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In this note we establish a local version of the following result: a locally compact connected partially ordered non-degenerate semigroup S with unit contains a non-degenerate linearly ordered local subsemigroup (containing the unit). This is an extension of a result of Gleason [2; 664] who proved a similar theorem under the additional hypotheses that

(1) S is a semigroup with right invariant uniform structure and

(2) for any compact neighborhood U of the identity there are nets $\{x_i\}$ in S and $\{n_i\}$ integers such that $x_i \to e$ and $x_i^{n_i} \notin U$. A consequence of our theorem is the fact that a nondegenerate compact connected partially ordered semigroup with unit contains a standard thread joining the unit to the minimal ideal.

By a local semigroup S we mean a Hausdorff space with an open subset U and a multiplication $m: U \times U \to S$ which is continuous and associative insofar as is meaningful. A unit is an (unique, if it exists) element u of U satisfying $ux = xu = x$ for all $x \in U$. A local subsemigroup of S is a subset L containing the unit such that for some open set V about the unit, $(V \cap L)^2 \subset L$. We say that the local semigroup S is partially ordered if the relation $\leqq$ defined by $a \leqq b$ if and only if $a = bc$ is reflexive and antisymmetric. In case S is a semigroup, S is partially ordered if and only if each principal right ideal has a unique generator, i.e. (assuming a unit) that $aS = bS$ implies $a = b$. In this case, $\leqq$ is also transitive.

Closure is denoted by *, the null set by $\square$, the boundary of V by $F(V)$, and the complement of B in A by $A \backslash B$.

As in [4] we use the following topolopy for the space $\mathscr{S}(X)$ of non-empty closed subsets of the space X: for open sets U and V of X, let $N(U, V) = \{A \mid A \in \mathscr{S}(X), A \subset U, A \cap V \neq \square\}$; take $\{N(U, V) \mid U, V \text{ open}\}$ for a sub-basis for the open sets of $\mathscr{S}(X)$. It is easy to see that if X is compact Hausdorff, so is $\mathscr{S}(X)$.

THEOREM 1. *Let S be a locally compact partially ordered local semigroup with unit u, and let U_0 be a non-degenerate open connected set about u with U_0^6 defined. Then S contains a non-degenerate compact connected linearly ordered local sub-semigroup L with $u \in L \subset U_0$.*

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Proof. Let U_1 be an open set containing u with U_1^* compact and $U_1^* \subset U_0$. Define $\leqq$ on $U_1^* \times U_1^*$ by: $a \leqq b$ if and only if $a = bc$ for some $c \in U_1^*$. From the compactness of U_1^* it is easily seen that Graph $(\leqq)$ is closed in $U_1^* \times U_1^*$. We show first that $\leqq$ is transitive on some neighborhood of u. Let U_2 be an open set about u with $U_2^2 \subset U_1$. We claim there is an open set U containing u, $U \subset U_2$, such that if $a, b \in U^*$ with $a = bc$ for some $c \in U_1^*$, then $c \in U_2$. If this is false, then for any open set U with $u \in U \subset U_2$, there are elements a and b of U^* with $a = bc$ for some $c \in U_1^* \backslash U_2$. Hence there are nets a_α and b_α converging to u with $a_\alpha = b_\alpha \cdot c_\alpha$ where $c_\alpha \in U_1^* \backslash U_2$. It follows that c_α must also converge to u, a contradiction. Since $U_2^2 \subset U_1$ it follows that $\leqq$ is transitive on U^*. Also the restriction of $\leqq$ on $U^* \times U^*$ is closed and hence U^* is locally convex [6]. We show next that there exists an open set V_1 with $u \in V_1 \subset U$ such that $e^2 = e \in V_1$ implies $eU_0e \neq e$. Suppose the contrary; we can then find a net of idempotents $e_\alpha \rightarrow u$ with $e_\alpha U_0 e_\alpha = e_\alpha$. Let $x \in U_0$; then $e_\alpha = e_\alpha x e_\alpha$ converges to $uxu = x$, so that $x = u$ and U_0 is degenerate, a contradiction. Let V be a convex open set with $u \in V \subset V^* \subset (V^*)^2 \subset V_1$. Then $e^2 = e \in V$ implies $eU_0e \neq e$.

Let $\mathscr{C}$ denote the collection of all closed chains C in U^* with $u \in C$, $C \cap S \backslash V \neq \square$, and $(C \cap V)^2 \subset C$. Note that $\mathscr{C} \neq \square$, for if $a \in F(V)$, then the elements u and a constitute an element of $\mathscr{C}$.

