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ASYMPTOTIC PROPERTIES OF DERIVATIVES OF STATIONARY MEASURES

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1. Introduction. Let X be a non-empty set and $\mathscr{S}$ be a σ-algebra of subsets of X. Consider the infinite product space $\Omega = \prod_{n=-\infty}^{\infty} X_n$ where $X_n = X$ for $n = 0, \pm 1, \pm 2, \cdots$ and the infinite product σ-algebra $\mathscr{F} = \prod_{n=-\infty}^{\infty} \mathscr{S}_n$ where $\mathscr{S}_n = \mathscr{S}$ for $n = 0, \pm 1, \pm 2, \cdots$. Elements of Ω are bilateral infinite sequences $\{\cdots, x_{-1}, x_0, x_1, \cdots\}$ with $x_n \in X$. Let us denote the elements of Ω by w. If $w = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$ x_n is called the nth coordinate of w and shall be considered as a function on Ω to X. Let T be the shift transformation on Ω to Ω: the nth coordinate of Tw is equal to the $n+1$th coordinate of w. For any function g on Ω, Tg is the function defined by $Tg(w) = g(Tw)$ so that $Tx_n = x_{n+1}$ for any integer n. We shall consider two probability measures μ, ν defined on $\mathscr{F}$. For $n = 1, 2, \cdots$ let $\Omega_n = \prod_{i=1}^{n} X_i$ where $X_i = X, i = 1, 2 \cdots, n$ and $\mathscr{F}_n = \prod_{i=1}^{n} \mathscr{S}_i$ where $\mathscr{S}_i = \mathscr{S}, i = 1, 2, \cdots, n$. Then $\Omega_1 = X$ and $\mathscr{F}_1 = \mathscr{S}$. Let $\mathscr{F}_{m\,n}, m \leqq n, n = 0, \pm 1, \pm 2, \cdots,$ be the σ-algebra of subsets of Ω consisting of sets of the form

$$[w = \{\cdots, x_{-1}, x_0, x_1 \cdots\} \colon (x_m, x_{m+1}, \cdots, x_n) \in E]$$

Where $E \in \mathscr{F}_{n-m+1}$. Then $\mathscr{F}_{m\,n} \subset \mathscr{F}_{m\,n+1} \subset \mathscr{F}$. Let $\mu_{m\,n}, \nu_{m\,n}$ be the contractions of μ, ν, respectively to $\mathscr{F}_{m\,n}$. If $\nu_{m\,n}$ is absolutely continuous with respect to $\mu_{m\,n}$, the derivative of $\nu_{m\,n}$ with respect to $\mu_{m\,n}$ is a function of $x_m, \cdots, x_n$ and shall be designated by $f_{m\,n}(x_m, \cdots, x_n)$. Since $f_{m\,n}(x_m, \cdots, x_n)$ is positive with ν-probability one $1/f_{m\,n}(x_m, \cdots x_n)$ is well defined with ν-probability one. We shall let the function $1/f_{m\,n}(x_m, \cdots, x_n)$ take on the value 0 when $f_{m\,n}(x_m, \cdots, x_n) \leqq 0$. Thus $1/f_{m\,n}(x_m, \cdots, x_n)$ is well defined everywhere. In fact $1/f_{m\,n}(x_m, \cdots, x_n)$ is the derivative of $\nu_{m\,n}$-continuous part of $\mu_{m\,n}$ with respect to $\nu_{m\,n}$. According to the celebrated theorem of E. S. Anderson and B. Jessen [1] and J. L. Doob ([2]), pp. 343) $1/f_{m\,n}(x_m, \cdots, x_n)$ converges with ν-probability one as $n \to \infty$. If we assume that μ, ν are stationary, i.e., μ, ν are T invariant, more precise results may be expected. A fundamental theorem of Information Theory, first proved by C. Shannon for stationary Markovian measures [5] and later generalized to any stationary measure by B. McMillan [4], may be considered as a theorem of this sort. In their theorem X is assumed to be a finite set. In this paper we shall first treat Markovian stationary measures μ, ν with X being

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any set, finite or infinite, and $\mathscr{S}$ any σ-algebra of subsets of X. It will be proved that $n^{-1}\log f_{m\,n}(x_m, \cdots, x_n)$ converges as $n \to \infty$ with ν-probability one and also in $L_1(\nu)$ under some integrability conditions. The case that ν is only stationary is also treated. Similar convergence theorem is proved under the assumption that X is countable.

2. Asymptotic properties of derivatives of a Markovian measure with stationary transition probabilities with respect to another such measure.

