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ON UNIVALENCE OF A CONTINUED FRACTION

EDWARD PETER MERKES AND W. T. SCOTT

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1. Introduction. For a fixed positive integer α let K_α denote the class of functions $f(z)$ which are regular at $z = 0$ and which have C -fraction expansions of the form

$$(1.1) \quad f(z) \sim \frac{z}{1 + \frac{a_1 z^\alpha}{1 + \frac{a_2 z^\alpha}{1 + \dots + \frac{a_n z^\alpha}{1 + \dots}}}, |a_n| \leq 1/4.$$

From an elementary convergence theorem for continued fractions [4, p.42], it follows that each function of the class K_α is regular for $|z| < 1$. This and the one-to-one correspondence between C -fractions and power series [4, p. 400] permit a replacement of the correspondence symbol in (1.1) by equality for $|z| < 1$.

The purpose of this paper is to determine for K_α the radius of univalence, $U(\alpha)$, and bounds for the starlike radius, $S(\alpha)$, and the radius of convexity, $C(\alpha)$. In the case of S -fractions it was shown by Thale [3] that $U(1) \geq 12\sqrt{2}-16$ and Perron [2] established the fact that actual equality holds. This result is a special case of Theorem 2.1 whose proof employs value region techniques similar to those used by Thale and Perron. Moreover, the result $S(1) \geq 8/9$ in [3] is improved in Theorem 4.2.

The developments in this depend on the following value region theorem which is an immediate consequence of a result of Paydon and Wall [1]:

THEOREM 1.1. *If $f(z) \in K_\alpha$ and $|z|^\alpha = \rho^\alpha \leq 4r(1 - r)$, $0 \leq r \leq 1/2$, then*

$$(1.2) \quad \left| \frac{f(z)}{z} - \frac{1}{1 - r^2} \right| \leq \frac{r}{1 - r^2}.$$

Moreover, for $z = \sqrt[\alpha]{4r(1 - r)} e^{im\pi/\alpha}$, ($m = 1, 2, \dots, \alpha$), there is a value of $f(z)/z$ on the boundary of the disc (1.2) if and only if there exists a φ , $0 \leq \varphi < 2\pi$, such that $f(z) \equiv f(z; \varphi)$, where

$$(1.3) \quad f(z; \varphi) = \frac{z}{1 + \frac{\frac{1}{4}e^{i\varphi}z^\alpha}{1 + \frac{\frac{1}{4}z^\alpha}{1 + \dots + \frac{\frac{1}{4}z^\alpha}{1 + \dots}}}}.$$

2. Determination of $U(\alpha)$. For $f(z) \in K_\alpha$ and for a fixed positive integer n put

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$$(2.1) \quad \begin{aligned} f_{0,n}(z) &= z, \\ f_{p+1,n}(z) &= \frac{z}{1 + a_{n-p}z^{\alpha-1}f_{p,n}(z)}, \quad (p = 0, 1, \dots, n - 1), \end{aligned}$$

where the numbers a_j are the coefficients in the C -fraction expansion (1.1) of $f(z)$. It is easily seen that $f_{n,n}(z)$ is the approximant of (1.1) of order $n + 1$, and that $f_{p,n}(z) \in K_\alpha$ for each p .

For non-negative integers s, t , and for non-zero numbers z_1, z_2 , (2.1) may be used to show that

$$(2.2) \quad \begin{aligned} & z_1^s z_2^t f_{p+1,n}(z_1) - z_1^t z_2^s f_{p+1,n}(z_2) \\ &= \frac{f_{p+1,n}(z_1)f_{p+1,n}(z_2)}{z_1 z_2} \{ z_1^{s+1} z_2^t - z_1^t z_2^{s+1} - a_{n-p} [z_1^{t+\alpha-1} z_2^{s+1} f_{p,n}(z_1) \\ & \quad - z_1^{s+1} z_2^{t+\alpha-1} f_{p,n}(z_2)] \}, \quad (p = 0, 1, \dots, n - 1). \end{aligned}$$

This identity plays a fundamental role in the proof of the following theorem.

