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## **ASYMPTOTIC PROPERTIES OF DERIVATIVES OF STATIONARY MEASURES**

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**1. Introduction.** Let  $X$  be a non-empty set and  $\mathcal{S}$  be a  $\sigma$ -algebra of subsets of  $X$ . Consider the infinite product space  $\Omega = \prod_{n=-\infty}^{\infty} X_n$  where  $X_n = X$  for  $n = 0, \pm 1, \pm 2, \dots$  and the infinite product  $\sigma$ -algebra  $\mathcal{F} = \prod_{n=-\infty}^{\infty} \mathcal{S}_n$  where  $\mathcal{S}_n = \mathcal{S}$  for  $n = 0, \pm 1, \pm 2, \dots$ . Elements of  $\Omega$  are bilateral infinite sequences  $\{\dots, x_{-1}, x_0, x_1, \dots\}$  with  $x_n \in X$ . Let us denote the elements of  $\Omega$  by  $w$ . If  $w = \{\dots, x_{-1}, x_0, x_1, \dots\}$   $x_n$  is called the  $n$ th coordinate of  $w$  and shall be considered as a function on  $\Omega$  to  $X$ . Let  $T$  be the shift transformation on  $\Omega$  to  $\Omega$ : the  $n$ th coordinate of  $Tw$  is equal to the  $n + 1$ th coordinate of  $w$ . For any function  $g$  on  $\Omega$ ,  $Tg$  is the function defined by  $Tg(w) = g(Tw)$  so that  $Tx_n = x_{n+1}$  for any integer  $n$ . We shall consider two probability measures  $\mu, \nu$  defined on  $\mathcal{F}$ . For  $n = 1, 2, \dots$  let  $\Omega_n = \prod_{i=1}^n X_i$  where  $X_i = X, i = 1, 2, \dots, n$  and  $\mathcal{F}_n = \prod_{i=1}^n \mathcal{S}_i$  where  $\mathcal{S}_i = \mathcal{S}, i = 1, 2, \dots, n$ . Then  $\Omega_1 = X$  and  $\mathcal{F}_1 = \mathcal{S}$ . Let  $\mathcal{F}_{m,n}, m \leq n, n = 0, \pm 1, \pm 2, \dots$ , be the  $\sigma$ -algebra of subsets of  $\Omega$  consisting of sets of the form

$$[w = \{\dots, x_{-1}, x_0, x_1, \dots\}: (x_m, x_{m+1}, \dots, x_n) \in E]$$

Where  $E \in \mathcal{F}_{n-m+1}$ . Then  $\mathcal{F}_m \subset \mathcal{F}_{m+1} \subset \mathcal{F}$ . Let  $\mu_{m,n}, \nu_{m,n}$  be the contractions of  $\mu, \nu$ , respectively to  $\mathcal{F}_{m,n}$ . If  $\nu_{m,n}$  is absolutely continuous with respect to  $\mu_{m,n}$ , the derivative of  $\nu_{m,n}$  with respect to  $\mu_{m,n}$  is a function of  $x_m, \dots, x_n$  and shall be designated by  $f_{m,n}(x_m, \dots, x_n)$ . Since  $f_{m,n}(x_m, \dots, x_n)$  is positive with  $\nu$ -probability one  $1/f_{m,n}(x_m, \dots, x_n)$  is well defined with  $\nu$ -probability one. We shall let the function  $1/f_{m,n}(x_m, \dots, x_n)$  take on the value 0 when  $f_{m,n}(x_m, \dots, x_n) \leq 0$ . Thus  $1/f_{m,n}(x_m, \dots, x_n)$  is well defined everywhere. In fact  $1/f_{m,n}(x_m, \dots, x_n)$  is the derivative of  $\nu_{m,n}$ -continuous part of  $\mu_{m,n}$  with respect to  $\nu_{m,n}$ . According to the celebrated theorem of E. S. Anderson and B. Jessen [1] and J. L. Doob ([2]), pp. 343)  $1/f_{m,n}(x_m, \dots, x_n)$  converges with  $\nu$ -probability one as  $n \rightarrow \infty$ . If we assume that  $\mu, \nu$  are stationary, i.e.,  $\mu, \nu$  are  $T$  invariant, more precise results may be expected. A fundamental theorem of Information Theory, first proved by C. Shannon for stationary Markovian measures [5] and later generalized to any stationary measure by B. McMillan [4], may be considered as a theorem of this sort. In their theorem  $X$  is assumed to be a finite set. In this paper we shall first treat Markovian stationary measures  $\mu, \nu$  with  $X$  being