Let X, $\mathscr{S}$, Ω, $\mathscr{F}$, Ω_n, $\mathscr{F}_n$, $\mathscr{F}_{m\,n}$, $\mu_{m\,n}$, $\nu_{m\,n}$ $f_{m\,n}(x_m, \cdots, x_n)$ be as in § 1. x_n, $n = 0, \pm 1, \pm 2, \cdots$, are considered as functions or random variables on Ω to X. Notations for conditional prababilities and conditional expectations relative to one or several random variables will be as in [2], chapter 1, § 7. Since we have two probability measures we shall use subscripts μ, ν to indicate conditional probabilities and conditional expectations taken under measures μ, ν respectively. In this section μ, ν *are assumed to be Markovian* i.e., for any $A \in \mathscr{S}$, $m < n$, $n = 0 \pm 1, \pm 2, \cdots$,

(1) $P_\mu[x_n \in A \mid x_m, \cdots, x_{n-1}] = P_\mu[x_n \in A \mid x_{n-1}]$ with μ-probability one and

(2) $P_\nu[x_n \in A \mid x_m, \cdots, x_{n-1}] = P_\nu[x_n \in A \mid x_{n-1}]$ with ν-probability one. For any set $E \subset \Omega$ let I_E be the real valued function on Ω defined by

$$I_E(w) = 1 \text{ if } w \in E$$
$$= 0 \text{ if } w \notin E .$$

LEMMA 1. *If $\nu_{n-1\,n}$ is absolutely continuous with respect to $\mu_{n-1\,n}$ then for any $A \in \mathscr{S}$*

$$(3) \qquad P_\nu[x_n \in A \mid x_{n-1}] f_{n-1\,n-1}(x_{n-1}) = E_\mu[I_{(x_n \in A)} f_{n-1\,n}(x_{n-1}, x_n) \mid x_{n-1}] \text{ with } \mu\text{-probability one.}$$

Proof. For any $A, B \in \mathscr{S}$

$$\nu[x_n \in A,\, x_{n-1} \in B] = \int_{[x_{n-1} \in B]} P_\nu[x_n \in A \mid x_{n-1}] d\nu = \int_{[x_{n-1} \in B]} P_\nu[x_n \in A \mid x_{n-1}] f_{n-1\,n-1}(x_{n-1}) d\mu .$$

On the other hand

$$\nu[x_n \in A,\, x_{n-1} \in B] = \int_{[x_{n-1} \in B]} I_{x_n \in A} f_{n-1\,n}(x_{n-1}, x_n) \mid x_{n-1}) d\mu$$

$$= \int_{[x_{n-1} \in B]} E_\mu[I_{x_n \in A} f_{n-1\, n}(x_{n-1}, x_n) \mid x_{n-1}] d\mu \ .$$

Hence for any $B \in \mathscr{S}$

$$\int_{[x_{n-1} \in B]} P_\nu[x_n \in A \mid x_{n-1}] f_{n-1\, n-1}(x_{n-1}) d\mu$$
$$= \int_{[x_{n-1} \in B]} E_\mu[I_{x_n \in A} f_{n-1\, n}(x_{n-1}, x_n) \mid x_{n-1}] d\mu \ ,$$

therefore (3) is true with μ-probability one. Dividing both sides of (3) by $f_{n-1\, n-1}(x_{n-1})$ we then have

$$(4) \qquad P_\nu[x_n \in A \mid x_{n-1}] = \frac{E_\mu[I_{x_n \in A} f_{n-1\, n}(x_{n-1}, x_n) \mid x_{n-1}]}{f_{n-1\, n-1}(x_{n-1})} \ .$$

With μ-probability one on the set $[f_{n-1\, n-1}(x_{n-1}) > 0]$. Since $\nu[f_{n-1\, n-1}(x_{n-1}) > 0] = 1$, (4) is true with ν-probability one.

THEOREM 1. *If $\nu_{n-1\, n}$ is absolutely continuous with respect to $\mu_{n-1\, n}$ for $n = 0, \pm 1, \pm 2, \cdots$ then $\nu_{m\, n}$ is absolutely continuous with respect to $\mu_{m\, n}$ for $n = 0, \pm 1, \pm 2, \cdots$ and $m \leqq n$ with*

$$(5) \qquad f_{m\, n}(x_m, \cdots, x_n) = f_{m\, m+1}(x_m, x_{m+1}) \frac{f_{m+1\, m+2}(x_{m+1}, x_{m+2})}{f_{m+1\, m+1}(x_{m+1})} \cdots \frac{f_{n-1\, n}(x_{n-1}, x_n)}{f_{n-1\, n-1}(x_{n-1})}$$

with μ-probability one.

Proof. We shall prove the theorem for the case that $m = 1, n = 2, 3, \cdots$. The proof for the general case that m is any integer is similar. Since $\nu_{1\, 2}$ is absolutely continuous with respect to $\mu_{1\, 2}$ by hypothesis, (5) is trivially true for $m = 1, n = 2$. Suppose $\nu_{1\, k}$ ($k \geqq 2$) is absolutely continuous with respect to $\mu_{1\, k}$ and $f_{1\, k}(x_1, \cdots, x_k)$ is given by (5) with μ-probability one. For any $A \in \mathscr{S}, B \in \mathscr{F}_k$

$$\nu[x_{k+1} \in A, (x_1, \cdots, x_k) \in B]$$
$$= \int_{[(x_1, \cdots, x_k) \in B]} P_\nu[x_{k+1} \in A \mid x_1, \cdots, x_k] d\nu \ .$$

