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**ANALYTIC AUTOMORPHISMS OF BOUNDED SYMMETRIC
COMPLEX DOMAINS**

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In a former paper [2] I determined the full group of one-to-one analytic mappings of a bounded symmetric Cartan domain [1]. Those investigations were incomplete, because it was impossible to treat the second Cartan-type of $n(n-1)/2$ complex dimensions for odd n by this method. The present note is devoted to a new shorter proof of the former result (n even), which furthermore covers the remaining case of odd n .

Take the complex $n(n-1)/2$ -dimensional space of skew symmetric n -rowed matrices Z . The irreducible bounded symmetric Cartan space in question is the set \mathcal{E}_n of those matrices Z , for which

$$I + Z\bar{Z} > 0, \quad Z' = -Z,$$

is positive definite. Here I is the n by n unit matrix. Obviously \mathcal{E}_2 is the unit circle. It is easy to see that analytic automorphisms of \mathcal{E}_n are described by the group ϕ of the mappings

$$(1) \quad W = (AZ + B)(-\bar{B}Z + \bar{A})^{-1},$$

where the n -rowed matrices A, B fulfill

$$M^*KM = K \quad \text{with} \quad M = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \quad K = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Here M^* denotes the conjugate transpose of M . For $n = 4$

$$W = \tilde{Z}$$

is a further analytic automorphism, where \tilde{Z} arises from Z by interchanging the elements z_{14} and z_{23} ,

$$\tilde{Z} = \begin{pmatrix} 0 & z_{12} & z_{13} & z_{23} \\ -z_{12} & 0 & z_{14} & z_{24} \\ -z_{13} & -z_{14} & 0 & z_{34} \\ -z_{23} & -z_{24} & -z_{34} & 0 \end{pmatrix}.$$

For $W\bar{W}$ and $\tilde{Z}\bar{\tilde{Z}}$ have the same characteristic roots. But this mapping is not contained in ϕ , since $CZ = \tilde{Z}D$ cannot be satisfied identically in Z by non-singular constant matrices C, D . On the other hand the following theorem holds.

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THEOREM. *Each analytic automorphism of \mathcal{E}_n can be written as $W = f(Z)$ or $W = f(\tilde{Z})$ (only for $n = 4$) with $f \in \phi$.*

Therefore the group ϕ is already the full group of analytic automorphisms for $n \neq 4$. Only in the exceptional case $n = 4$ there are the further mappings $W = f(\tilde{Z})$, which together with ϕ form the full group of analytic automorphisms. The proof of this theorem consists of two parts. The first analytic part is a reproduction of my former proof [2], which will be given here again for completeness, the second part is of algebraic character.

The group ϕ acts transitively on \mathcal{E}_n . For take an arbitrary point Z_1 of \mathcal{E}_n , choose the matrix A such that

$$A(I + Z_1 \bar{Z}_1)A^* = I$$

and define $B = -AZ_1$. Then (1) maps Z into 0. Therefore it is sufficient to investigate the stability group of the zero matrix.

First we show that each analytic one-to-one mapping $W = W(Z)$ of \mathcal{E}_n with the fixed point 0 is linear. For an arbitrary point $Z_1 \in \mathcal{E}_n$ let $r_1, \dots, r_n, 0 \leq r_1 \leq \dots \leq r_n < 1$, be the characteristic roots of $Z_1 Z_1^*$. Then also tZ_1 belongs to \mathcal{E}_n , if t is a complex number with $t\bar{t}r_n < 1$. Consequently there exists a power series expansion

$$(2) \quad W(tZ_1) = \sum_{k=1}^{\infty} t^k W_k(Z_1), \quad t\bar{t}r_n < 1.$$

The elements of the skew-symmetric matrices $W_k(Z_1)$ are homogeneous polynomials of degree k in the independent elements of Z_1 . Because of $I + W(tZ_1)\bar{W}(tZ_1) > 0$ for $\bar{t}t = 1$, one obtains from (2)

$$(3) \quad \frac{1}{2\pi i} \int_{|\bar{t}t|=1} (I + W(tZ_1)\bar{W}(tZ_1)) \frac{dt}{t} = I + \sum_{k=1}^{\infty} W_k(Z_1)\bar{W}_k(Z_1) > 0$$

and in particular $I + \bar{W}_1(Z_1)W_1(Z_1) > 0$. Therefore the linear function $W_1(Z)$ is an analytic mapping of \mathcal{E}_n into itself. Its determinant D is at the same time the Jacobian of the function $W(Z)$ with respect to Z . By interchanging Z and W it can be assumed $D\bar{D} \geq 1$. Consequently $W(Z)$ is an analytic automorphism of \mathcal{E}_n and even maps the boundary onto itself. Take now in particular

$$(4) \quad Z_1 = U'PU, \quad P = [(0), p_1 F, \dots, p_m F], \quad F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with an unitary matrix U , $m = [n/2]$. P shall be the matrix, which is built up by the two-rowed blocks $p_1 F, \dots, p_m F$ and possibly by the element 0 along the main diagonal. Z_1 belongs to the interior of \mathcal{E}_n , if $-1 < p_k < 1$ ($k = 1, \dots, m$), and to the boundary, if $-1 \leq p_k \leq 1$ ($k =$