THEOREM 2.1. *The radius of univalence of K_α is given by*

$$(2.3) \quad \begin{aligned} U(2) &= 2\sqrt{2/3}, \\ [U(\alpha)]^\alpha &= \left[\frac{6\sqrt{\alpha^2 - 2\alpha + 9} - 2(\alpha + 7)}{(\alpha - 2)^2} \right], \quad (\alpha = 1, 3, 4, \dots). \end{aligned}$$

There is no larger region, containing the disc $|z| < U(\alpha)$, in which all functions of K_α are univalent.

Proof. For $f(z) \in K_\alpha$ and for a fixed positive odd integer $n=2m+1$ it follows from (2.2) that

$$(2.4) \quad \begin{aligned} & f_{n,n}(z_1) - f_{n,n}(z_2) \\ &= \frac{f_{n,n}(z_1)f_{n,n}(z_2)}{z_1 z_2} \{ z_1 - z_2 - a_1 [z_1^{\alpha-1} z_2 f_{n-1,n}(z_1) - z_1 z_2^{\alpha-1} f_{n-1,n}(z_2)] \}. \end{aligned}$$

Repeated application of (2.2) yields

$$(2.5) \quad \begin{aligned} & a_1 [z_1^{\alpha-1} z_2 f_{n-1,n}(z_1) - z_1 z_2^{\alpha-1} f_{n-1,n}(z_2)] \\ &= \sum_{j=1}^{m+1} (z_1 z_2)^{(j-1)\alpha+1} (z_1^{\alpha-1} - z_2^{\alpha-1}) \prod_{p=1}^{2j-1} a_p \frac{f_{n-p,n}(z_1)f_{n-p,n}(z_2)}{z_1 z_2} \\ & \quad - \sum_{j=1}^m (z_1 z_2)^{j\alpha} (z_1 - z_2) \prod_{p=1}^{2j} a_p \frac{f_{n-p,n}(z_1)f_{n-p,n}(z_2)}{z_1 z_2}. \end{aligned}$$

For z_1 and z_2 in the disc $|z| < 1$, r can be chosen with $0 < r < 1/2$ such that $|z_i|^\alpha \leq 4r(1-r)$, ($i = 1, 2$), and by Theorem 1.1, $|f_{p,n}(z_i)/z_i| \leq 1/(1-r)$, ($i = 1, 2; p = 0, 1, \dots, n$). When the triangle inequality is applied to the right member of (2.5) and the indicated bounds are used, there

results

$$\begin{aligned}
 & |a_1| |z_1^{\alpha-1} z_2 f_{n-1,n}(z_1) - z_1 z_2^{\alpha-1} f_{n-1,n}(z_2)| \\
 & \leq |z - z_2| \left[\sum_{j=1}^{m+1} (\alpha - 1) \left(\frac{r}{1-r} \right)^{2j-1} + \sum_{j=1}^m \left(\frac{r}{1-r} \right)^{2j} \right] \\
 & < |z_1 - z_2| \frac{r}{1-2r} [\alpha - 1 - (\alpha - 2)r].
 \end{aligned}$$

This inequality and (2.4) give

$$(2.6) \quad \begin{aligned}
 & |f_{n,n}(z_1) - f_{n,n}(z_2)| \\
 & \geq \frac{|f_{n,n}(z_1)f_{n,n}(z_2)|}{|z_1 z_2|} |z_1 - z_2| \left\{ 1 - \frac{r[\alpha - 1 - (\alpha - 2)r]}{1 - 2r} \right\}.
 \end{aligned}$$

Since Theorem 1.1 shows that neither of the factors $|f_{n,n}(z_i)/z_i|$, $(i=1, 2)$, is zero, it follows from (2.6) that $f_{n,n}(z_1) \neq f_{n,n}(z_2)$ for $z_1 \neq z_2$ if r is such that $1 - 2r > r[\alpha - 1 - (\alpha - 2)r]$. This is equivalent to the condition $r < r_0(\alpha)$ where

$$\begin{aligned}
 r_0(2) &= 1/3 \\
 r_0(\alpha) &= \frac{\alpha + 1 - \sqrt{\alpha^2 - 2\alpha + 9}}{2(\alpha - 2)}, \quad (\alpha = 1, 3, 4, \dots),
 \end{aligned}$$

and it is easily seen that $f_{2m+1,2m+1}(z)$ is univalent for $|z|^\alpha < [U(\alpha)]^\alpha = 4r_0(\alpha)[1 - r_0(\alpha)]$.