Since ν is Markovian and by (4)

$$\nu[x_{k+1} \in A, (x_1, \cdots, x_k) \in B]$$
$$= \int_{[(x_1, \cdots, x_k) \in B]} P_\nu[x_{k+1} \in A \mid x_k] d\nu$$

$$= \int_{[(x_1,\cdots,x_k)\in B]} \frac{E_\mu[I_{x_{k+1}\in A} f_{k\,k+1}(x_k, x_{k+1}) \mid x_k]}{f_{k\,k}(x_k)} d\nu$$

$$= \int_{[(x_1,\cdots,x_k)\in B]} \frac{E_\mu[I_{x_{k+1}\in A} f_{k\,k+1}(x_k, x_{k+1}) \mid x_k]}{f_{k\,k}(x_k)} f_{1\,k}(x_1, \cdots, x_k) d\mu .$$

Since μ is Markovian

$$E_\mu[I_{x_{k+1}\in A} f_{k\,k+1}(x_k, x_{k+1}) \mid x_k] = E_\mu[I_{x_{k+1}\in A} f_{k\,k+1}(x_k, x_{k+1}) \mid x_1, \cdots, x_k]$$

with μ-probability one. Hence

$$\nu[x_{k+1} \in A, (x_1, \cdots, x_k) \in B] = \int_{(x_1,\cdots,x_k)\in B} E_\mu\left[I_{x_{k+1}\in A} \frac{f_{k\,k+1}(x_k, x_{k+1})}{f_{k\,k}(x_k)} f_{1\,k}(x_1, \cdots, x_k) \mid x_1, \cdots, x_k \right] d\mu$$

$$= \int_{(x_1,\cdots,x_k)\in B} I_{x_{n+1}\in A} f_{1\,k}(x_1, \cdots, x_k) \frac{f_{k\,k+1}(x_k, x_{k+1})}{f_{k\,k}(x_k)} d\mu .$$

Hence

$$\nu[x_{k+1} \in A, (x_1, \cdots, x_k) \in B] = \int_{[x_{k+1}\in A, (x_1,\cdots,x_k)\in B]} f_{1\,k}(x_1, \cdots, x_k) \frac{f_{k\,k+1}(x_k, x_{k+1})}{f_{k\,k}(x_k)} d\mu$$

for any $A \in \mathscr{S}, B \in \mathscr{F}_k$. Hence for any $E \in \mathscr{F}_{1\,k+1}$

$$\nu(E) = \int_E f_{1\,k}(x_1, \cdots, x_k) \frac{f_{k\,k+1}(x_k, x_{k+1})}{f_{k\,k}(x_k)} d\mu ,$$

Therefore $\nu_{1\,k+1}$ is absolutely continuous with respect to $\mu_{1\,k+1}$ and

$$(6) \qquad f_{1\,k+1}(x_1, \cdots, x_{k+1}) = f_{1\,k}(x_1, \cdots, x_k) \frac{f_{k\,k+1}(x_k, x_{k+1})}{f_{k\,k}(x_k)}$$

with μ-probability one. (6) together with the supposition that (5) holds true for $m = 1, n = k$ implies that (5) holds true for $m = 1, n = k + 1$. Thus the theorem for the case that $m = 1$ is proved.

Any Markovian probability measure on $\mathscr{F}$ is said to have *stationary transition probabilities* if E being a set of probability one implies that TE, $T^{-1}E$ are also of probability one and for any $A \in \mathscr{S}$ and any n

$$P[x_{n+1} \in A \mid x_n] = TP[x_n \in A \mid x_{n-1}]$$

with probability one. Thus for a Markovian probability measure with stationary transition probabilities we have for any pair of integers m, n and any $A \in \mathscr{S}$

(7) $P[x_n \in A \mid x_{n-1}] = T^{n-m}P[x_m \in A \mid x_{m-1}]$ with probability one and

(8) $E[g(x_{n-1}, x_n) \mid x_{n-1}] = T^{n-m}E[g(x_{m-1}, x_m) \mid x_{m-1}]$ wtih probability one for any real valued $\mathscr{F}_2$-measurable function g on Ω_2.

THEOREM 2. *Let both μ, ν have stationary transition probabilities. If $\nu_{n\,n}$ is absolutely continuous with respect to $\mu_{n\,n}$ for $n = 0, \pm 1, \pm 2, \cdots$ and $\nu_{1\,2}$ is absolutely continuous with respect to $\mu_{1\,2}$ then $\nu_{m\,n}$ is absolutely continuous with respect to $\mu_{m\,n}$ for $m \leqq n, n = 0, \pm 1, \pm 2, \cdots$ and*

(9)
$$f_{m\,n}(x_m, \cdots, x_n) = f_{m\,m}(x_m)\frac{f_{1\,2}(x_m, x_{m+1})}{f_{1\,1}(x_m)} \cdots \cdots \frac{f_{1\,2}(x_{n-1}, x_n)}{f_{1\,1}(x_{n-1})}$$

with μ-probability one.