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any set, finite or infinite, and  $\mathcal{S}$ , any  $\sigma$ -algebra of subsets of  $X$ . It will be proved that  $n^{-1} \log f_m(x_m, \dots, x_n)$  converges as  $n \rightarrow \infty$  with  $\nu$ -probability one and also in  $L_1(\nu)$  under some integrability conditions. The case that  $\nu$  is only stationary is also treated. Similar convergence theorem is proved under the assumption that  $X$  is countable.

## 2. Asymptotic properties of derivatives of a Markovian measure with stationary transition probabilities with respect to another such measure.

Let  $X, \mathcal{S}, \Omega, \mathcal{F}, \Omega_n, \mathcal{F}_n, \mathcal{F}_{m,n}, \mu_{m,n}, \nu_{m,n} f_m(x_m, \dots, x_n)$  be as in §1.  $x_n, n = 0, \pm 1, \pm 2, \dots$ , are considered as functions or random variables on  $\Omega$  to  $X$ . Notations for conditional probabilities and conditional expectations relative to one or several random variables will be as in [2], chapter 1, §7. Since we have two probability measures we shall use subscripts  $\mu, \nu$  to indicate conditional probabilities and conditional expectations taken under measures  $\mu, \nu$  respectively. In this section  $\mu, \nu$  are assumed to be Markovian i.e., for any  $A \in \mathcal{S}, m < n, n = 0, \pm 1, \pm 2, \dots$ ,

(1)  $P_\mu[x_n \in A | x_m, \dots, x_{n-1}] = P_\mu[x_n \in A | x_{n-1}]$  with  $\mu$ -probability one and

(2)  $P_\nu[x_n \in A | x_m, \dots, x_{n-1}] = P_\nu[x_n \in A | x_{n-1}]$  with  $\nu$ -probability one. For any set  $E \subset \Omega$  let  $I_E$  be the real valued function on  $\Omega$  defined by

$$\begin{aligned} I_E(w) &= 1 \text{ if } w \in E \\ &= 0 \text{ if } w \notin E. \end{aligned}$$

LEMMA 1. If  $\nu_{n-1,n}$  is absolutely continuous with respect to  $\mu_{n-1,n}$  then for any  $A \in \mathcal{S}$

$$\begin{aligned} (3) \quad P_\nu[x_n \in A | x_{n-1}] f_{n-1,n-1}(x_{n-1}) \\ = E_\mu[I_{(x_n \in A)} f_{n-1,n}(x_{n-1}, x_n) | x_{n-1}] \text{ with } \mu\text{-probability one.} \end{aligned}$$

*Proof.* For any  $A, B \in \mathcal{S}$

$$\begin{aligned} \nu[x_n \in A, x_{n-1} \in B] \\ &= \int_{[x_{n-1} \in B]} P_\nu[x_n \in A | x_{n-1}] d\nu \\ &= \int_{[x_{n-1} \in B]} P_\nu[x_n \in A | x_{n-1}] f_{n-1,n-1}(x_{n-1}) d\mu. \end{aligned}$$

On the other hand

$$\begin{aligned} \nu[x_n \in A, x_{n-1} \in B] \\ = \int_{[x_{n-1} \in B]} I_{x_n \in A} f_{n-1,n}(x_{n-1}, x_n) | x_{n-1}) d\mu \end{aligned}$$

$$= \int_{[x_{n-1} \in B]} E_{\mu}[I_{x_n \in A} f_{n-1, n}(x_{n-1}, x_n) | x_{n-1}] d\mu .$$

Hence for any  $B \in \mathcal{S}$

$$\begin{aligned} & \int_{[x_{n-1} \in B]} P_{\nu}[x_n \in A | x_{n-1}] f_{n-1, n-1}(x_{n-1}) d\mu \\ &= \int_{[x_{n-1} \in B]} E_{\mu}[I_{x_n \in A} f_{n-1, n}(x_{n-1}, x_n) | x_{n-1}] d\mu , \end{aligned}$$

therefore (3) is true with  $\mu$ -probability one. Dividing both sides of (3) by  $f_{n-1, n-1}(x_{n-1})$  we then have

$$(4) \quad P_{\nu}[x_n \in A | x_{n-1}] = \frac{E_{\mu}[I_{x_n \in A} f_{n-1, n}(x_{n-1}, x_n) | x_{n-1}]}{f_{n-1, n-1}(x_{n-1})} .$$

With  $\mu$ -probability one on the set  $[f_{n-1, n-1}(x_{n-1}) > 0]$ . Since  $\nu[f_{n-1, n-1}(x_{n-1}) > 0] = 1$ , (4) is true with  $\nu$ -probability one.