If the function $f(z)$ has a non-terminating C -fraction (1.1), the univalence of $f(z)$ for $|z| < U(\alpha)$ is an immediate consequence of the fact that $f(z)$ is the uniform limit of its sequence of even approximants, $f_{2m+1,2m+1}(z)$, for $|z| \leq \rho < 1$. The case where $f(z)$ has a C -fraction expansion (1.1) terminating with an odd number of partial quotients may be reduced to the previously considered case for even approximants by adding a partial quotient, $a_{2m} z^\alpha / 1$ with $a_{2m} = 0$, and noting that $f_{2m-1,2m-1}(z) = f_{2m,2m}(z)$ in this case.

In order to complete the proof that the radius of univalence of K_α is the value $U(\alpha)$ given in (2.3), it suffices to exhibit a function of K_α which is not univalent in $|z| < \rho$ for any $\rho > U(\alpha)$. Such a function is the function $f(z, \pi)$ of (1.3), that is,

$$f(z, \pi) = \frac{2z}{3 - \sqrt{1 + z^\alpha}},$$

where the branch of the radical with positive real part for $|z| < 1$ is used. This function is not univalent at the points $e^{im\pi/\alpha} U(\alpha)$, $(m = 1, 2, \dots, \alpha)$, where its derivative vanishes.

The final statement in Theorem 2.1 may be verified by applying to the function $f(z, \pi)$ the observation that, for every real θ , $e^{-i\theta} f(e^{i\theta} z) \in K_\alpha$

whenever $f(z) \in K_\alpha$.

3. A covering theorem. The value region inequality (1.2) can be rewritten as

$$(3.1) \quad \left| \frac{f(z)}{z} - \frac{4}{2 + \rho^\alpha + 2\sqrt{1 - \rho^\alpha}} \right| \leq \frac{2(1 - \sqrt{1 - \rho^\alpha})}{2 + \rho^\alpha + 2\sqrt{1 - \rho^\alpha}},$$

where $|z| = \rho$ and $f(z) \in K_\alpha$. Thus for $|z| = \rho$ the following inequalities, which provide a means of comparison between K_α and various classes of univalent functions, are obtained:

$$(3.2) \quad \frac{2}{3 - \sqrt{1 - \rho^\alpha}} \leq \Re \left\{ \frac{f(z)}{z} \right\} \leq \frac{2}{1 + \sqrt{1 - \rho^\alpha}},$$

$$(3.3) \quad \left| \Im \frac{f(z)}{z} \right| \leq \frac{2(1 - \sqrt{1 - \rho^\alpha})}{2 + \rho^\alpha + 2\sqrt{1 - \rho^\alpha}},$$

$$(3.4) \quad \frac{2\rho}{3 - \sqrt{1 - \rho^\alpha}} \leq |f(z)| \leq \frac{2(1 - \sqrt{1 - \rho^\alpha})}{\rho^{\alpha-1}},$$

$$(3.5) \quad \left| \arg \frac{f(z)}{z} \right| \leq \arcsin \frac{1 - \sqrt{1 - \rho^\alpha}}{2}.$$

Each of the inequalities (3.2)–(3.5) is sharp. This fact follows at once from Theorem 1.1 since equality in any one of (3.2)–(3.5) depends on the attainment by $f(z)/z$ of a suitable boundary value for the disc (3.1) or (1.2).

The following theorem is an immediate consequence of (3.4) and Theorem 2.1:

THEOREM 3.1. *If $f(z) \in K_\alpha$, then the image of $|z| < U(\alpha)$ by $w = f(z)$ contains the disc*

$$(3.6) \quad |w| < \frac{2U(\alpha)}{3 - \sqrt{1 - [U(\alpha)]^\alpha}},$$

and is contained in the disc

$$(3.7) \quad |w| < 2 \frac{1 - \sqrt{1 - [U(\alpha)]^\alpha}}{[U(\alpha)]^{\alpha-1}}.$$

These results are sharp.

4. A lower bound for $S(\alpha)$. An upper bound for $S(\alpha)$, the starlike radius for the class K_α , is evidently the value $U(\alpha)$ determined in § 2. In this section a lower bound for $S(\alpha)$ is found by determining a number

$\rho_1(\alpha)$ such that every function of K_α is starlike in the disc $|z| < \rho_1(\alpha)$.

