

Pacific Journal of Mathematics

**ON A COMMUTATOR RESULT OF TAUSSKY AND
ZASSENHAUS**

MARVIN DAVID MARCUS AND N. A. KHAN

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1. Introduction and results. Let M_n denote the set of n -square matrices over a field F . For A, B in M_n let $[A, B] = AB - BA'$, where A' is the transpose of A and define inductively

$$(1.1) \quad [A, B]_k = [A, [A, B]_{k-1}].$$

If $P^{-1}JP = A$, then

$$[A, X] = [P^{-1}JP, X] = P^{-1}[J, PXP'](P^{-1})',$$

and similarly

$$(1.2) \quad [A, X]_k = P^{-1}[J, PXP']_k(P^{-1})'.$$

Now for a fixed A let T be the linear map of M_n into itself defined by

$$(1.3) \quad T(Y) = [A, Y]$$

and (1.1) implies that

$$T^k(Y) = [A, Y]_k.$$

In a recent paper [1], Taussky and Zassenhaus showed that A is non-derogatory if and only if any nonsingular X in the null space of T is symmetric. In this note we investigate the structure of the null space of both T and T^2 for arbitrary A .

Enlarge the field F to include $\lambda_i, i = 1, \dots, p$, the distinct eigenvalues of A , and let $(x - \lambda_i)^{e_{ij}}, j = 1, \dots, n_i, e_{i1} > \dots > e_{in_i}, i = 1, \dots, p$ be the distinct elementary divisors of A where $(x - \lambda_i)^{e_{ij}}$ appears with multiplicity r_{ij} . Set $m_i = \sum_{j=1}^{n_i} r_{ij}e_{ij}$, the algebraic multiplicity of λ_i . Let $\eta(T)$ denote the null space of T , $\sigma(T)$ denote the subspace of symmetric matrices in $\eta(T)$, and $\gamma(T)$ denote the subspace of skew-symmetric matrices in $\eta(T)$. We show that

$$(1.4) \quad \dim \gamma(T) = \sum_{i=1}^p \left[\sum_{j=1}^{n_i} \left(r_{ij}^2 e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right],$$

$$(1.5) \quad \dim \sigma(T) = \frac{1}{2} \sum_{i=1}^p \left[\sum_{j=1}^{n_i} \left\{ r_{ij}(r_{ij} + 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right\} \right],$$

Received December 17, 1959. The work of this author was supported by U. S. National Science Foundation Grant, NSF-G5416. The second author is a Postdoctorate Fellow of the National Research Council of Canada. The authors are grateful to Professor O. Taussky for her helpful suggestions.

$$(1.6) \quad \dim \eta(T^2) = \sum_{i=1}^p \left[\sum_{j=1}^{n_i} \left\{ r_{ij}^2(2e_{ij} - 1) + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right],$$

$$(1.7) \quad \dim \sigma(T^2) = \frac{1}{2} \sum_{i=1}^p \left[\sum_{j=1}^{n_i} \left\{ r_{ij}^2(2e_{ij} - 1) + r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right].$$

In case A is nonderogatory, $n_i = 1, r_{ij} = 1, i=1, \dots, p$ and (1.4) and (1.5) reduce to

$$\dim \eta(T) = n = \dim \sigma(T).$$

Thus every matrix X satisfying

$$(1.8) \quad AX = XA'$$

where A is non-derogatory is symmetric, the result in [1]. Moreover, if every matrix X satisfying (1.8) is symmetric then $\dim \eta(T) = \dim \sigma(T)$. Using the formulas (1.4) and (1.5) we see that this condition implies that

$$\sum_{i=1}^p \sum_{j=1}^{n_i} (r_{ij}^2 - r_{ij})e_{ij} + 2 \sum_{i=1}^p r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} = 0.$$

Now since r_{ij}, e_{ij} and n_i are all positive integers we conclude that $r_{ij} = 1, j = 1, \dots, n_i$ and $n_i = 1$. That is, there is only one elementary divisor corresponding to each eigenvalue. Hence, if every matrix X satisfying (1.8) is symmetric then A is non-derogatory, a result also found in [1].