**THEOREM 1.** *If  $\nu_{n-1, n}$  is absolutely continuous with respect to  $\mu_{n-1, n}$  for  $n = 0, \pm 1, \pm 2, \dots$  then  $\nu_{m, n}$  is absolutely continuous with respect to  $\mu_{m, n}$  for  $n = 0, \pm 1, \pm 2, \dots$  and  $m \leq n$  with*

$$(5) \quad \begin{aligned} f_{m, n}(x_m, \dots, x_n) &= f_{m, m+1}(x_m, x_{m+1}) \frac{f_{m+1, m+2}(x_{m+1}, x_{m+2})}{f_{m+1, m+1}(x_{m+1})} \\ &\dots \frac{f_{n-1, n}(x_{n-1}, x_n)}{f_{n-1, n-1}(x_{n-1})} \end{aligned}$$

with  $\mu$ -probability one.

*Proof.* We shall prove the theorem for the case that  $m = 1, n = 2, 3, \dots$ . The proof for the general case that  $m$  is any integer is similar. Since  $\nu_{1, 2}$  is absolutely continuous with respect to  $\mu_{1, 2}$  by hypothesis, (5) is trivially true for  $m = 1, n = 2$ . Suppose  $\nu_{1, k} (k \geq 2)$  is absolutely continuous with respect to  $\mu_{1, k}$  and  $f_{1, k}(x_1, \dots, x_k)$  is given by (5) with  $\mu$ -probability one. For any  $A \in \mathcal{S}, B \in \mathcal{F}_k$

$$\begin{aligned} & \nu[x_{k+1} \in A, (x_1, \dots, x_k) \in B] \\ &= \int_{[(x_1, \dots, x_k) \in B]} P_{\nu}[x_{k+1} \in A | x_1, \dots, x_k] d\nu . \end{aligned}$$

Since  $\nu$  is Markovian and by (4)

$$\begin{aligned} & \nu[x_{k+1} \in A, (x_1, \dots, x_k) \in B] \\ &= \int_{[(x_1, \dots, x_k) \in B]} P_{\nu}[x_{k+1} \in A | x_k] d\nu \end{aligned}$$

$$\begin{aligned}
&= \int_{[(x_1, \dots, x_k) \in B]} \frac{E_\mu[I_{x_{k+1} \in A} f_{k, k+1}(x_k, x_{k+1}) | x_k]}{f_{k, k}(x_k)} d\nu \\
&= \int_{[(x_1, \dots, x_k) \in B]} \frac{E_\mu[I_{x_{k+1} \in A} f_{k, k+1}(x_k, x_{k+1}) | x_k]}{f_{k, k}(x_k)} f_{1, k}(x_1, \dots, x_k) d\mu.
\end{aligned}$$

Since  $\mu$  is Markovian

$$\begin{aligned}
&E_\mu[I_{x_{k+1} \in A} f_{k, k+1}(x_k, x_{k+1}) | x_k] \\
&= E_\mu[I_{x_{k+1} \in A} f_{k, k+1}(x_k, x_{k+1}) | x_1, \dots, x_k]
\end{aligned}$$

with  $\mu$ -probability one. Hence

$$\begin{aligned}
&\nu[x_{k+1} \in A, (x_1, \dots, x_k) \in B] \\
&= \int_{(x_1, \dots, x_k) \in B} E_\mu \left[ I_{x_{k+1} \in A} \frac{f_{k, k+1}(x_k, x_{k+1})}{f_{k, k}(x_k)} f_{1, k}(x_1, \dots, x_k) \mid x_1, \dots, x_k \right] d\mu \\
&= \int_{(x_1, \dots, x_k) \in B} I_{x_{k+1} \in A} f_{1, k}(x_1, \dots, x_k) \frac{f_{k, k+1}(x_k, x_{k+1})}{f_{k, k}(x_k)} d\mu.
\end{aligned}$$