LEMMA 4.1. *If $f(z) \in K_\alpha$ and $|a| \leq 1/4$, then*

$$(4.1) \quad w(z) = -\frac{az^{\alpha-1}f(z)}{1+az^{\alpha-1}f(z)}$$

satisfies

$$(4.2) \quad \left| w - \frac{r^2}{1-r^2} \right| \leq \frac{r}{1-r^2}$$

whenever $|z|^\alpha \leq 4r(1-r)$, $0 \leq r \leq 1/2$.

Proof. The lemma is obvious when $a = 0$. For $0 < |a| \leq 1/4$, (4.1) yields

$$\frac{f(z)}{z} = \frac{1}{az^\alpha} \cdot \frac{-w(z)}{1+w(z)},$$

and the desired result is easily obtained by applying the inequality $|f(z)/z| \leq 1/(1-r)$, which is a consequence of Theorem 1.1.

LEMMA 4.2. *If α is a positive integer and if for fixed r , $0 < r < 1/2$, c and d are numbers such that*

$$(4.3) \quad 0 \leq c \leq \frac{1 + (\alpha - 2)r^2}{1 - 2r^2}, \quad 0 < d = \frac{1 + (\alpha - 2)r}{1 - 2r} - c,$$

then $\sigma = 1$ satisfies

$$(4.4) \quad |\sigma - c| \leq d.$$

Moreover, if w is a parameter satisfying (4.2) and if σ_0 satisfies (4.4), then σ_1 satisfies (4.4) where

$$(4.5) \quad \sigma_1 = 1 + w(\sigma_0 + \alpha - 1).$$

Proof. It is obvious that $1 - c \leq d$ holds for all r , $0 < r < 1/2$, and that $-d \leq 1 - c$ holds provided

$$c \leq \frac{2 + (\alpha - 4)r}{2(1 - 2r)}.$$

The fact that $\sigma = 1$ satisfies (4.4) may be verified by noting that the upper bound of c in this last inequality exceeds the upper bound on c in (4.3) for all r , $0 < r < 1/2$.

The proof of the second statement is obtained by using (4.2), (4.3),

(4.4), (4.5), and the triangle inequality to show that

$$\begin{aligned} |\sigma_1 - c| &\leq \left| 1 - c + \frac{(c + \alpha - 1)r^2}{1 - r^2} \right| \\ &\quad + (c + \alpha - 1) \left| w - \frac{r^2}{1 - r^2} \right| + |w| |\sigma_0 - c| \\ &\leq \frac{1 + (\alpha - 2)r^2 - (1 - 2r^2)c}{1 - r^2} + \frac{(c + \alpha - 1)r}{1 - r^2} + \frac{rd}{1 - r^2} = d. \end{aligned}$$

LEMMA 4.3. *If (4.3) holds for $0 < r < 1/2$, there is a value of c satisfying $c \geq d$ if and only if $0 < r \leq r_1(\alpha)$, where $r_1(\alpha)$ is the smallest positive root of*

$$(4.6) \quad 1 - (\alpha + 2)r + 2(\alpha - 1)r^2 - 2(\alpha - 2)r^3 = 0.$$

Proof. By (4.3) the inequality $c \geq d$ holds if and only if

$$\frac{1 + (\alpha - 2)r^2}{1 - 2r^2} \geq \frac{1 + (\alpha - 2)r}{2(1 - 2r)},$$

which is equivalent to the statement that the left member of (4.6) is nonnegative. Clearly $r_1(\alpha) < 1/2$.

THEOREM 4.1. *If $f(z) \in K_\alpha$ and c, d satisfy (4.3), where $|z|^\alpha = \rho^\alpha \leq 4r(1 - r)$, then*

$$(4.7) \quad \left| z \frac{f'(z)}{f(z)} - c \right| \leq d.$$

Proof. For the functions $f_{p,n}(z)$ of (2.1) put

$$\sigma_{p,n} = z \frac{f'_{p,n}}{f_{p,n}}, \quad w_{p,n} = - \frac{a_{n-p} z^{\alpha-1} f_{p+1,n}}{1 + a_{n-p} z^{\alpha-1} f_{p+1,n}},$$