Proof. By Lemma 1, for any $A \in \mathscr{S}$

(10)
$$P_\nu[x_2 \in A \mid x_1] = \frac{E_\mu[I_{x_2 \in A} f_{1\,2}(x_1, x_2) \mid x_1]}{f_{1\,1}(x_1)}$$

with ν-probability one. For any $A, B \in \mathscr{S}$

$$\begin{aligned}
&\nu[x_n \in A, x_{n-1} \in B] \\
&= \int_{[x_{n-1} \in B]} P_\nu[x_n \in A \mid x_{n-1}] d\nu \\
&= \int_{[x_{n-1} \in B]} T^{n-2} P_\nu[x_2 \in A \mid x_1] d\nu \\
&= \int_{[x_{n-1} \in B]} \{T^{n-2} P_\nu[x_2 \in A \mid x_1]\} f_{n-1\,n-1}(x_{n-1}) d\mu .
\end{aligned}$$

Hence by (10) and (8)

$$\begin{aligned}
&\nu[x_n \in A, x_{n-1} \in B] \\
&= \int_{[x_{n-1} \in B]} T^{n-2} \left\{ \frac{E_\mu[I_{x_2 \in A} f_{1\,2}(x_1, x_2) \mid x_1]}{f_{1\,1}(x_1)} \right. f_{n-1\,n-1}(x_{n-1}) d\mu \\
&= \int_{[x_{n-1} \in B]} \frac{E_\mu[I_{x_n \in A} f_{1\,2}(x_{n-1}, x_n) \mid x_{n-1}]}{f_{1\,1}(x_{n-1})} f_{n-1\,n-1}(x_{n-1}) d\mu \\
&= \int_{[x_{n-1} \in B]} I_{x_n \in A} f_{n-1\,n-1}(x_{n-1}) \frac{f_{1\,2}(x_{n-1}, x_n)}{f_{1\,1}(x_{n-1})} d\mu \\
&= \int_{[x_n \in A, x_{n-1} \in B]} f_{n-1\,n-1}(x_{n-1}) \frac{f_{1\,2}(x_{n-1}, x_n)}{f_{1\,1}(x_{n-1})} d\mu .
\end{aligned}$$

Thus for any $E \in \mathscr{F}_{n-1\,n}$

$$\nu(E) = \int_E f_{n-1\,n-1}(x_{n-1}) \frac{f_{1\,2}(x_{n-1}, x_n)}{f_{1\,1}(x_{n-1})} d\mu \,. \tag{11}$$

Hence for any integer n, $\nu_{n-1\,n}$ is absolutely continuous with respect to $\mu_{n-1\,n}$ and Theorem 1 is applicable. (11) also implies that

$$f_{n-1\,n}(x_{n-1}, x_n) = f_{n-1\,n-1}(x_{n-1}) \frac{f_{1\,2}(x_{n-1}, x_n)}{f_{1\,1}(x_{n-1})} \tag{12}$$

with μ-probability one. Hence

$$\frac{f_{n-1\,n}(x_{n-1}, x_n)}{f_{n-1\,n-1}(x_{n-1})} = \frac{f_{1\,2}(x_{n-1}, x_n)}{f_{1\,1}(x_{n-1})} \tag{13}$$

with μ-probability one on the set $[f_{n-1\,n-1}(x_{n-1}) > 0]$. However, except that w belongs to a set of μ-probability 0, $n > 1$, $f_{n-1\,n-1}(x_{n-1}(w)) = 0$ imply that $f_{1\,n-1}(x_1(w), \cdots, x_{n-1}(w)) = 0$, hence

$$f_{1\,n-1}(x_1, \cdots, x_{n-1}) \frac{f_{n-1\,n}(x_{n-1}, x_n)}{f_{n-1\,n-1}(x_{n-1})} = f_{1\,n-1}(x_1, \cdots, x_{n-1}) \frac{f_{1\,2}(x_{n-1}, x_n)}{f_{1\,1}(x_{n-1})}$$

with μ-probability one. Thus by (6)

$$f_{1\,n}(x_1, \cdots, x_n) = f_{1\,n-1}(x_1, \cdots, x_{n-1}) \frac{f_{1\,2}(x_{n-1}, x_n)}{f_{1\,1}(x_{n-1})} \tag{14}$$

with μ-probability one. Combining (12) (13) and by induction, if $n > 1$

$$f_{1\,n}(x_1, \cdots, x_n) = f_{1\,1}(x_1) \frac{f_{1\,2}(x_1, x_2)}{f_{1\,1}(x_1)} \cdots \frac{f_{1\,2}(x_{n-1}, x_n)}{f_{1\,1}(x_{n-1})}$$

with μ-probability one. Thus we have proved the theorem for the case that $m = 1$. For the general case the proof is similar.

THEOREM 3. *If μ has stationary transition probabilities and ν is stationary and if*

$$\int |\log f_{m\,m+1}(x_m, x_{m+1})| \, d\nu < \infty \quad \textit{then}$$

$$\int |\log f_{m\,n}(x_m, \cdots, x_n)| \, d\nu < \infty \quad \textit{for } n = m, m+1, m+2, \cdots$$

and $n^{-1} \log f_{m\,n}(x_m, \cdots, x_n)$ converges as $n \to \infty$ with ν-probability one and also in $L_1(\nu)$ to a function g with $\int g \, d\nu = a$ where

$$a = \int [\log f_{1\,2}(x_1, x_2) - \log f_{1\,1}(x_1)] d\nu \geqq 0$$

In particular, if ν is ergodic, $g = a$ with ν-probability one.