We also show in this case that $\eta(T)$ consists of matrices of the form PXP' where P is fixed (depending on A) and X is persymmetric, (i.e. all the entries of X on each line perpendicular to the main diagonal are equal).

We next note that $\eta(T) = \sigma(T) + \gamma(T)$ (direct) and $\eta(T^2) = \sigma(T^2) + \gamma(T^2)$ (direct). The first statement is easy to show; we indicate the brief proof of the second statement:

Since $X = \frac{X + X'}{2} + \frac{X - X'}{2}$, if $X \in \eta(T^2)$, then

$$\begin{aligned} T^2(X + X') &= [A, [A, X + X']] \\ &= [A, [A, X] + [A, X']] \\ &= [A, [A, X]] + [A, [A, X']] \\ &= T^2(X) - [A, [A, X']] \\ &= [A, [A, X]]' \\ &= (T^2(X))' = 0. \end{aligned}$$

Similarly, $T^2(X - X') = 0$. Thus any $X \in \eta(T^2)$ is expressible uniquely as a sum of two elements, one in $\sigma(T^2)$ and the other in $\gamma(T^2)$. Hence

$$(1.9) \quad \begin{aligned} \dim \gamma(T) &= \dim \eta(T) - \dim \sigma(T) \\ &= \frac{1}{2} \sum_{i=1}^p \left[\sum_{j=1}^{n_i} \left\{ r_{ij}(r_{ij} - 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right], \end{aligned}$$

$$(1.10) \quad \begin{aligned} \dim \gamma(T^2) &= \dim \eta(T^2) - \dim \sigma(T^2) \\ &= \frac{1}{2} \sum_{i=1}^p \left[\sum_{j=1}^{n_i} \left\{ r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right]. \end{aligned}$$

In case A is non-derogatory, (1.6), (1.7) and (1.10) reduce to

$$\dim \eta(T^2) = 2n - p,$$

$$\dim \sigma(T^2) = n,$$

$$\dim \gamma(T^2) = n - p.$$

We thus conclude that *unless all the eigenvalues of A are distinct ($p = n$) there exist skew-symmetric matrices X satisfying*

$$(1.11) \quad A^2X - 2AXA' + X(A')^2 = 0.$$

If $p = n$, and A is non-derogatory

$$\dim \eta(T^2) = n = \dim \sigma(T^2)$$

and any matrix X satisfying (1.11) is symmetric.

On the other hand suppose

$$\dim \eta(T^2) = \dim \sigma(T^2).$$

From (1.6) and (1.7) we conclude that

$$\sum_{i=1}^p \left[\sum_{j=1}^{n_i} \left\{ r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right] = 0.$$

Hence $n_i = 1$, $r_{ij} = 1$, $e_{ik} = 1$ and we conclude that $p = n$. That is, *if every matrix X satisfying (1.11) is symmetric then the eigenvalues of A are distinct.*

We show finally (Theorem 2) that *if A is an n -square matrix with p distinct eigenvalues then both $\dim \gamma(T)$ and $\dim \gamma(T^2)$ are at most $\frac{1}{2}(n - p)(n - p + 1)$. Moreover, for each p this bound is best possible.*

Thus if there exists a skew-symmetric solution of (1.8) or (1.11), then A has multiple eigenvalues, without the assumption that A is non-derogatory.

II. *Proofs.* Let $E_{ij} \in M_n$ be the matrix with 1 in position i, j and 0 elsewhere. With respect to this basis, ordered lexicographically, it may be checked that T has the matrix representaion

$$(2.1) \quad T = I \otimes A - A \otimes I$$

where \otimes indicates Kronecker product.