Hence

$$\begin{aligned}
&\nu[x_{k+1} \in A, (x_1, \dots, x_k) \in B] \\
&= \int_{[x_{k+1} \in A, (x_1, \dots, x_k) \in B]} f_{1, k}(x_1, \dots, x_k) \frac{f_{k, k+1}(x_k, x_{k+1})}{f_{k, k}(x_k)} d\mu
\end{aligned}$$

for any  $A \in \mathcal{S}, B \in \mathcal{F}_k$ . Hence for any  $E \in \mathcal{F}_{1, k+1}$

$$\nu(E) = \int_E f_{1, k}(x_1, \dots, x_k) \frac{f_{k, k+1}(x_k, x_{k+1})}{f_{k, k}(x_k)} d\mu,$$

Therefore  $\nu_{1, k+1}$  is absolutely continuous with respect to  $\mu_{1, k+1}$  and

$$(6) \quad f_{1, k+1}(x_1, \dots, x_{k+1}) = f_{1, k}(x_1, \dots, x_k) \frac{f_{k, k+1}(x_k, x_{k+1})}{f_{k, k}(x_k)}$$

with  $\mu$ -probability one. (6) together with the supposition that (5) holds true for  $m = 1, n = k$  implies that (5) holds true for  $m = 1, n = k + 1$ . Thus the theorem for the case that  $m = 1$  is proved.

Any Markovian probability measure on  $\mathcal{S}$  is said to have *stationary transition probabilities* if  $E$  being a set of probability one implies that  $TE, T^{-1}E$  are also of probability one and for any  $A \in \mathcal{S}$  and any  $n$

$$P[x_{n+1} \in A | x_n] = TP[x_n \in A | x_{n-1}]$$

with probability one. Thus for a Markovian probability measure with stationary transition probabilities we have for any pair of integers  $m, n$  and any  $A \in \mathcal{S}$

(7)  $P[x_n \in A | x_{n-1}] = T^{n-m}P[x_m \in A | x_{m-1}]$  with probability one and

(8)  $E[g(x_{n-1}, x_n) | x_{n-1}] = T^{n-m}E[g(x_{m-1}, x_m) | x_{m-1}]$  with probability one for any real valued  $\mathcal{F}_2$ -measurable function  $g$  on  $\Omega_2$ .

**THEOREM 2.** *Let both  $\mu, \nu$  have stationary transition probabilities. If  $\nu_{n,n}$  is absolutely continuous with respect to  $\mu_{n,n}$  for  $n = 0, \pm 1, \pm 2, \dots$  and  $\nu_{1,2}$  is absolutely continuous with respect to  $\mu_{1,2}$  then  $\nu_{m,n}$  is absolutely continuous with respect to  $\mu_{m,n}$  for  $m \leq n, n = 0, \pm 1, \pm 2, \dots$  and*

$$(9) \quad f_{m,n}(x_m, \dots, x_n) = f_{m,m}(x_m) \frac{f_{1,2}(x_m, x_{m+1})}{f_{1,1}(x_m)} \dots \\ \dots \frac{f_{1,2}(x_{n-1}, x_n)}{f_{1,1}(x_{n-1})}$$

with  $\mu$ -probability one.

*Proof.* By Lemma 1, for any  $A \in \mathcal{S}$

$$(10) \quad P_\nu[x_2 \in A | x_1] = \frac{E_\mu[I_{x_2 \in A} f_{1,2}(x_1, x_2) | x_1]}{f_{1,1}(x_1)}$$

with  $\nu$ -probability one. For any  $A, B \in \mathcal{S}$

$$\begin{aligned} \nu[x_n \in A, x_{n-1} \in B] &= \int_{[x_{n-1} \in B]} P_\nu[x_n \in A | x_{n-1}] d\nu \\ &= \int_{[x_{n-1} \in B]} T^{n-2} P_\nu[x_2 \in A | x_1] d\nu \\ &= \int_{[x_{n-1} \in B]} \{T^{n-2} P_\nu[x_2 \in A | x_1]\} f_{n-1, n-1}(x_{n-1}) d\mu. \end{aligned}$$