Proof. We shall first prove the theorem for the case that $m = 1$. Since for any $A \in \mathscr{S}$

$$\nu[x_1 \in A] = \int_{[x_1 \in A]} f_{1\,1}(x_1) d_\mu = \int_{[x_1 \in A]} f_{1\,2}(x_1, x_2) d\mu ,$$

hence

$$E_\mu[f_{1\,2}(x_1, x_2) \mid x_1] = f_{1\,1}(x_1) .$$

Since $\int \mid \log f_{1\,2}(x_1, x_2) \mid d\nu < \infty$ hence

$$\int \mid f_{1\,2}(x_1, x_2) \log f_{1\,2}(x_1, x_2) \mid d\mu = \int \mid \log f_{1\,2}(x_1, x_2) \mid d\nu < \infty .$$

The real valued function $L(\xi) = \xi \log \xi$ defined for all real $\xi \geqq 0[L(0)$ is taken to be 0] is convex. By Jensen's inequality for conditional expectations ([2], pp. 33)

$$E_\mu[L\{f_{1\,2}(x_1 x_2)\} \mid x_1] \geqq L\{f_{1\,1}(x_1)\} . \tag{15}$$

By (15) and the fact that $L(\xi)$ is a function bounded below by a constant, we have

$$\int \mid L\{f_{1\,1}(x_1)\} \mid d\mu = \int \mid \log f_{1\,1}(x_1) \mid d\nu < \infty$$

and

$$\int \log f_{1\,2}(x_1, x_2) d_\nu - \int \log f_{1\,1}(x_1) d_\nu = a \geqq 0 .$$

Now by Theorem 2

$$\log f_{1\,n}(x_1, \cdots, x_n) = \log f_{1\,1}(x_1) + \sum_{i=2}^{n} \{\log f_{1\,2}(x_{i-1}, x_i) - \log f_{1\,1}(x_{i-1})\} .$$

Since ν is stationary, $\log f_{1\,n}(x_1, \cdots, x_n)$ is ν-integrable. Applying the ergodic theorem $n^{-1} \log f_{1\,n}(x_1, \cdots, x_n)$ converges with ν-probability one and also in $L_1(\nu)$ to a function g with

$$\int g d\nu = \int [\log f_{1\,2}(x_1, x_2) - \log f_{1\,1}(x_1)] d\nu = a \geqq 0 .$$

For m being any integer, we only need to mentioned that by (13),

$$\log f_{m\,m+1}(x_m, x_{m+1}) - \log f_{m\,m}(x_m) = \log f_{1\,2}(x_1, x_2) - \log f_{1\,1}(x_1)$$

with ν-probability one and therefore the same conclusion follows with a similar proof.

COROLLARY 1. *Suppose μ, ν satisfy the hypothesis of Theorem 3 for $m = 1$. If ν is ergodic and if there is an $A \in \mathscr{S}$ such that*

$$\nu\{P_\nu[x_2 \in A \mid x_1] \neq P_\mu[x_2 \in A \mid x_1]\} > 0 \tag{16}$$

then ν is singular with respect to μ.

Proof. First we shall show that follows from (16)

$$\mu[f_{1\,1}(x_1) \neq f_{1\,2}(x_1, x_2)] > 0 . \tag{17}$$

For, if $f_{1\,1}(x_1) = f_{1\,2}(x_1, x_2)$ with μ-probability one then by Lemma 1 $P_\nu[x_2 \in A \mid x_1]f_{1\,1}(x_1) = P_\mu[x_2 \in A \mid x_1]f_{1\,1}(x_1)$ with μ-probability one. Thus $P_\nu[x_2 \in A \mid x_1] = P_\mu[x_2 \in A \mid x_1]$ with ν-probability one for every $A \in \mathscr{S}$. Now the function $L(\xi) = \xi \log \xi$ is strictly convex, hence it follows from (17) that

$$a = \int [L\{f_{1\,2}(x_1, x_2)\} - L\{f_{1\,1}(x_1)\}]d_\mu > 0 .$$

Applying Theorem 3 $f_{1\,n}(x_1, \cdots, x_n) \rightarrow \infty$ with ν-probability one as $n \rightarrow \infty$. Hence $1/f_n(x_1, \cdots, x_n) \rightarrow 0$ with ν-probability one as $n \rightarrow \infty$. Let $\mathscr{F}'$ be the σ-algebra generated by $\bigcup_{n=1}^\infty \mathscr{F}_{1\,n}$ and μ', ν' be the contractions of μ, ν to $\mathscr{F}'$ respectively. Since $1/f_{1\,n}(x_1, \cdots, x_n)$ is the derivative of $\nu_{1\,n}$-continuous part of $\mu_{1\,n}$ with respect to $\nu_{1\,n}$, $1/f_{1\,n}(x, \cdots, x_n)$ converges with ν-probability one as $n \rightarrow \infty$ to the derivative of ν'-continuous part of μ' with respect to ν' ([2], pp. 343). Now $1/f_{1\,n}(x_1, \cdots, x_n)$ converges to 0 with ν-probability one, hence the ν'-continuous part of μ' is 0 and μ', ν' are mutually singular. Hence μ, ν are mutually singular.