From (1.2) we may take A to be in Jordan canonical form J , since $[A, X]_k = 0$ if and only if $[J, PXP']_k = 0$ and PXP' is symmetric if and only if X is. We write

$$(2.2) \quad J = \sum_{s=1}^p J_s$$

where

$$(2.3) \quad J_s = \lambda_s I_{m_s} + \sum_{t=1}^{n_s} \sum_1^{r_{st}} U_{e_{st}} ;$$

\sum indicates direct sum, I_t is a t -square identity matrix, U_t is t -square auxiliary unit matrix (i.e. 1 in the superdiagonal and 0 elsewhere) and $\sum_1^{r_{st}} U_{e_{st}}$ is the direct sum of $U_{e_{st}}$ with itself r_{st} times.

By a routine computation we see that

$$T^k(Y) = 0$$

if and only if

$$(2.4) \quad \sum_{\alpha=0}^k \binom{k}{\alpha} (-1)^\alpha J_s^{k-\alpha} Y_{st} (J_t')^\alpha = 0, \quad s, t = 1, \dots, p,$$

where $Y = (Y_{st})$, $s, t = 1, \dots, p$ is a partitioning of Y conformal with the partitioning of J given by (2.2).

For $s \neq t$, it is clear that the matrix representation of (2.4),

$$(I_{m_t} \otimes J_s - J_t \otimes I_{m_s})^k$$

has the single nonzero eigenvalue $(\lambda_s - \lambda_t)^k$ and thus $Y_{st} = 0$. Hence we need only consider the equation (2.4) for $s = t$. We may again partition Y_{ss} conformally with J_s in (2.3). We are thus led to consider the null space of the mapping

$$(2.5) \quad (I_{e_{st}} \otimes U_{e_{sj}} - U_{e_{st}} \otimes I_{e_{sj}})^k .$$

LEMMA 1. Let $T = I_m \otimes U_n - U_m \otimes I_n$. Then

$$(2.6) \quad \dim \eta(T) = \min(m, n),$$

$$(2.7) \quad \dim \eta(T^2) = \begin{cases} 2 \min(m, n), & \text{if } m \neq n \\ 2n - 1, & \text{if } m = n. \end{cases}$$

Proof. Suppose $n \leq m$ and that $T(X) = 0$. Let x_1, \dots, x_n be the column n -vectors of X . Then we have

$$(2.8) \quad U_n x_j - x_{j+1} = 0, \quad j = 1, 2, \dots, m-1,$$

$$U_n x_m = 0.$$

For $r = 1, 2, \dots, n-1$ consider the $(r-j+1)$ coordinate of (2.8) for $j = 1, \dots, r$ and we conclude that

$$x_{r+1,1} = x_{r,2} = \dots = x_{1,r+1} = c_{r+1}.$$

Next consider the $(n-j+1)$ coordinate of (2.8) for $j = 1, \dots, n$ to obtain

$$0 = x_{n2} = x_{n-1,3} = \dots = x_{1,n+1}.$$

Similarly we see that the remaining elements of X are zero. Hence we find that the j th column of the $n \times m$ matrix X is the transpose of the n -vector

$$[c_j, c_{j+1}, \dots, c_n, 0, \dots, 0]$$

for $j = 1, 2, \dots, n$. The other $m-n$ columns are zero.

In case $n \geq m$, it is easy to check that the j th row of X is the m -vector

$$[c_j, c_{j+1}, \dots, c_m, 0, \dots, 0]$$

for $j = 1, 2, \dots, m$. The other $n-m$ rows are zero.

This establishes (2.6). To prove (2.7) let $T^2(X) = 0$ and x_1, x_2, \dots, x_m be the column n -vectors of X . Let us consider the following cases:

(i) $m = n$.

