a, c)(b, c)]^2 \\ A &= (b, c; b, d)(b, c; c, d)(b, d; c, d) \\ B &= (a, c; a, d)(a, c; c, d)(a, d; c, d) \\ C &= (a, b; a, d)(a, b; b, d)(a, d; b, d) \\ Y &= (a, b; c, d)(a, c; b, d)(a, d; b, c) . \end{aligned}$$

Now modulo G_5, $X = 1$, while $A = B = C = 1$ by (2′) and (3′). Hence, $1 \equiv (a, b; c, d)(a, c; b, d)(a, d; b, c) \bmod G_5$, which is (9).

COROLLARY 1. *If $x_1, \cdots, x_k$ and a are elements of order 2 in a group G of exponent 4, then for $i = 2, \cdots, k$,*

$$(x_1, a, x_2, a, \cdots, x_i, \cdots, x_k) \equiv (x_1, x_2, \cdots, a, x_i, a, \cdots, x_k) \bmod G_{k+3} .$$

Hence, if two of $x_1, \cdots, x_k, a$ are equal, $(x_1, a, x_2, a, \cdots, x_k) \equiv 1 \bmod G_{k+3}$.

Proof. Let a, b, c and d be elements of order 2 in G. Then modulo G_6,

$$\begin{aligned}(b, a, c, a, d) &= (b, a;, c, a; d)\\ &= (b, a, d; c, a)(c, a, d; b, a)\\ &= (b, a, c; d, a)(c, a, b; d, a)\\ &= (b, c, a; d, a)\\ &= (b, c, a, d, a)\,.\end{aligned}$$

The first statement follows. Now the second statement is clearly true if a appears a third time, since then $(x_1, a, x_2, a, \cdots, a, \cdots, x_k) = (x_1, x_2, \cdots, a, a, a, \cdots, x_k) = 1$. If some x_i appears twice, then modulo $G_{k+3}(x_1, a, x_2, a, \cdots, x_i, \cdots, x_k) = (x_1, \cdots, a, x_i, a, \cdots, x_k) = (x_1, x_2, \cdots, x_i, a, x_i, \cdots, x_k) = (x_1, x_i\, x_2, x_i, \cdots, a, \cdots, x_k)$ (the second step following from (5)), and we are back to the case of three appearances of a. Thus the corollary is proved.

COROLLARY 2. *If a, b, c, d and f are elements of order 2 in a group G of exponent 4, then*

$$1 \equiv (a, f, b; c, d)(a, f, c; b, d)(a, f, d; b, c) \bmod G_6 \tag{10}$$

$$(a, c; d, f; b)(a, d; c, f; b) \equiv (c, d; a, f; b) \bmod G_6\,. \tag{11}$$

Proof. These follow from (9).

THEOREM 4.1. $G(4)_6 = 1$.

Proof. Let the generators of $G(4)$ be a, b, c and d and consider any commutator $C = (x_1, x_2, x_3, x_4, x_5, x_6)$ in a, b, c and d. It will be sufficient to prove $C = 1$ under the assumption that $G(4)_7 = 1$. As in the proof of Theorem 3.1, we may suppose that $C = (a, x_2, x_3, x_4, x_5, a)$. Moreover, if x_2, x_3, x_4 or x_5 is a, then by Theorem 2.1 or Corollary 1 of Lemma 4.1, $C = 1$. It will thus be sufficient to prove $(a, b, c, b, d, a) = 1$, $(a, b, c, d, b; a) = 1$, and $(a, c, b, d, b, a) = 1$. Now by Corollary 1 of Lemma 4.1, $(a, b, c, b, d, a) = (a, c, b, d, b, a) = 1$, while by (1′), $(a, b, c; b, d, a) = (a, c, b; b, d; a)(b, c, a; b, d; a)$, so that by (6) $(a, b, c; b, d; a) = 1$. Thus $(a, b, c, d, b, a) = (a, b, c, b, d, a)(a, b, c; b, d; a) = 1$, and the theorem is proved.