Hence by (10) and (8)

$$\begin{aligned} \nu[x_n \in A, x_{n-1} \in B] &= \int_{[x_{n-1} \in B]} T^{n-2} \left\{ \frac{E_\mu[I_{x_2 \in A} f_{1,2}(x_1, x_2) | x_1]}{f_{1,1}(x_1)} f_{n-1, n-1}(x_{n-1}) \right\} d\mu \\ &= \int_{[x_{n-1} \in B]} \frac{E_\mu[I_{x_n \in A} f_{1,2}(x_{n-1}, x_n) | x_{n-1}]}{f_{1,1}(x_{n-1})} f_{n-1, n-1}(x_{n-1}) d\mu \\ &= \int_{[x_{n-1} \in B]} I_{x_n \in A} f_{n-1, n-1}(x_{n-1}) \frac{f_{1,2}(x_{n-1}, x_n)}{f_{1,1}(x_{n-1})} d\mu \\ &= \int_{[x_n \in A, x_{n-1} \in B]} f_{n-1, n-1}(x_{n-1}) \frac{f_{1,2}(x_{n-1}, x_n)}{f_{1,1}(x_{n-1})} d\mu. \end{aligned}$$

Thus for any  $E \in \mathcal{F}_{n-1, n}$

$$(11) \quad \nu(E) = \int_E f_{n-1 \ n-1}(x_{n-1}) \frac{f_{1 \ 2}(x_{n-1}, x_n)}{f_{1 \ 1}(x_{n-1})} d\mu.$$

Hence for any integer  $n$ ,  $\nu_{n-1 \ n}$  is absolutely continuous with respect to  $\mu_{n-1 \ n}$  and Theorem 1 is applicable. (11) also implies that

$$(12) \quad f_{n-1 \ n}(x_{n-1}, x_n) = f_{n-1 \ n-1}(x_{n-1}) \frac{f_{1 \ 2}(x_{n-1}, x_n)}{f_{1 \ 1}(x_{n-1})}$$

with  $\mu$ -probability one. Hence

$$(13) \quad \frac{f_{n-1 \ n}(x_{n-1}, x_n)}{f_{n-1 \ n-1}(x_{n-1})} = \frac{f_{1 \ 2}(x_{n-1}, x_n)}{f_{1 \ 1}(x_{n-1})}$$

with  $\mu$ -probability one on the set  $[f_{n-1 \ n-1}(x_{n-1}) > 0]$ . However, except that  $w$  belongs to a set of  $\mu$ -probability 0,  $n > 1$ ,  $f_{n-1 \ n-1}(x_{n-1}(w)) = 0$  imply that  $f_{1 \ n-1}(x_1(w), \dots, x_{n-1}(w)) = 0$ , hence

$$f_{1 \ n-1}(x_1, \dots, x_{n-1}) \frac{f_{n-1 \ n}(x_{n-1}, x_n)}{f_{n-1 \ n-1}(x_{n-1})} = f_{1 \ n-1}(x_1, \dots, x_{n-1}) \frac{f_{1 \ 2}(x_{n-1}, x_n)}{f_{1 \ 1}(x_{n-1})}$$

with  $\mu$ -probability one. Thus by (6)

$$(14) \quad f_{1 \ n}(x_1, \dots, x_n) = f_{1 \ n-1}(x_1, \dots, x_{n-1}) \frac{f_{1 \ 2}(x_{n-1}, x_n)}{f_{1 \ 1}(x_{n-1})}$$

with  $\mu$ -probability one. Combining (12) (13) and by induction, if  $n > 1$

$$f_{1 \ n}(x_1, \dots, x_n) = f_{1 \ 1}(x_1) \frac{f_{1 \ 2}(x_1, x_2)}{f_{1 \ 1}(x_1)} \dots \frac{f_{1 \ 2}(x_{n-1}, x_n)}{f_{1 \ 1}(x_{n-1})}$$

with  $\mu$ -probability one. Thus we have proved the theorem for the case that  $m = 1$ . For the general case the proof is similar.