$1, \dots, m$) and $p_k = \pm 1$ for at least one k . Now $|I + W_1(Z_1)\bar{W}_1|$ is a polynomial in p_1, \dots, p_m of total degree $4m$ and on the other hand (see [2], Lemma 4) the square of a polynomial. As $|I + W_1(Z_1)\bar{W}_1|$ vanishes on the boundary of \mathcal{E}_n , this polynomial is divisible by

$$|I + Z_1\bar{Z}_1| = \prod_{k=1}^m (1 - p_k^2)^2.$$

Because the constant terms and the degrees of both polynomials are equal, one obtains

$$(5) \quad |I + W_1(Z_1)\bar{W}_1| = |I + Z_1\bar{Z}_1|$$

even identically in Z_1 ; for each skew-symmetric matrix Z_1 permits a representation (4) (see [2], Lemma 3). On account of (5) and the linearity of W_1 the matrices $W_1\bar{W}_1$ and $Z\bar{Z}$ always have the same characteristic roots and this implies

$$(6) \quad W_1(Z) = U'ZU$$

with unitary U , which for the present still depends on Z .

Put now

$$Z = uX, \quad X = U'_1, [e^{i\zeta_1}F, \dots, e^{i\zeta_r}F, (0)]U_1, \quad 0 \leq u \leq 1,$$

with real variables ζ_1, \dots, ζ_r . Then $Z \in \mathcal{E}_n$ and by (6)

$$W_1W_1^* = u^2U'U'_1 \begin{pmatrix} I^{(n-1)} & 0 \\ 0 & (0) \end{pmatrix} \bar{U}_1\bar{U}$$

for all u between 0 and 1. Because of (3) one obtains

$$\bar{U}_1\bar{U}(I + W_1\bar{W}_1 + W_k\bar{W}_k)U'U'_1 > 0 \quad (k = 2, 3, \dots).$$

If u tends to 1, one gets

$$\begin{pmatrix} 0 & 0 \\ 0 & (1) \end{pmatrix} + \bar{U}_1\bar{U}W_k\bar{W}_kU'U'_1 > 0,$$

hence $W_k(X) = 0$. As W_k is a polynomial, $W_k(Z)$ even vanishes identically in Z . Therefore the stability group of \mathcal{E}_n is linear.

The investigation of $W = W_1(Z)$ is now a purely algebraic problem. The representation (6) shows that $\text{rank } W = \text{rank } Z$ and beyond this the equality of the characteristic roots of $W\bar{W}$ and $Z\bar{Z}$. These properties will be used in order to determine $W(Z)$ explicitly. We have to prove

$$(7) \quad W(Z) = U'ZU \quad \text{or} \quad W(Z) = U'\tilde{Z}U$$

with unitary constant U , where the second type only occurs for $n = 4$. The proof of this fact will be given by induction. The assertion (7) is trivial for the unit circle ($n = 2$). Let us assume its correctness for $2, 3, \dots, n - 1$ and consider \mathcal{E}_n . Write the linear mapping $W(Z)$ of \mathcal{E}_n onto itself as

$$W = \sum_{k < l} z_{kl} A_{kl}$$

with constant skew-symmetric n by n matrices A_{kl} . Because of the equality of the characteristic roots of WW^* and ZZ^* the hermitian matrix $A_{kl}A_{kl}^*$ has $1, 1, 0, \dots, 0$ as characteristic roots. Therefore after unitary transformation of W we can assume $A_{12} = E_{12}$, where in general E_{kl} denotes the skew-symmetric matrix the elements of which are all zero besides the element in the k th row and l th column and the element in the l th row and k th column, which are 1 respectively -1 . Since $\text{tr}(A_{12}\bar{A}_{kl}) = 0$ for $(k, l) \neq (1, 2)$, one obtains

$$A_{kl} = \begin{pmatrix} 0^{(2)} & * \\ * & * \end{pmatrix} \quad (k, l) \neq (1, 2).$$

$A_{12} = E_{12}$ does not change, if W is transformed by

$$\begin{pmatrix} U^{(2)} & 0 \\ 0 & V \end{pmatrix}$$

with unitary $U, V, |U| = 1$. Therefore

$$A_{13} = \begin{pmatrix} 0^{(2)} & B \\ -B' & C \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & \end{pmatrix}$$

can be assumed. From $\text{rank } W = \text{rank } Z$ identically in Z one obtains possibly after unitary transformation $A_{13} = E_{13}$.