3. Extension to k-Markovian measures.

The results of the preceding section can be extended to k-Markovian measures immediately. We shall state the theorems only since the proofs in the preceding section with obvious modifications apply as well.

THEOREM 4. *Let μ, ν be any two k-Markovian measures on $\mathscr{F}$. If $\nu_{n-k\,n}$ is absolutely continuous with respect to $\mu_{n-k,\,n}$ for $n = 0, \pm 1, \pm 2, \cdots$, then $\nu_{m\,n}$ is absolutely continuous with respect to $\mu_{m\,n}$ for $n = 0, \pm 1, \pm 2, \cdots$ and $m \leqq n$ with*

$$f_{m\,n}(x_m, \cdots, x_n) = f_{m\,m+k}(x_m, \cdots, x_{m+k}) \frac{f_{m+1,m+1+k}(x_{m+1}, \cdots, x_{m+1+k})}{f_{m+1,m+k}(x_{m+1}, \cdots, x_{m+k})} \cdots \frac{f_{n-k\,n}(x_{n-k}, \cdots, x_n)}{f_{n-k\,n-1}(x_{n-k}, \cdots, x_{n-1})} \tag{18}$$

with μ-probability one.

THEOREM 5. *Let μ, ν be two k-Markovian measures on $\mathscr{F}$ with stationary transition probabilities. If $\nu_{n-k+1,n}$ is absolutely continuous with respect to $\mu_{n-k+1,n}$ for $n = 0, \pm 1, \pm 2, \cdots$ and $\nu_{1\,k+1}$ is absolutely continuous with respect to $\mu_{1\,k+1}$ then $\nu_{m\,n}$ is absolutely continuous with respect to $\mu_{m\,n}$ for $n = 0, \pm 1, \pm 2, \cdots, m \leqq n$ and*

$$(19) \quad f_{m\,n}(x_m, \cdots, x_n) = f_{m\,m+k-1}(x_m, \cdots, x_{m+k-1}) \frac{f_{1\,k+1}(x_{m+1}, \cdots, x_{m+k+1})}{f_{1,k}(x_{m+1}, \cdots, x_{m+k})} \frac{f_{1\,k+1}(x_{n-k}, \cdots, x_n)}{f_{1\,k}(x_{n-k}, \cdots, {}_{n-1})}$$

with μ-probability one.

THEOREM 6. *Let μ, ν be two k-Markovian measures such that ν is stationary and μ has stationary transition probabilities. If*

$$\int |\log f_{m\,m+k}(x_m, \cdots, x_{m+k})| \, d\nu < \infty$$

then $\int |\log f_{m\,n}(x_m, \cdots, x_n)| \, d\nu < \infty$ for $n = m, m+1, m+2, \cdots$ and $n^{-1} \log f_{m\,n}(x_m, \cdots, x_n)$ converges as $n \to \infty$ with ν-probability one to a function g with $\int g d\nu = a \geqq 0$ where

$$a = \int |\log f_{1\,k+1}(x_1, \cdots, x_{k+1}) - \log f_{1\,k}(x_1, \cdots, x_k)| \, d\nu \geqq 0 \, .$$

In particular, if ν is ergodic, $g = a$ with ν-probability one.

COROLLARY 2. *Suppose μ, ν satisfy the hypothesis of Theorem 6 for $m = 1$. If ν is ergodic and if there is a set $A \in \mathscr{S}$ such that*

$$(20) \qquad \nu\{[P_\nu[x_{k+1} \in A \mid x_1, \cdots, x_k] \neq \mathrm{P}_\mu[x_{k+1} \in A] \mid x_1, \cdots, x_k]\} > 0$$

Then ν is singular with respect to μ.

4. A generalization of McMillan's theorem. In the setting of this paper, McMillan's Theorem may be stated as the following. Let X be a finite set of K points and $\mathscr{S}$ be the σ-algebra of all subsets of X. Let ν be any stationary probability measure on $\mathscr{F}$ and μ be the measure on $\mathscr{F}$ such that $\mu[X_m = a_0, X_{m+1} = a_1, \cdots, X_n = a_{n-m}] = K^{-(n-m+1)}$ for any intergers m, n and $a_0, a_1 \cdots a_{n-m}$ in X. μ may be described as the equally distributed independent measure on $\mathscr{F}$. Then $n^r f_{1\,n}(x_1, \cdots, x_n)$ converges as $n \to \infty$ in $L_1(\nu)$. In particular, if ν is ergodic, the limit function is equal to $\log K - H$ with ν-probability one where H is the entropy of ν measure [4]. We shall generalize this theorem to the case that X is countable and μ is Markovian with stationary transition probabilities.

