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In [1] Fuchs has asked (problem 3) the cardinality of the set of all pure subgroups of an Abelian group. The purpose of this paper is to settle the question for nondenumerable Abelian groups. $|A|$ will denote the cardinality of the set A.

THEOREM. *Let G be a nondenumerable Abelian group, and let $\mathscr{P}$ be the collection of all pure subgroups, P, of G with $|P| = |G|$. Then $|\mathscr{P}| = 2^{|G|}$.*

Proof. Let T be the torsion subgroup of G. If $|T| < |G|$, then $|G/T| = |G|$ and by a result of Walker [3, Theorem 4], G/T, and hence G, has $2^{|G|}$ pure subgroups of order $|G|$.

If $|T| = |G|$, then we write T in the form $T = \sum_{i,\alpha} \oplus Z_\alpha(p_i^\infty) \oplus \sum_p \oplus R_p$, where the R_p are reduced primary groups and $\sum_{i,\alpha} \oplus Z_\alpha(p_i^\infty)$ is the maximal divisible subgroup of T.

If the above decomposition of T has $|G|$ summands then the theorem follows.

If the above decomposition has fewer than $|G|$ summands, then $|\sum_p \oplus R_p| = |G|$.

We first consider the case that there exists a prime, p, such that $|R_p| = |G|$. Let B be a basic subgroup of R_p. If $|B| < |R_p|$, then $|R_p/B| = |G|$ and $R_p/B = \sum_{\alpha \in A} \oplus Z_\alpha(p^\infty)$ with $|A| = |G|$. Thus the theorem holds for R_p/B, and hence also for G. If $|B| = |R_p|$, then since B is the direct sum of cyclic groups, $B = \sum_{\alpha \in A} \oplus C_\alpha$, it follows that $|A| = |G|$. Thus the theorem follows for B and hence for G. Finally, if $|R_p| < |G|$ for all p, we let[1] $R' = \sum_{p_i} \oplus R_{p_i}$, where the sum is taken over all primes, p_i, such that $|R_{p_i}| > \aleph_0$. Then $|R'| = |G| = \sum |R_{p_i}|$. We have proved above that for each p_i, R_{p_i} has $2^{|R_{p_i}|}$ pure subgroups, $P(i)$ of order $|R_{p_i}|$. For each i, choose $P(i) \subset R_{p_i}$ with $|P(i)| = |R_{p_i}|$. Then $P = \sum \oplus P(i)$ is a pure subgroup of R' with $|P| = |R'|$, and the number of subgroups formed in this way is $2^{|G|}$.

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2. W. Scott, *Groups and cardinal numbers*, Amer. J. Math., **74** (1952), 187–197.
3. E. Walker, *Subdirect sums and infinite Abelian groups*, Pacific J. Math., **9** (1959), 287–291.

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[1] This is exactly the method used by Scott, [2].

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