**THEOREM 3.** *If  $\mu$  has stationary transition probabilities and  $\nu$  is stationary and if*

$$\begin{aligned} & \int |\log f_{m \ m+1}(x_m, x_{m+1})| d\nu < \infty \text{ then} \\ & \int |\log f_{m \ n}(x_m, \dots, x_n)| d\nu < \infty \text{ for } n = m, m + 1, m + 2, \dots \end{aligned}$$

and  $n^{-1} \log f_{m \ n}(x_m, \dots, x_n)$  converges as  $n \rightarrow \infty$  with  $\nu$ -probability one and also in  $L_1(\nu)$  to a function  $g$  with  $\int g d\nu = a$  where

$$a = \int [\log f_{1 \ 2}(x_1, x_2) - \log f_{1 \ 1}(x_1)] d\nu \geq 0$$

In particular, if  $\nu$  is ergodic,  $g = a$  with  $\nu$ -probability one.

*Proof.* We shall first prove the theorem for the case that  $m = 1$ . Since for any  $A \in \mathcal{S}$

$$\nu[x_1 \in A] = \int_{[x_1 \in A]} f_{11}(x_1) d\mu = \int_{[x_1 \in A]} f_{12}(x_1, x_2) d\mu,$$

hence

$$E_\mu[f_{12}(x_1, x_2) | x_1] = f_{11}(x_1).$$

Since  $\int |\log f_{12}(x_1, x_2)| d\nu < \infty$  hence

$$\int |f_{12}(x_1, x_2) \log f_{12}(x_1, x_2)| d\mu = \int |\log f_{12}(x_1, x_2)| d\nu < \infty.$$

The real valued function  $L(\xi) = \xi \log \xi$  defined for all real  $\xi \geq 0$  [ $L(0)$  is taken to be 0] is convex. By Jensen's inequality for conditional expectations ([2], pp. 33)

$$(15) \quad E_\mu[L\{f_{12}(x_1, x_2)\} | x_1] \geq L\{f_{11}(x_1)\}.$$

By (15) and the fact that  $L(\xi)$  is a function bounded below by a constant, we have

$$\int |L\{f_{11}(x_1)\}| d\mu = \int |\log f_{11}(x_1)| d\nu < \infty$$

and

$$\int \log f_{12}(x_1, x_2) d\nu - \int \log f_{11}(x_1) d\nu = a \geq 0.$$

Now by Theorem 2

$$\log f_{1n}(x_1, \dots, x_n) = \log f_{11}(x_1) + \sum_{i=2}^n \{\log f_{12}(x_{i-1}, x_i) - \log f_{11}(x_{i-1})\}.$$

Since  $\nu$  is stationary,  $\log f_{1n}(x_1, \dots, x_n)$  is  $\nu$ -integrable. Applying the ergodic theorem  $n^{-1} \log f_{1n}(x_1, \dots, x_n)$  converges with  $\nu$ -probability one and also in  $L_1(\nu)$  to a function  $g$  with

$$\int g d\nu = \int [\log f_{12}(x_1, x_2) - \log f_{11}(x_1)] d\nu = a \geq 0.$$

For  $m$  being any integer, we only need to mention that by (13),

$$\log f_{m, m+1}(x_m, x_{m+1}) - \log f_{m, m}(x_m) = \log f_{12}(x_1, x_2) - \log f_{11}(x_1)$$

with  $\nu$ -probability one and therefore the same conclusion follows with a similar proof.



**COROLLARY 1.** *Suppose  $\mu, \nu$  satisfy the hypothesis of Theorem 3 for  $m = 1$ . If  $\nu$  is ergodic and if there is an  $A \in \mathcal{S}$  such that*

$$(16) \quad \nu\{P_\nu[x_2 \in A \mid x_1] \neq P_\mu[x_2 \in A \mid x_1]\} > 0$$

*then  $\nu$  is singular with respect to  $\mu$ .*

*Proof.* First we shall show that follows from (16)

$$(17) \quad \mu\{f_{1_1}(x_1) \neq f_{1_2}(x_1, x_2)\} > 0 .$$

For, if  $f_{1_1}(x_1) = f_{1_2}(x_1, x_2)$  with  $\mu$ -probability one then by Lemma 1