5. The main result, that $G(n)_{n+2} = 1$, has now been proved for $n = 2, 3$ and 4. In this section we derive an identity analogous to (1) and (9) for five generators. This identity enables us to prove, in § 6, that $G(n)_{n+2} = 1$ for $n \geq 5$.

LEMMA 5.1. *If a, b, c, d and f are elements of order 2 in a group G of exponent 4, then*

$$(12) \qquad (a, b; c, d; f) \equiv (c, b; f, d; a)(f, b; a, d; c) \bmod G_6 .$$

COROLLARY. *If* $(x_1, \cdots, x_k)$, $(y_1, \cdots, y_j)$, $(z_1, \cdots, z_m)$, a *and* b $(k, j, m \geq 1)$ *are elements of order* 2 *in a group* G *of exponent* 4, *then*

$$(13) \qquad (x_1, \cdots, x_k, a; y_1, \cdots, y_j, b; z_1, \cdots, z_m) \equiv C_1C_2 \bmod G_{k+j+m+3} ,$$

where

$$C_1 = (y_1, \cdots, y_j; z_1, \cdots, z_m; x_1, \cdots, x_k, b; a)$$
$$C_2 = (x_1, \cdots, x_k; z_1, \cdots, z_m; y_1, \cdots, y_j, a; b) .$$

Proof of Lemma 5.1. First, working moduio G_5, we collect f's in the expression $(abcdf)^2$ to get $(abcdf)^2 = (abcd)a(a, f)b(b, f)c(c, f)d(d, f)$. Then collecting b, c and d in that order we obtain $(abcdf)^2 = (abcd)^2S_2S_3S_4$ where

$$S_2 = (a, f)(b, f)(c, f)(d, f)$$
$$S_3 = (a, f, d)(a, f, c)(a, f, b)(b, f, d)(b, f, c)(c, f, d)$$
$$S_4 = (a, f, c, d)(a, f, b, d)(a, f, b, c)(b, f, c, d) .$$

But as in the proof of Lemma 4.1, $(abcd)^2 \equiv T_2T_3T_4 \bmod G_5$, where

$$T_2 = (a, b)(a, c)(a, d)(b, d)(c, d)$$
$$T_3 = (a, c, b)(a, d, c)(a, d, b)(b, d, c)$$
$$T_4 = (a, d, b, c) .$$

Thus, modulo G_5, $(abcdf)^2 = T_2T_3T_4S_2S_3S_4$. But then, modulo G_6,

$$\begin{aligned} 1 = (abcdf)^4 &= T_2T_3T_4S_2S_3T_2T_3T_4S_2S_3 \\ &= T_2T_3T_4T_2S_2(S_2, T_2)S_3(S_3, T_2)T_3T_4S_2S_3 \\ &= (T_2T_3T_4)^2S_2(S_2, T_3)(S_2, T_2)S_3(S_3, T_2)S_2S_3 \\ &= S_2(S_2, T_3)(S_2, T_2)S_3(S_3, T_2)S_2S_3 \\ &= S_2^2(S_2, T_3)(S_2, T_2)S_3(S_3, S_2)(S_3, T_2)S_3 \\ &= S_2^2(S_2, T_3)(S_2, T_2)S_3^2(S_3, S_2)(S_3, T_2) . \end{aligned}$$

But modulo G_6, $S_3^2 = 1$, while S_2^2 is a product of commutators of weight 4. Thus the last relation may be rewritten as $1 \equiv A \bmod G_6$ where A is a product of commutators in a, b, c, d and f of weight 4 or 5; hence the factors of A commute modulo G_6. Let A'_a be the product of all factors of A which do *not* contain a as argument, and let A_a be the product of the remaining factors of A. Then $1 \equiv A'_aA_a \bmod G_6$, so that, setting $a = 1$, $1 \equiv A'_a \bmod G_6$, and hence $1 \equiv A_a \bmod G_6$. Continuing this argument we finally arrive at $1 \equiv A_{abcdf} \bmod G_6$, where A_{abcdf} is the product of all factors of A which contain each of a, b, c, d and f. But what are these factors? Clearly S_2^2 and (S_2, T_2) do not contain any such factors; and since each factor of S_2 and S_3 contains f, (S_3, S_2) cannot contain any such factors. We are left with (S_2, T_3) and (S_3, T_2). The product of the desired factors of (S_2, T_3) is clearly

$$(a, f; b, d, c)(b, f; a, d, c)(c, f; a, d, b)(d, f; a, c, b) ,$$

while the product of the desired factors of (S_3, T_2) is

$$(a, f, d; b, c)(a, f, c; b, d)(a, f, b; c, d)(b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) .$$