We have

$$U_n^2 x_n = 0, \quad U_n^2 x_{n-1} = 2U_n x_n$$

and

$$U_n^2 x_j - 2U_n x_{j+1} + x_{j+2} = 0, \quad j = 1, 2, \dots, n-2.$$

Solving these equations recursively we find that the 1st, 2nd and j th rows of X are respectively

$$[x_{11}, x_{12}, \dots, x_{1,n-2}, x_{1,n-1}, x_{1n}],$$

$$[x_{21}, x_{22}, \dots, x_{2,n-2}, x_{2,n-1}, 0]$$

and

$$(j-1)[x_{2,j-1}, x_{2,j}, \dots, x_{2,n-1}, 0, \dots, 0]$$

$$- (j-2)[x_{1,j}, x_{1,j+1}, \dots, x_{1,n}, 0, \dots, 0],$$

for $j = 3, 4, \dots, n$.

The number of arbitrary parameters in X is $2n-1$.

(ii) $n < m$.

Here we have the following equations:

$$(2.9) \quad \begin{aligned} U_n^2 x_j - 2U_n x_{j+1} + x_{j+2} &= 0, \quad j = 1, 2, \dots, m-2 \\ U_n^2 x_{m-1} - 2U_n x_m &= 0 \\ U_n^2 x_m &= 0 \end{aligned}$$

and by solving recursively again we find that the 1st, 2nd and j th rows of X are respectively the m -vectors

$$\begin{aligned} [x_{11}, \dots, x_{1,n-1}, x_{1,n}, nx_{n,2}, 0, \dots, 0], \\ [x_{21}, \dots, x_{2,n-1}, (n-1)x_{n,2}, 0, 0, \dots, 0] \end{aligned}$$

and

$$\begin{aligned} [(j-1)x_{2,j-1}, \dots, (j-1)x_{2,n-1}, (n-j+1)x_{n,2}, 0, \dots, 0] \\ - (j-2)[x_{1,j}, \dots, x_{1,n}, 0, 0, \dots, 0] \end{aligned}$$

for $j = 3, 4, \dots, n$.

In case $n > m$, by similar computation we find that the 1st, 2nd and j th rows of X are respectively

$$\begin{aligned} [x_{11}, \dots, x_{1,m-2}, x_{1,m-1}, x_{1m}], \\ [x_{21}, \dots, x_{2,m-2}, x_{2,m-1}, x_{2m}] \end{aligned}$$

and

$$\begin{aligned} (j-1)[x_{2,j-1}, \dots, x_{2,m-1}, x_{2m}, 0, \dots, 0] \\ - (j-2)[x_{1,j}, \dots, x_{1,m}, 0, 0, \dots, 0] \end{aligned}$$

for $j = 3, 4, \dots, m+1$. The remaining $n-m-1$ rows are zero.

From case (ii), we observe that the number of parameters in X is $2 \min(m, n)$.

We now state and prove the following

LEMMA 2. *Let A be an n -square matrix with the single eigenvalue λ and let $(x - \lambda)^{n_i}$ be an elementary divisor of A of multiplicity r_i , $i = 1, \dots, p$, $n_1 > \dots > n_p$. Then the most general matrix X satisfying (1.11) has*

$$(2.10) \quad \sum_{i=1}^p \left[r_i^2 (2n_i - 1) + 4r_i \sum_{j=i+1}^p r_j e_j \right]$$

arbitrary parameters.

Moreover if X is symmetric it contains

$$(2.11) \quad \frac{1}{2} \sum_{i=1}^p \left[r_i^2(2n_i - 1) + r_i + 4r_i \sum_{j=i+1}^p r_j n_j \right]$$

parameters.

Proof. Without any loss of generality we can assume that

$$(2.12) \quad A = \sum_{i=1}^p \sum_{j=1}^{r_i} U_i$$

where $\sum U_i$ indicates the direct sum of U_i with itself r_i times. We partition X conformally with A in (2.12) and observe that the equation

$$U_i^2 X_{ij} - 2U_i X_{ij} U_j + X_{ij}(U_j)^2 = 0$$

determines the structure of any block X_{ij} in the partitioning of X .