and note by differentiation that $\sigma_{p+1,n} = 1 + w_{p,n}(\sigma_{p,n} + \alpha - 1)$. For $|z| = \rho$ inductive application of Lemmas 4.1 and 4.2 shows that (4.7) holds for $f_{n,n}$, and the validity of (4.7) in this case for $|z| \leq \rho$ follows from the maximum property for harmonic functions. Inasmuch as $f_{n,n}$ is the $(n + 1)$ th approximant of (1.1) the theorem holds for functions of K_α having terminating C -fraction expansions. The validity of the theorem in the case of non-terminating C -fractions (1.1) is an immediate consequence of the uniform convergence of $f_{n,n}$ to f on any closed subset of $|z| < 1$.

THEOREM 4.2. *The starlike radius of K_α satisfies $S(\alpha) \geq \rho_1(\alpha)$ where*

$[\rho_1(\alpha)]^\alpha = 4r_1(\alpha)[1 - r_1(\alpha)]$ and where $r_1(\alpha)$ is the smallest positive root of (4.6).

Proof. For $r \leq r_1(\alpha)$ Lemma 4.3 shows that Theorem 4.1 can be applied to any function $f(z) \in K_\alpha$ with $c \geq d$, and hence that

$$\operatorname{Re} z \frac{f'(z)}{f(z)} \geq 0, \quad |z| \leq \rho_1(\alpha).$$

Since this inequality insures that $f(z)$ is starlike for $|z| < \rho_1(\alpha)$ the proof is complete.

In particular, $r_1(1) = (\sqrt{3} - 1)/2$ and $S(1) \geq 4\sqrt{3} - 6$ which improves the lower bound of 8/9 obtained for $S(1)$ in [3].

5. A lower bound for $C(\alpha)$. It is clear that $S(\alpha)$ and $U(\alpha)$ are upper bounds for $C(\alpha)$, the radius of convexity of K_α . In this section a lower bound for $C(\alpha)$ is found by determining a number $\rho_2(\alpha)$ such that every function of K_α is convex for $|z| < \rho_2(\alpha)$.

LEMMA 5.1. *Let α denote a positive integer and let $r_2(\alpha)$ be the smallest positive root of the equation:*

$$(5.1) \quad 1 - (\alpha^2 + 2\alpha + 6)r + 6(\alpha^2 + \alpha + 2)r^2 - 4(3\alpha^2 + 2)r^3 + 12(\alpha - 1)\alpha r^4 - 4\alpha(\alpha - 2)r^5 = 0.$$

If for fixed r , $0 < r \leq r_2(\alpha)$, σ_0 and σ_1 are numbers which satisfy

$$(5.2) \quad |\sigma_0 - c| \leq d, \quad |\sigma_1 - c| \leq d,$$

where

$$(5.3) \quad \frac{1 + (\alpha - 2)r}{2(1 - 2r)} \leq c \leq \frac{1 + (\alpha - 2)r^2}{1 - 2r^2}, \quad d = \frac{1 + (\alpha - 2)r}{1 - 2r} - c,$$

and if

$$(5.4) \quad \gamma_1 = 2(\sigma_1 - 1) + \frac{\sigma_1 - 1}{\sigma_1} \left[\gamma_0 \frac{\sigma_0}{\sigma_0 + \alpha - 1} + (\alpha - 1) \frac{2\sigma_0 + \alpha - 2}{\sigma_0 + \alpha - 1} \right],$$

then $|\gamma_0| \leq 1$ implies $|\gamma_1| \leq 1$.

Proof. For $0 < r < r_1(\alpha)$, where $r_1(\alpha)$ is as determined in Theorem 4.2, $0 < d < c$ and

$$c^2 - d^2 - c \leq -\frac{\alpha r^2 [(\alpha - 1) - 2(\alpha - 2)r + 2(\alpha - 2)r^2]}{(1 - 2r)^2(1 - 2r^2)} \leq 0.$$

Thus by (5.2)

$$\left| \frac{\sigma_1 - 1}{\sigma_1} - \frac{c^2 - d^2 - c}{c^2 - d^2} \right| \leq \frac{d}{c^2 - d^2}$$

and it follows that

$$\left| \frac{\sigma_1 - 1}{\sigma_1} \right| \leq \frac{1}{c - d} - 1.$$

Similarly, (5.2) can be used to show that

$$\left| \frac{\sigma_0}{\sigma_0 + \alpha - 1} \right| \leq \frac{c + d}{c + d + \alpha - 1},$$

$$\left| (\alpha - 1) \frac{2\sigma_0 + \alpha - 2}{\sigma_0 + \alpha - 1} \right| \leq (\alpha - 1) \frac{2(c + d) + \alpha - 2}{c + d + \alpha - 1}.$$