Hence, modulo G_6,

$$\begin{aligned} 1 = (a, f; b, d, c)(b, f; a, d, c)(c, f; a, d, b)(d, f; a, c, b) \\ \cdot (a, f, d; b, c)(a, f, c; b, d)(a, f, b; c, d) \\ \cdot (b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) . \end{aligned}$$

so that by (10)

$$\begin{aligned} 1 = (a, f; b, d, c)(b, f; a, d, c)(c, f; a, d, b)(d, f; a, c, b) \\ \cdot (b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) . \end{aligned}$$

Using (7) on the first four factors gives, modulo G_6,

$$\begin{aligned} 1 &= (a, f, c; b, d)(a, f; b, d; c)(b, f, c; a, d)(b, f; a, d; c) \\ &\quad \cdot (c, f, b; a, d)(c, f; a, d; b)(d, f, b; a, c)(d, f; a, c; b) \\ &\quad \cdot (b, f, d; a, c)(b, f, c; a, d)(c, f, d; a, b) \\ &= (a, f, c; b, d)(a, f; b, d; c)(b, f; a, d; c)(c, f, b; a, d)(c, f; a, d; b) \\ &\quad \cdot (d, f, b; a, c)(d, f; a, c; b)(b, f, d; a, c)(c, f, d; a, b) \\ &= (a, f, c; b, d)(a, f; b, d; c)(b, f; a, d; c)(c, f, b; a, d) \\ &\quad \cdot (c, f; a, d; b)(d, f; a, c; b)(b, d, f; a, c)(c, f, d; a, b) , \end{aligned}$$

where the last step follows from (8). Now applying (11) twice gives

$$\begin{aligned} 1 = (a, f, c; b, d)(a, b; d, f; c)(c, f, b; a, d)(a, f; c, d; b) \\ \cdot (b, d, f; a, c)(c, f, d; a, b) , \end{aligned}$$

so that by (10)

$$1 = (a, f, c; b, d)(a, b; d, f; c)(a, f; c, d; b)(b, d, f; a, c)(c, f, a; b, d)$$

and hence by (8)

$$1 = (a, b; d, f; c)(a, f; c, d; b)(b, d, f; a, c)(a, c, f; b, d) .$$

Thus, by (7)

$$1 \equiv (a, b; d, f; c)(a, f; c, d; b)(a, c; b, d; f) \bmod G_6 ,$$

so that interchanging a with b and c with f we get

$$1 \equiv (a, b; c, d; f)(c, b; f, d; a)(f, b; a, d; c) \bmod G_6$$

which is (12). Thus the lemma is proved.

The corollary follows immediately.

6. Having proved the crucial relation (12), we are now in a position to prove the main theorem.

THEOREM 6.1. *Let $G(n)$, $(n = 1, 2, \cdots)$ be the freest group of exponent 4 generated by n elements of order 2. Then $G(n)_{n+2} = 1$.*

Proof. The proof is by induction on n. We have the result for $n = 1, 2, 3$ and 4. Assuming the result true for n we now prove it for $n + 1$. As before, we may assume $G(n + 1)_{n+4} = 1$. Consider a commutator $C = (y_1, y_2, \cdots, y_{n+3})$ in the generators $x_1, \cdots, x_n$, a and b of $G(n + 1)$. As before, we may restrict attention to the case $C = (a, y_2, \cdots, y_{n+2}, a)$. There are two possibilities to consider—*Case* 1: a appears again; *Case* 2: b appears twice. In either case we may assume that every x_i appears once, since otherwise, by the inductive assumption, $C = 1$.