From case (i) of Lemma 1, we conclude that any block X_{ij} corresponding to equal U_i 's contains $2n_i - 1$ arbitrary parameters and there are r_i^2 such blocks. Also from case (ii) any block in X that corresponds to U_i and $U_j, i < j$, contains $2n_j$ arbitrary parameters. Hence the total number of parameters in X is given by (2.10).

In order to find the number of parameters in a symmetric X we first consider a diagonal block. Its structure has been discussed in Lemma 1, case (i). We observe that if this matrix is symmetric, the number of parameters in it reduces from $2n_i - 1$ to n_i .

Then we consider two symmetrically placed off-diagonal blocks X_{ij} and X_{ji} of orders $n_i \times n_j$ and $n_j \times n_i$ respectively. If X is to be symmetric then by equating the terms of X_{ij} and X_{ji} which are symmetrically placed about the main diagonal of X , the number of arbitrary parameters in X_{ij} and X_{ji} reduces from $2(2n_j)$ to $2n_j$. If X_{ij} and X_{ji} are of order $n_i \times n_i$ then the number of parameters reduces from $2(2n_i - 1)$ to $2n_i - 1$.

We are now in a position to sum the number of parameters in X if it is symmetric and satisfies (1.11). There are r_i blocks in the main diagonal, each of order $n_i, i = 1, \dots, p$. The number of parameters in each of these blocks is n_i . There are $r_i(r_i - 1)/2$ other square blocks of order n_i . Each of them contains $(2n_i - 1)$ parameters. Thus

$$\frac{1}{2} \sum_{i=1}^p \{r_i^2(2n_i - 1) + r_i\}$$

is the number of parameters in all those blocks of X which are square. Since any block of order $n_i \times n_j$ where $n_i > n_j$ contains $2n_j$ parameters, and since we are considering X to be symmetric, we conclude that the total number of arbitrary parameters in X is given by (2.11).

We can similarly prove the following

LEMMA 3. *Let A be the matrix given in Lemma 2. Then the most*

general matrix X satisfying (1.8) has

$$\sum_{i=1}^p \left(r_i^2 n_i + 2r_i \sum_{j=i+1}^p r_j n_j \right)$$

arbitrary parameters.

Moreover if X is symmetric, it contains

$$\frac{1}{2} \sum_{i=1}^p \left[r_i (r_i + 1) n_i + 2r_i \sum_{j=i+1}^p r_j n_j \right]$$

parameters.

We now state and prove the following

THEOREM 1. *Let A be an n -square matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_p$ and let $(x - \lambda_i)^{e_{ij}}$, $j = 1, \dots, n_i$, $e_{i1} > \dots > e_{in_i}$ be the elementary divisors of A corresponding to λ_i , where each $(x - \lambda_i)^{e_{ij}}$ has been repeated r_{ij} times. Then (1.4), (1.5), (1.6) and (1.7) hold.*

Proof. It was pointed out earlier that if $Y = (Y_{rs})$, $r, s = 1, \dots, p$ is the partitioning of Y conformal with the partitioning of J in (2.2), then all the off-diagonal blocks are zero. Hence we have simply to find the number of parameters in Y_{ii} , $i = 1, \dots, p$.

As proved in Lemma 2, the number of parameters in Y_{ii} is

$$\sum_{j=1}^{n_i} \left[r_{ij}^2 (2e_{ij} - 1) + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right].$$

Summing the above with respect to i we obtain the formula (1.6). In case Y is symmetric, the number of parameters in Y_{ii} is

$$\frac{1}{2} \sum_{j=1}^{n_i} \left[r_{ij}^2 (2e_{ij} - 1) + r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right].$$

Summing the above on i we obtain (1.7).

Similarly, we can make use of Lemma 3 in proving (1.4) and (1.5).