THEOREM 7. *Let the totality of elements of X be $a_1, a_2, \cdots$ and ν be a stationary probability measure on $\mathscr{F}$ such that $\int -\log \nu_1(x_1) d\nu < \infty$ where ν_1 is the function defined on X by $\nu_1(a_i) = \nu[x_1 = a_i]$. Let μ be a Markovian measure on $\mathscr{F}$ with stationary transition probabilities. Let $p(a_i, a_j)$ be the value of $P_\mu[x_1 = a_j \mid x_0]$ when $x_0 = a_i$. Let $\nu_{1\,n}$ be absolutely continuous with respect to $\mu_{1\,n}$ for $n = 1, 2, \cdots$. If*

$$\int -\log p(x_1, x_2) d\nu < \infty$$

and $\int |\log f_{1\,1}(x_1)| d\nu < \infty$ then $\int |\log f_{1\,n}(x_1, \cdots, x_n)| d\nu < \infty$ for $n = 1, 2, \cdots$ and $n^{-1} \log f_{1\,n}(x_1, \cdots, x_n)$ converges as $n \to \infty$ in $L_1(\nu)$. In particular, if ν is ergodic, the limit is equal to a constant with ν-probability one.

Proof. Let

$$\nu_n(a_{i_1}, a_{i_2}, \cdots, a_{i_n}) = \nu[x_1 = a_{i_1}, x_2 = a_{i_2}, \cdots, x_n = a_{i_n}]$$

and

$$\mu_n(a_{i_1}, a_{i_2}, \cdots, a_{i_n}) = \mu[x_1 = a_{i_1}, x_2 = a_{i_2}, \cdots, x_n = a_{i_n}] .$$

Then

$$f_{1\,n}(x_1, \cdots, x_n) = \frac{\nu_n(x_1, \cdots, x_n)}{\mu_n(x_1, \cdots, x_n)}$$

with μ-proba[illegible] and

[illegible] $$= a_i \mid x_{n-1}, \cdots, x_1] = \frac{\nu_n(x_1, \cdots, x_{n-1}, a_i)}{\nu_{n-1}(x_1, \cdots, x_{n-1})}$$

with ν-probability one and

$$P_\mu[x_n = a_i \mid x_{n-1}] = \frac{\mu_n(x_1, \cdots, x_{n-1}, a_i)}{\mu_n(x_1, \cdots, x_{n-1})}$$

with μ-probability one. Hence

$$\frac{f_{1\,n}(x_1, \cdots, x_n)}{f_{1\,n-1}(x_1, \cdots, x_{n-1})} = \sum_{i=1}^{\infty} \frac{P_\nu[x_n = a_i \mid x_{n-1}, \cdots, x_1]}{P_\mu[x_n = a_i \mid x_{n-1}]} I_{x_n = a_i}$$

with ν-probability one and

$$\text{(21)} \quad \log \frac{f_{1\,n-1}(x_1, \cdots, x_n)}{f_{1\,n-1}(x_1, \cdots, x_{n-1})} = \sum_{i=1}^{\infty} \log P_\nu[x_n = a_i \mid x_{n-1}, \cdots, x_1] I_{x_n = a_i} - \log p(x_{n-1}, x_n) = T^n g_n$$

with ν-probability one where

$$g_n = \sum_{i=1}^{\infty} \log P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}] I_{x_0 = a_i} - \log p(x_{-1}, x_0) . \tag{22}$$

We know that $P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}]$ converges with ν-probability one as $n \to \infty$ to $P_\nu[x_0 = a_i \mid x_{-1}, x_{-2}, \cdots]$ by Doob's Martingale Convergence Theorem. Hence $L\{P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}]\}$ converges with ν-probability one to $L\{P_\nu[x_0 = a_i \mid x_{-1}, x_{-2}, \cdots]\}$. But $L(\xi)$ is a bounded function for $0 \leqq \xi \leqq 1$, hence $L\{P_\nu[x_0 = a_i \mid x_{-1}, x_{-(n-1)}\}$ are uniformly bounded with ν-probability one. Hence $L\{P_\nu[x_0 = x_i \mid x_{-1}, \cdots, x_{-(n-1)}]\}$ also converges in $L_1(\nu)$ to $L\{P_\nu[x_0 = a_i \mid x_{-1}, x_{-2}, \cdots]\}$ as $n \to \infty$. Now by Jensen's inequality $\int - L\{P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}]\} d\nu \leqq - L\{P_\nu[x_0 = a_i]\}$. Since

$$\sum_{i=1}^{\infty} - L\{P_\nu[x_0 = a_i]\} = \int - \log \nu_1(x_0) d\nu < \infty$$

$$\sum_{i=1}^{m} - L\{P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}]\}$$

converges in $L_1(\nu)$, as $m \to \infty$, to

$$\sum_{i=1}^{\infty} - L\{P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}]\}$$

uniformly in n. Hence

$$\sum_{i=1}^{\infty} - L\{P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}$$

converges in $L_1(\nu)$ to

$$\sum_{i=1}^{\infty} - L\{P_\nu[x_0 = a_i \mid x_{-1}, x_{-2}, \cdots]\} \text{ as } n \to \infty. \quad \text{Now}$$

$$\int - \sum_{i=1}^{\infty} \log P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}] I_{x_0 = a_i} d\nu$$
$$= \int - \sum_{i=1}^{\infty} L\{P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}]\} d\nu \text{ and}$$

$$\int - \sum_{i=1}^{\infty} \log P_\nu[x_0 = a_i \mid x_{-1}, x_{-2}, \cdots] I_{x_0 = a_i} d\nu$$
$$= \int - \sum_{i=1}^{\infty} L\{P_\nu[x_0 = a_i \mid x_{-1}, x_{-2}, \cdots]\} d\nu, \text{ hence}$$

$$\lim_{n \to \infty} \int - \sum_{i=1}^{\infty} \log P_\nu[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}] I_{x_0 = a_i} d\nu = \int - \sum_{i=1}^{\infty} \log P_\nu[x_0 = a_i \mid x_{-1}, x_{-2}, \cdots] I_{x_0 = a_i} d\nu . \tag{23}$$