Case 1. The proof in this case is by induction on the position of the middle a. Clearly $(a, y_2, a, \cdots, a) = 1$. Assume that for some $i \geq 3$, $(a, y_2, \cdots, y_{i-1}, a, \cdots, a) = 1$. Then

$$\begin{aligned}(a, y_2, \cdots, y_i, a, y_{i+1}, \cdots, y_{n+2}, a) \\ &= (a, y_2, \cdots, y_{i-1}; y_i, a; y_{i+1}; \cdots; y_{n+2}; a) \\ &= (a, y_2, \cdots, y_{i-1}; y_i, a; a, y_{n+2}, \cdots, y_{i+1}) ,\end{aligned}$$

where the last step follows from $G(n)_{n+2} = 1$. But by (13),

$$(a, y_2, \cdots, y_{i-1}; y_i, a; a, y_{n+2}, \cdots, y_{i+1}) = C_1C_2$$

where

$$\begin{aligned}C_1 &= (a, y_2, \cdots, y_{i-2}, y_i; a, y_{n+2}, \cdots, y_{i+1}, a; y_{i-1}) \\ C_2 &= (a, y_2, \cdots, y_{i-2}; a, y_{n+2}, \cdots, y_{i+1}; y_{i-1}, a; y_i) .\end{aligned}$$

Since y_i and y_{i-1} appear only once, by the assumption that $G(n)_{n+2} = 1$ we have $C_1 = C_2 = 1$. Hence, by induction, $C = 1$ if a appears three times.

Case 2. In this case also the proof is by induction, this time on the distance between the b's. Let

$$C = (a, z_1, \cdots, z_i, b, z_{i+1}, \cdots, z_j, b, z_{j+1}, \cdots, z_{n-1}, a) ,$$

where $0 \leq i < j \leq n - 1$ (that is, there might be no entries between the a's and the b's). If $j - i = 1$, then clearly $C = 1$. Assume that $C = 1$ for $j - i = k \geq 1$. Then as in Case 1,

$$\begin{aligned}(a, z_1, \cdots, z_i, b, z_{i+1}, \cdots, z_{i+k+1}, b, z_{i+k+2}, \cdots, z_{n-1}, a)\\ = (a, z_1, \cdots, z_i, b, z_{i+1}, \cdots, z_{i+k}; z_{i+k+1}, b; a, z_{n-1}, \cdots, z_{i+k+2})\\ = C_1C_2\end{aligned}$$

where

$$C_1 = (a, \cdots, b, \cdots, z_{i+k-1}, z_{i+k+1}; a, z_{n-1}, \cdots, z_{i+k+2}, b; z_{i+k}) = 1$$
$$C_2 = (a, \cdots, b, \cdots, z_{i+k-1}; a, z_{n-1}, \cdots, z_{i+k+2}; z_{i+k}, b; z_{i+k+1}) = 1$$

Thus $C = 1$ for $j - i = k + 1$, so that by induction $C = 1$ if b appears twice.

Since $C = 1$ in both cases, we conclude that $G(n+1)_{n+3} = 1$, so that by induction $G(n)_{n+2} = 1$ for $n = 1, 2, \cdots$.

7. The author conjectures that the class of $G(n)$ is precisely $n + 1$ for $n > 2$. As supporting evidence, he has constructed $G(n)/G(n)''$ and shown that its class is exactly n. Moreover, for $n = 3$ and $n = 4$, $G(n)''$ is fairly large, and $G(n)_{n+1} \neq 1$.

BIBLIOGRAPHY

1. M. Hall, Jr., *The Theory of Groups*, the Macmillan Co., 1959.

2. I. N. Sanov, *Solution of Burnside's problem for exponent* 4, Leningrad State Univ. Ann. **10**, (1940), 166–170.

UNIVERSITY OF WISCONSIN
CALIFORNIA INSTITUTE OF TECHNOLOGY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 10, No. 3 November, 1960