We now prove

THEOREM 2. *Let A be as given in Theorem 1. Then the maximum number of linearly independent skew-symmetric matrices satisfying (1.8) or (1.11) is*

$$\frac{1}{2} (n - p)(n - p + 1).$$

Proof. In order to prove our result for $\dim \gamma(T^2)$, let $m_i = \sum_{j=1}^{n_i} r_{ij} e_{ij}$ and consider

$$\begin{aligned}
 m_i^2 - m_i &- \sum_{j=1}^{n_i} \left[r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \\
 &= \sum_{j=1}^{n_i} \left[r_{ij}^2e_{ij}^2 + 2r_{ij}e_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} - r_{ij}e_{ij} \right] \\
 &\quad - \sum_{j=1}^{n_i} \left[r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \\
 &= \sum_{j=1}^{n_i} \left[r_{ij}^2(e_{ij} - 1)^2 - r_{ij}(e_{ij} - 1) + 2r_{ij}(e_{ij} - 2) \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right].
 \end{aligned}$$

Now, it is clear that $r_{ij}^2(e_{ij} - 1) \geq r_{ij}(e_{ij} - 1)$. The last term in the above expression will be negative only when $e_{ij} = 1$. But we know that $e_{i1} > e_{i2} > \dots > e_{in_i}$, so that e_{ij} will be 1 only for $j = n_i$. In that case $\sum_{k=j+1}^{n_i}$ does not appear, and we have

$$\frac{1}{2} \sum_{j=1}^{n_i} \left[r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \leq \frac{1}{2} (m_i^2 - m_i).$$

This holds for $i = 1, \dots, p$.

To determine a bound on $\gamma(T)$, consider

$$\begin{aligned}
 m_i^2 - m_i &- \sum_{j=1}^{n_i} \left[r_{ij}(r_{ij} - 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \\
 &= \sum_{j=1}^{n_i} \left[r_{ij}^2e_{ij}(e_{ij} - 1) + 2r_{ij}(e_{ij} - 1) \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \\
 &\geq 0, \text{ since } e_{ij} \geq 1.
 \end{aligned}$$

Thus we have

$$\frac{1}{2} \sum_{j=1}^{n_i} \left[r_{ij}(r_{ij} - 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \leq \frac{1}{2} (m_i^2 - m_i).$$

It may be observed that the upper bound is attained for $r_{i1} = m_i, e_{i1} = 1$ and the remaining e 's and r 's all zero.

We have thus proved that

$$\dim \gamma(T^2) \leq \frac{1}{2} \sum_{i=1}^p (m_i^2 - m_i)$$

and

$$\dim \gamma(T) \leq \frac{1}{2} \sum_{i=1}^p (m_i^2 - m_i),$$

where m_i is the multiplicity of the eigenvalue λ_i of A .

Now we have to maximize $\sum_{i=1}^p (m_i^2 - m_i)$ under the condition that

$m_1 + \cdots + m_p = n$, the order of A . Note that

$$m_i^2 - m_i = (m_i - 1)^2 + (m_i - 1)$$

and each $m_i - 1 \geq 0$. Hence, we have

$$\sum_{i=1}^p (m_i - 1)^2 \leq \left[\sum_{i=1}^p (m_i - 1) \right]^2 = (n - p)^2.$$