For $A_{14} = (a_{kl})$ we get two possibilities. First the equation $\text{tr}(A_{12}\bar{A}_{14}) = \text{tr}(A_{13}\bar{A}_{14}) = 0$ implies $a_{12} = a_{13} = 0$. After unitary transformation all the elements of the first row besides a_{14} are zero. Then take only the elements z_{12}, z_{13}, z_{14} of Z distinct from zero; from $\text{rank } W = \text{rank } Z = 2$ one sees

$$A_{14} = E_{14} \quad \text{or} \quad A_{14} = E_{23}.$$

By a similar consideration $A_{1\nu}$ turns out to be $E_{1\nu}$ or E_{23} . But actually for $\nu > 4$ the second possibility $A_{1\nu} = E_{23}$ may not occur. For $A_{14} = A_{1\nu} = E_{23}$ is impossible because of $\text{tr}(A_{14}\bar{A}_{1\nu}) = 0$. If $A_{14} = E_{14}, A_{1\nu} = E_{23}$, choose only the elements $z_{1\nu}, z_{14} \neq 0$, then $\text{rank } Z = 2$ but $\text{rank } W = 4$. Therefore $A_{1\nu} = E_{1\nu}$ ($\nu \neq 4$), $A_{14} = E_{14}$ or E_{23} . Furthermore $A_{14} = E_{23}$ may only happen if $n = 4$. For assume $A_{14} = E_{23}, A_{15} = E_{15}$ and take only the elements $z_{14}, z_{15} \neq 0$. This implies $\text{rank } Z = 2$ but $\text{rank } W = 4$.

Let us summarize our results. After a suitable unitary transformation W can be written as

$$W = \begin{pmatrix} 0 & z' \\ -z & L(QZ_0) \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & z' \\ -z & Z_0 \end{pmatrix},$$

besides the exceptional case $n = 4, A_{14} = E_{23}$. Now $L(Z_0)$ is an analytic automorphism of \mathcal{E}_{n-1} with the fixed point 0. For $n = 3$ we know $L(Z_1) = e^{\zeta}Z_1$ with a real constant ζ . Therefore $W = U'ZU$ with a constant unitary matrix U , which is the theorem for $n = 3$. For $n > 5$ the induction hypothesis shows

$$W = \begin{pmatrix} 0 & z'U' \\ -Uz & Z_0 \end{pmatrix}$$

with constant unitary U . From the equality

$$\text{rank } W = \text{rank } Z$$

U turns out to be a diagonal matrix. Finally consider the sum of the two-rowed principal minors of $W\bar{W}$ and $Z\bar{Z}$. These two quantities are equal identically in Z because of the fact that $W\bar{W}$ and $Z\bar{Z}$ have the same characteristic roots. By this identity one obtains $U = aI$ with a complex number a of absolute value 1, which again proves our theorem.

There still remain the cases $n = 4$ and 5. For $n = 4, A_{14} = E_{14}$ we can use the reasoning above. Let $A_{14} = E_{23}$; since

$$\text{tr}(A_{1\nu}\bar{A}_{23}) = \text{tr}(A_{1\nu}\bar{A}_{24}) = \text{tr}(A_{1\nu}\bar{A}_{34}) = 0 \quad (\nu = 2, 3, 4)$$

W only differs from \tilde{Z} in the last row, where a linear combination of z_{23}, z_{24}, z_{34} appears. The identity between the ranks of Z and W shows $w_{14} = a_1z_{23}, w_{24} = a_2z_{24}, w_{34} = a_3z_{34}$. Now it is easy to compute the sum of the two-rowed principal minors of $W\bar{W}$ and $Z\bar{Z}$. This computation shows again the assertion for $n = 4$.

For $n = 5$ we know by the induction hypothesis

$$L(Z_0) = U'Z_0U \quad \text{or} \quad L(Z_0) = U'\tilde{Z}_0U$$

with constant unitary U . The first case can be treated as above. In the second case one obtains

$$W = \begin{pmatrix} 0 & z'U' \\ -Uz & Z_0 \end{pmatrix}.$$

Choose once only $z_{14}, z_{24} \neq 0$, then only $z_{14}, z_{34}, z_{45} \neq 0$. In any case $\text{rank } Z = 2$, hence $\text{rank } W = 2$. But this implies that all the elements of the third column of U vanish, which is a contradiction to the unitary character of U . This final remark completes the proof.

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