(23) together with the facts that the sequence

$$\left\{-\sum_{i=1}^{\infty}\log P_{\nu}[x_0 = x_i \mid x_{-1}, \cdots, x_{-(n-1)}]I_{x_0=a_i}\right\}$$

is also convergent with ν-probability one and that the functions

$$-\sum_{i=1}^{\infty}\log P_{\nu}[x_0 = x_i \mid x_{-1}, \cdots, x_{-(n-1)}]I_{x_0=a_i}$$

are non negative with ν-probability one imply that

$$\sum_{i=1}^{\infty} P_{\nu}[x_0 = a_i \mid x_{-1}, \cdots, x_{-(n-1)}]I_{x_0=a_i}$$

converges as $n \to \infty$ in $L_1(\nu)$ to

$$\sum_{i=1}^{\infty} P_{\nu}[x_0 = a_i \mid x_{-1}, x_{-2}, \cdots]I_{x_0=a_i} .$$

Thus we have $\{g_n\}$ to be an $L_1(\nu)$ convergent sequence. Let the limit of the sequence *be* h. Let $\bar{h}$ be the $L_1(\nu)$ limit of $1/n(h + Th + \cdots + T^n h)$ as $n \to \infty$. Now by (21)

$$\log f_{1\,2}(x_1, \cdots, x_n) = \log f_{1\,1}(x_1) + \sum_{i=2}^{n} T^i g_i. \quad \text{Thus}$$

$$\begin{aligned}\int\left|\frac{1}{n}\log f_{1\,n}(x_1, \cdots, x_n) - \bar{h}\right|d\nu & \\ \leqq \frac{1}{n}\int|\log f_{1\,1}(x_1)|\,d\nu &+ \int\left|\frac{1}{n}\left(\sum_{i=2}^{n} T^i g_i - \sum_{i=2}^{n} T^i h\right)\right|d\nu \\ &+ \int\left|\frac{1}{n}\sum_{i=2}^{n} T^i h - \bar{h}\right|d\nu \\ = \frac{1}{n}\int\left|\log f_{1\,1}(x_1)\right|d\nu &+ \frac{1}{n}\sum_{i=2}^{n}\int|g_i - h|\,d\nu \\ &+ \int\left|\frac{1}{n}\sum_{i=2}^{n} T^i h - \bar{h}\right|d\nu \to 0 \text{ as } n \to \infty .\end{aligned}$$

COROLLARY 3. *Under the hypothesis of Theorem* 7, *if* ν *is ergodic and not Markovian then* ν *is singular to* μ.

Proof. If ν is ergodic then the $L_1(\nu)$ limit, $\bar{h}$, of $\{1/n \log f_{1\,n}(x_1, \cdots, x_n)\}$ is equal with ν probability one to

$$\int\sum_{i=1}^{\infty} L\{P_{\nu}[x_0 = a_i \mid x_{-1}, x_{-2}, \cdots]\}d\nu - \int\log p(x_{-1}, x_0)d\nu$$

which is greater or equal to

$$\int \sum_{i=1}^{\infty} L\{P_\nu[x_0 = a_i \mid x_{-1}, x_{-2}]\} d\nu - \int \log p(x_{-1}, x_0) d\nu \ .$$

Hence by (21)

$$\begin{aligned} \bar{h} &\geqq \int \sum_{i=1}^{\infty} \log P_\nu[x_0 = a_i \mid x_{-1}, x_{-2}] I_{x_0=a_i} d\nu - \int \log p(x_{-1}, x_0) d\nu \\ &= \int \log f_{1\,3}(x_1, x_2, x_2) d\nu - \int \log f_{1\,2}(x_1, x_2) d\nu \ . \end{aligned}$$

However $\int \log f_{1\,3}(x_1, x_2, x_3) d\nu - \int \log f_{1\,2}(x_1, x_2) d\nu = 0$ if and only if

$$\mu[f_{1\,2}(x_1, x_2) \neq f_{1\,3}(x_1, x_2, x_3)] = 0 \ . \tag{24}$$

(24) implies that

$$P_\nu[x_3 \in A \mid x_1, x_2] = P_\mu[x_3 \in A \mid x_1, x_2]$$

with ν-probability one for any $A \in \mathscr{S}$. This is impossible since μ is Markovian and ν is not. Hence $\bar{h} > 0$ with ν-probability one. Hence $f_{1\,n}(x_1, \cdots, x_n) \rightarrow \infty$ with ν probability one and ν is singular to μ by the same argument used in the proof in Corollary 1.

The extensions of Theorem 7 and Corollary 3 to k-Markovian μ is obvious.

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