Thus the maximum value of both $\dim \gamma(T^2)$ and $\dim \gamma(T)$ is

$$\frac{1}{2}[(n - p)^2 + (n - p)].$$

The bounds are achieved when $m_1 = \cdots = m_{p-1} = 1$ and $m_p = n - p + 1$.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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| | |
|---|------|
| M. Altman, <i>An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space</i> | 1107 |
| Nesmith Cornett Ankeny, <i>Criterion for rth power residuacity</i> | 1115 |
| Julius Rubin Blum and David Lee Hanson, <i>On invariant probability measures I</i> | 1125 |
| Frank Featherstone Bonsall, <i>Positive operators compact in an auxiliary topology</i> | 1131 |
| Billy Joe Boyer, <i>Summability of derived conjugate series</i> | 1139 |
| Delmar L. Boyer, <i>A note on a problem of Fuchs</i> | 1147 |
| Hans-Joachim Bremermann, <i>The envelopes of holomorphy of tube domains in infinite dimensional Banach spaces</i> | 1149 |
| Andrew Michael Bruckner, <i>Minimal superadditive extensions of superadditive functions</i> | 1155 |
| Billy Finney Bryant, <i>On expansive homeomorphisms</i> | 1163 |
| Jean W. Butler, <i>On complete and independent sets of operations in finite algebras</i> | 1169 |
| Lucien Le Cam, <i>An approximation theorem for the Poisson binomial distribution</i> | 1181 |
| Paul Civin, <i>Involutions on locally compact rings</i> | 1199 |
| Earl A. Coddington, <i>Normal extensions of formally normal operators</i> | 1203 |
| Jacob Feldman, <i>Some classes of equivalent Gaussian processes on an interval</i> | 1211 |
| Shaul Foguel, <i>Weak and strong convergence for Markov processes</i> | 1221 |
| Martin Fox, <i>Some zero sum two-person games with moves in the unit interval</i> | 1235 |
| Robert Pertsch Gilbert, <i>Singularities of three-dimensional harmonic functions</i> | 1243 |
| Branko Grünbaum, <i>Partitions of mass-distributions and of convex bodies by hyperplanes</i> | 1257 |
| Sidney Morris Harmon, <i>Regular covering surfaces of Riemann surfaces</i> | 1263 |
| Edwin Hewitt and Herbert S. Zuckerman, <i>The multiplicative semigroup of integers modulo m</i> | 1291 |
| Paul Daniel Hill, <i>Relation of a direct limit group to associated vector groups</i> | 1309 |
| Calvin Virgil Holmes, <i>Commutator groups of monomial groups</i> | 1313 |
| James Fredrik Jakobsen and W. R. Utz, <i>The non-existence of expansive homeomorphisms on a closed 2-cell</i> | 1319 |
| John William Jewett, <i>Multiplication on classes of pseudo-analytic functions</i> | 1323 |
| Helmut Klingen, <i>Analytic automorphisms of bounded symmetric complex domains</i> | 1327 |
| Robert Jacob Koch, <i>Ordered semigroups in partially ordered semigroups</i> | 1333 |
| Marvin David Marcus and N. A. Khan, <i>On a commutator result of Taussky and Zassenhaus</i> | 1337 |
| John Glen Marica and Steve Jerome Bryant, <i>Unary algebras</i> | 1347 |
| Edward Peter Merkes and W. T. Scott, <i>On univalence of a continued fraction</i> | 1361 |
| Shu-Teh Chen Moy, <i>Asymptotic properties of derivatives of stationary measures</i> | 1371 |
| John William Neuberger, <i>Concerning boundary value problems</i> | 1385 |
| Edward C. Posner, <i>Integral closure of differential rings</i> | 1393 |
| Marian Reichaw-Reichbach, <i>Some theorems on mappings onto</i> | 1397 |
| Marvin Rosenblum and Harold Widom, <i>Two extremal problems</i> | 1409 |
| Morton Lincoln Slater and Herbert S. Wilf, <i>A class of linear differential-difference equations</i> | 1419 |
| Charles Robson Storey, Jr., <i>The structure of threads</i> | 1429 |
| J. François Treves, <i>An estimate for differential polynomials in $\partial/\partial z_1, \dots, \partial/\partial z_n$</i> | 1447 |
| J. D. Weston, <i>On the representation of operators by convolutions integrals</i> | 1453 |
| James Victor Whittaker, <i>Normal subgroups of some homeomorphism groups</i> | 1469 |