

Pacific Journal of Mathematics

A NOTE ON A PROBLEM OF FUCHS

DELMAR L. BOYER

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In [1] Fuchs has asked (problem 3) the cardinality of the set of all pure subgroups of an Abelian group. The purpose of this paper is to settle the question for nondenumerable Abelian groups. $|A|$ will denote the cardinality of the set A .

THEOREM. *Let G be a nondenumerable Abelian group, and let \mathcal{P} be the collection of all pure subgroups, P , of G with $|P| = |G|$. Then $|\mathcal{P}| = 2^{|G|}$.*

Proof. Let T be the torsion subgroup of G . If $|T| < |G|$, then $|G/T| = |G|$ and by a result of Walker [3, Theorem 4], G/T , and hence G , has $2^{|G|}$ pure subgroups of order $|G|$.

If $|T| = |G|$, then we write T in the form $T = \sum_{i,\alpha} \oplus Z_\alpha(p_i^\infty) \oplus \sum_p \oplus R_p$, where the R_p are reduced primary groups and $\sum_{i,\alpha} \oplus Z_\alpha(p_i^\infty)$ is the maximal divisible subgroup of T .

If the above decomposition of T has $|G|$ summands then the theorem follows.

If the above decomposition has fewer than $|G|$ summands, then $|\sum_p \oplus R_p| = |G|$.

We first consider the case that there exists a prime, p , such that $|R_p| = |G|$. Let B be a basic subgroup of R_p . If $|B| < |R_p|$, then $|R_p/B| = |G|$ and $R_p/B = \sum_{\alpha \in A} \oplus Z_\alpha(p^\infty)$ with $|A| = |G|$. Thus the theorem holds for R_p/B , and hence also for G . If $|B| = |R_p|$, then since B is the direct sum of cyclic groups, $B = \sum_{\alpha \in A} \oplus C_\alpha$, it follows that $|A| = |G|$. Thus the theorem follows for B and hence for G . Finally, if $|R_p| < |G|$ for all p , we let $R' = \sum_{p_i} \oplus R_{p_i}$, where the sum is taken over all primes, p_i , such that $|R_{p_i}| > \aleph_0$. Then $|R'| = |G| = \sum |R_{p_i}|$. We have proved above that for each p_i , R_{p_i} has $2^{|R_{p_i}|}$ pure subgroups, $P(i)$ of order $|R_{p_i}|$. For each i , choose $P(i) \subset R_{p_i}$ with $|P(i)| = |R_{p_i}|$. Then $P = \sum \oplus P(i)$ is a pure subgroup of R' with $|P| = |R'|$, and the number of subgroups formed in this way is $2^{|G|}$.

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1. L. Fuchs, *Abelian groups*, Hungarian Academy of Science (1958), Budapest.
2. W. Scott, *Groups and cardinal numbers*, Amer. J. Math., **74** (1952), 187-197.
3. E. Walker, *Subdirect sums and infinite Abelian groups*, Pacific J. Math., **9** (1959), 287-291.

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¹ This is exactly the method used by Scott, [2].

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| | |
|---|------|
| M. Altman, <i>An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space</i> | 1107 |
| Nesmith Cornett Ankeny, <i>Criterion for rth power residuacity</i> | 1115 |
| Julius Rubin Blum and David Lee Hanson, <i>On invariant probability measures I</i> | 1125 |
| Frank Featherstone Bonsall, <i>Positive operators compact in an auxiliary topology</i> | 1131 |
| Billy Joe Boyer, <i>Summability of derived conjugate series</i> | 1139 |
| Delmar L. Boyer, <i>A note on a problem of Fuchs</i> | 1147 |
| Hans-Joachim Bremermann, <i>The envelopes of holomorphy of tube domains in infinite dimensional Banach spaces</i> | 1149 |
| Andrew Michael Bruckner, <i>Minimal superadditive extensions of superadditive functions</i> | 1155 |
| Billy Finney Bryant, <i>On expansive homeomorphisms</i> | 1163 |
| Jean W. Butler, <i>On complete and independent sets of operations in finite algebras</i> | 1169 |
| Lucien Le Cam, <i>An approximation theorem for the Poisson binomial distribution</i> | 1181 |
| Paul Civin, <i>Involutions on locally compact rings</i> | 1199 |
| Earl A. Coddington, <i>Normal extensions of formally normal operators</i> | 1203 |
| Jacob Feldman, <i>Some classes of equivalent Gaussian processes on an interval</i> | 1211 |
| Shaul Foguel, <i>Weak and strong convergence for Markov processes</i> | 1221 |
| Martin Fox, <i>Some zero sum two-person games with moves in the unit interval</i> | 1235 |
| Robert Pertsch Gilbert, <i>Singularities of three-dimensional harmonic functions</i> | 1243 |
| Branko Grünbaum, <i>Partitions of mass-distributions and of convex bodies by hyperplanes</i> | 1257 |
| Sidney Morris Harmon, <i>Regular covering surfaces of Riemann surfaces</i> | 1263 |
| Edwin Hewitt and Herbert S. Zuckerman, <i>The multiplicative semigroup of integers modulo m</i> | 1291 |
| Paul Daniel Hill, <i>Relation of a direct limit group to associated vector groups</i> | 1309 |
| Calvin Virgil Holmes, <i>Commutator groups of monomial groups</i> | 1313 |
| James Fredrik Jakobsen and W. R. Utz, <i>The non-existence of expansive homeomorphisms on a closed 2-cell</i> | 1319 |
| John William Jewett, <i>Multiplication on classes of pseudo-analytic functions</i> | 1323 |
| Helmut Klingen, <i>Analytic automorphisms of bounded symmetric complex domains</i> | 1327 |
| Robert Jacob Koch, <i>Ordered semigroups in partially ordered semigroups</i> | 1333 |
| Marvin David Marcus and N. A. Khan, <i>On a commutator result of Taussky and Zassenhaus</i> | 1337 |
| John Glen Marica and Steve Jerome Bryant, <i>Unary algebras</i> | 1347 |
| Edward Peter Merkes and W. T. Scott, <i>On univalence of a continued fraction</i> | 1361 |
| Shu-Teh Chen Moy, <i>Asymptotic properties of derivatives of stationary measures</i> | 1371 |
| John William Neuberger, <i>Concerning boundary value problems</i> | 1385 |
| Edward C. Posner, <i>Integral closure of differential rings</i> | 1393 |
| Marian Reichaw-Reichbach, <i>Some theorems on mappings onto</i> | 1397 |
| Marvin Rosenblum and Harold Widom, <i>Two extremal problems</i> | 1409 |
| Morton Lincoln Slater and Herbert S. Wilf, <i>A class of linear differential-difference equations</i> | 1419 |
| Charles Robson Storey, Jr., <i>The structure of threads</i> | 1429 |
| J. François Treves, <i>An estimate for differential polynomials in $\partial/\partial z_1, \dots, \partial/\partial z_n$</i> | 1447 |
| J. D. Weston, <i>On the representation of operators by convolutions integrals</i> | 1453 |
| James Victor Whittaker, <i>Normal subgroups of some homeomorphism groups</i> | 1469 |