For $|\gamma_0| \leq 1$ application to (5.4) of the triangle inequality, (5.2) and the bounds determined above lead to the inequality

$$(5.5) \quad |\gamma_1| \leq 2(c + d - 1) + \left[\frac{1}{c - d} - 1 \right] \frac{(2\alpha - 1)(c + d) + (\alpha - 1)(\alpha - 2)}{c + d + \alpha - 1}.$$

The desired inequality, $|\gamma_1| \leq 1$, will hold for those values of $r < r_1(\alpha)$ for which the right member of (5.5) does not exceed 1, or equivalently, for which

$$(5.6) \quad c - d \geq \frac{(2\alpha - 1)(c + d) + (\alpha - 1)(\alpha - 2)}{(2\alpha - 1)(c + d) + (\alpha - 1)(\alpha - 2) + [3 - 2(c + d)][c + d + \alpha - 1]} = D.$$

Since $2c = (c + d) + (c - d)$, (5.3) shows that the existence of a value of c satisfying (5.6) is insured for all $r < r_1(\alpha)$ for which

$$(5.7) \quad 2 \frac{1 + (\alpha - 2)r^2}{1 - 2r^2} \geq (c + d) + D.$$

This last inequality is equivalent to the requirement that the polynomial in the left member of (5.1) be non-negative.

The proof of the lemma will be completed by establishing the existence of a smallest positive zero, $r_2(\alpha)$ of (5.1) for which $r_2(\alpha) < r_1(\alpha)$. Since the equation (4.7) determining $r_1(\alpha)$ is equivalent to

$$2 \frac{1 + (\alpha - 2)r^2}{1 - 2r^2} = c + d,$$

and since $D > 0$ for $r = r_1(\alpha)$, it follows that (5.7) fails to hold for $r = r_1(\alpha)$. The desired conclusion about $r_2(\alpha)$ is then easily obtained by noting that (5.7) holds with strict inequality for $r = 0$.

THEOREM 5.1. *The radius of convexity of K_α satisfies*

$$(5.8) \quad [C(\alpha)]^\alpha \geq 4r_2(\alpha)[1 - r_2(\alpha)] = [\rho_2(\alpha)]^\alpha$$

where $r_2(\alpha)$ is the smallest positive root of (5.1)

Proof. For the functions $f_{p,n}(z)$ of (2.1) put

$$\sigma_{p,n} = z \frac{f'_{p,n}}{f_{p,n}}, \quad \gamma_{p,n} = z \frac{f''_{p,n}}{f'_{p,n}}.$$

It is easily verified from (2.1) that

$$\gamma_{p+1} = 2(\sigma_{p+1} - 1) + \frac{\sigma_{p+1} - 1}{\sigma_{p+1}} \left[\frac{\gamma_p \sigma_p}{\sigma_p + \alpha - 1} + (\alpha - 1) \frac{2\sigma_p + \alpha - 2}{\sigma_p + \alpha - 1} \right]$$

where the subscript n has been omitted. Theorem 4.1 and the fact that $\gamma_{0,n} = 0$ show that the hypotheses of Lemma 5.1 are satisfied, and inductive application of the lemma yields $|\gamma_{n,n}| \leq 1$. It follows that

$$\Re[1 + \gamma_{n,n}] \geq 0, \quad |z| \leq \rho_2(\alpha),$$

which insures the convexity of the $(n + 1)$ th approximant of any C -fraction (1.1) for $|z| < \rho_2(\alpha)$, and the proof of the theorem may be completed, as in Theorem 4.1, by reference to uniform convergence.

It is found that $\rho_2(1) > .641$. An upper bound for $C(\alpha)$ can be obtained by finding for the function $f(z, \pi)$ of (1.3) the zeros of $zf''(z, \pi) + f'(z, \pi)$ with smallest modulus. For $\alpha = 1$ this smallest modulus is approximately .707.

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