(i) *$\mathscr{C}$ is closed in $\mathscr{S}(U^*)$.* We will show that $\mathscr{C}$ is an intersection of closed set. Since the collection of all closed chains which contain u and meet $S \backslash V$ is closed [4], it remains to show that the collection of closed chains C satisfying $(C \cap V)^2 \subset C$ is closed. Suppose A is a closed chain with $(A \cap V)^2 \not\subset A$; then there are elements a and b of $A \cap V$ with $ab \in S \backslash A$. Hence there exist open sets U_a, U_b, and W containing a, b, and A respectively, with $U_a \cdot U_b \cap W = \square$. Now $N(W, U_a) \cap N(W, U_b)$ is an open set about A, and contains no chain C with $(C \cap V)^2 \subset C$. This establishes (i).

As in [4], we define $L(x) = \{y \mid y \leqq x\}$, $M(x) = \{y \mid x \leqq y\}$, and $(x, y) = \{z \mid x < z < y\}$. Let δ be an open cover of U^*, and define a subset M_δ of $\mathscr{S}(U^*)$ by: $C \in M_\delta$ if and only if C is a closed chain in U^*, and for any x and y in C with $x < y$ and $(x, y) \cap C = \square$, there exists $D \in \delta$ such that D^* meets both $L(x) \cap C$.

(ii) *$M_\delta \cap \mathscr{C} \neq \square$ for any open cover δ of U^*.* Let δ be an open cover of U^*, and let $\mathscr{D}$ be the collection of all closed chains C with $u \in C \subset U$, $C \in M_\delta$, and $(V \cap C)^2 \subset C$. Let τ be a maximal tower in $\mathscr{D}$, and let $T = U\tau$. Then T^* is a closed chain, $u \in T^* \subset U^*$, and $(V \cap T^*)^2 \subset T^*$. As in [4], $T^* \in M_\delta$, and it remains to show that $T^* \in \mathscr{C}$, i.e., that $T^* \cap S \backslash V \neq \square$. Suppose $T^* \subset V$; (note then that $T = T^*$) then since $(T \cap V)^2 \subset T$, T is a compact chain and a semigroup. Let $e = \inf T$. Since $e^2 \leqq e$ and $e^2 \in T$ we have $e^2 = e$. We show next that e is a zero for T. Let $y \in T$, then $ey \in T$ and $ey \leqq e$, so $ey = e$ and e is a left zero for T. Hence the minimal ideal K of T consists of left zeros for T [1]. Let $f \in K$; then $e \leqq f$ so there exists $c \in U_1^*$ with $e = fc$. Therefore $f = fe = e$, and e is the unique left zero, and hence a zero for T. Let $W \in \delta$ with $e \in W$. If $eU_0e \cap W \cap V$ contains an idempotent $g \neq e$, then $T \cup g$ is a semigroup: for if $x \in T$ then $xg = x(eg) = eg = g$ and $gx = (ge)x = g(ex) = ge = g$. Also $T \cup g$ is a chain, so by the maximality of τ, $T = T \cup g$, a contradiction.

Hence we may assume that $eU_0e \cap W \cap V$ has a unique idempotent e. Since $\leqq$ is antisymmetric, the maximal subgroup of S containing e is e. Also eU_0e is a local semigroup with unit e, $eU_0e \neq e$, and e is not isolated in eU_0e which is the continuous image of U_0 and hence connected. Hence [5; 122] there is a non-degenerate one parameter local semigroup A with $e \in A \subset eU_0e \cap W \cap V$; let $a \in A$ with $a \neq e$ and $a^2 \in A$. Define $a^0 = e$ and let $B_k = \bigcup_{n=0}^{k} a^n[a, e]$, $B_\infty = \bigcup_{n=0}^{\infty} a^n[a, e]$ where $[a, e]$ denotes the sub-arc of A from a to e. We assume temporarily that all products involved in forming B_k and B_∞ are defined. Each of the sets $a^n[a, e]$ is a compact connected chain (hence an arc) with minimal element a^{n+1} and maximal element a^n. Hence B_k is a compact connected chain from a^{k+1} to e. Also B_∞ is a connected chain, hence B_∞^* is a closed connected chain. Using the easily established commutativity of B_k and B_∞^* it follows that for $x \in T$ and $b \in B_k$ (or B_∞^*) then $xb = x(eb) = (xe)b = eb = b$, and similarly $bx = b$. Hence $[(T \cup B_k^2) \cap V]^2 \subset T \cup (B_k^2 \cap V)^2$ and similarly with B_k replaced by B_∞^*. We distinguish two cases:

Case 1: For some $k \geqq 0$, $a^{k+1} \in V$ and $a^{k+2} \notin V$. Then since V is convex, $a^0, a, \cdots, a^{k+1}$ are in V and all products involved in forming B_k are defined, so that $B_k \subset V$ and $B_{k+1} \not\subset V$. We show first that $B_k^2 \cap V \subset B_k$. Let $z \in B_k^2 \cap V$; then $z = xy$ with $x, y \in B_k$, so $x = a^n x'$ and $y = a^m y'$ with x' and y' in $[a, e]$. Hence $xy = a^{m+n}x'y'$. If $x'y' \in A$, then since $z \in V$ it follows that $m + n \leqq k$. If $x'y' \notin A$, then $x'y' = at$ for some $t \in A$, so $xy = a^{m+n+1}t$ and $m + n + 1 \leqq k$. In either case, then, $z \in B_k$. Note that $(T \cup B_k)^2 \in M_\delta$ since B_k^2 is a connected chain. Also $[(T \cup B_k^2) \cap V]^2 \subset T \cup (B_k^2 \cap V)^2 \subset T \cup B_k^2$, so that $T \cup B_k^2 \in \mathscr{D}$. This contradicts the maximality of τ.

Case 2: $a^k \in V$ for each $k \geqq 0$. Using the convexity of V we see that all products involved in forming B_∞ are defined, and $B_\infty = B_\infty^2 \subset V$, hence $B_\infty^* = B_\infty^{*2}$. Since B_∞^* is a connected chain, it follows that $T \cup B_\infty^* \in M_\delta$. Also $[(T \cup B_\infty^*) \cap V]^2 \subset T \cup B_\infty^*$, so that $T \cup B_\infty^* \in \mathscr{D}$, a contradiction to the maximality of τ. The proof of (ii) is now complete.

(iii) *$M_\delta \cap \mathscr{C}$ is closed for each finite open cover δ of U^*.*

This proof is similar to that in [4], and is omitted.

For any finite open cover δ of U^*, let $P_\delta = M_\delta \cap \mathscr{C}$. The collection of sets $\{P_\delta\}$ is a descending family, so $\bigcap P_\delta \neq \square$. If $C \in \bigcap P_\delta$, then as shown in [4], C is an arc. Clearly C is a local semigroup, and the proof is complete.

In what follows, a *standard thread* is a compact connected semigroup irreducibly connected between a zero and a unit. The structure of standard threads is known [5; 130]. The example in [4] shows that a compact connected semigroup with zero and unit need not contain a standard thread joining the zero to the unit. The problem of finding standard threads joining zero to unit has an affirmative solution in case either

(1) S is compact, connected, and one-dimensional [3], or

(2) S is compact, connected, and each element is idempotent [4].

A third solution is given by the following corollary.

COROLLARY 1. *If S is a non-degenerate compact connected partially ordered semigroup with unit u, then the minimal ideal K consists of left zeros for S, K consists of the set of minimal elements, and some elements of K can be joined by a standard thread to the unit.*

Proof. Note that Graph ($\leqq$) is closed since S is compact. Let G be a compact group in S, with unit e. Since $x^2 \leqq x$ for each $x \in S$, then for $x \in G$ we have $e \geqq x \geqq x^2 \geqq \cdots$, and $\{x^n\}$ clusters at an idempotent, which must be e. We conclude that $x = e$, and hence that each compact group in S is trivial. From this fact it is clear that K is proper, for otherwise $K = S$ would be a compact group [1]. From the fact that $aS = bS$ implies $a = b$ we conclude that each minimal right ideal is a single element, hence each element of K is a left zero for S [1]. Since a minimal element x of S is characterized by the equality $xS = x$, it is clear that K consists of the set of minimal elements of S, and hence that $S \backslash K$ is convex. In the proof of the Theorem, we take $S = U_0 = U_1 = U_2 = U$, and $V = S \backslash K$. Hence there is a compact connected linearly ordered local semigroup L containing u, with $L \cap S \backslash V \neq \square$. Since the elements of K are minimal it follows that L is a semigroup, and hence a standard thread.

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