$P_\nu[x_2 \in A \mid x_1]f_{1_1}(x_1) = P_\mu[x_2 \in A \mid x_1]f_{1_1}(x_1)$  with  $\mu$ -probability one. Thus  $P_\nu[x_2 \in A \mid x_1] = P_\mu[x_2 \in A \mid x_1]$  with  $\nu$ -probability one for every  $A \in \mathcal{S}$ . Now the function  $L(\xi) = \xi \log \xi$  is strictly convex, hence it follows from (17) that

$$a = \int [L\{f_{1_2}(x_1, x_2)\} - L\{f_{1_1}(x_1)\}]d_\mu > 0 .$$

Applying Theorem 3  $f_{1_n}(x_1, \dots, x_n) \rightarrow \infty$  with  $\nu$ -probability one as  $n \rightarrow \infty$ . Hence  $1/f_n(x_1, \dots, x_n) \rightarrow 0$  with  $\nu$ -probability one as  $n \rightarrow \infty$ . Let  $\mathcal{F}'$  be the  $\sigma$ -algebra generated by  $\bigcup_{n=1}^\infty \mathcal{F}_{1_n}$  and  $\mu', \nu'$  be the contractions of  $\mu, \nu$  to  $\mathcal{F}'$  respectively. Since  $1/f_{1_n}(x_1, \dots, x_n)$  is the derivative of  $\nu_{1_n}$ -continuous part of  $\mu_{1_n}$  with respect to  $\nu_{1_n}$ ,  $1/f_{1_n}(x, \dots, x_n)$  converges with  $\nu$ -probability one as  $n \rightarrow \infty$  to the derivative of  $\nu'$ -continuous part of  $\mu'$  with respect to  $\nu'$  ([2], pp. 343). Now  $1/f_{1_n}(x_1, \dots, x_n)$  converges to 0 with  $\nu$ -probability one, hence the  $\nu'$ -continuous part of  $\mu'$  is 0 and  $\mu', \nu'$  are mutually singular. Hence  $\mu, \nu$  are mutually singular.

**3. Extension to  $k$ -Markovian measures.** The results of the preceding section can be extended to  $k$ -Markovian measures immediately. We shall state the theorems only since the proofs in the preceding section with obvious modifications apply as well.

**THEOREM 4.** *Let  $\mu, \nu$  be any two  $k$ -Markovian measures on  $\mathcal{F}$ . If  $\nu_{n-k, n}$  is absolutely continuous with respect to  $\mu_{n-k, n}$  for  $n = 0, \pm 1, \pm 2, \dots$ , then  $\nu_{m, n}$  is absolutely continuous with respect to  $\mu_{m, n}$  for  $n = 0, \pm 1, \pm 2, \dots$  and  $m \leq n$  with*

$$(18) \quad f_{m, n}(x_m, \dots, x_n) = f_{m, m+k}(x_m, \dots, x_{m+k}) \frac{f_{m+1, m+1+k}(x_{m+1}, \dots, x_{m+1+k})}{f_{m+1, m+k}(x_{m+1}, \dots, x_{m+k})} \\ \dots \frac{f_{n-k, n}(x_{n-k}, \dots, x_n)}{f_{n-k, n-1}(x_{n-k}, \dots, x_{n-1})}$$

*with  $\mu$ -probability one.*

**THEOREM 5.** Let  $\mu, \nu$  be two  $k$ -Markovian measures on  $\mathcal{F}$  with stationary transition probabilities. If  $\nu_{n-k+1, n}$  is absolutely continuous with respect to  $\mu_{n-k+1, n}$  for  $n = 0, \pm 1, \pm 2, \dots$  and  $\nu_{1, k+1}$  is absolutely continuous with respect to  $\mu_{1, k+1}$  then  $\nu_{m, n}$  is absolutely continuous with respect to  $\mu_{m, n}$  for  $n = 0, \pm 1, \pm 2, \dots, m \leq n$  and

$$(19) \quad f_{m, n}(x_m, \dots, x_n) = f_{m, m+k-1}(x_m, \dots, x_{m+k-1}) \frac{f_{1, k+1}(x_{m+1}, \dots, x_{m+k+1})}{f_{1, k}(x_{m+1}, \dots, x_{m+k})} \\ \frac{f_{1, k+1}(x_{n-k}, \dots, x_n)}{f_{1, k}(x_{n-k}, \dots, x_{n-1})}$$

with  $\mu$ -probability one.

**THEOREM 6.** Let  $\mu, \nu$  be two  $k$ -Markovian measures such that  $\nu$  is stationary and  $\mu$  has stationary transition probabilities. If

$$\int |\log f_{m, m+k}(x_m, \dots, x_{m+k})| d\nu < \infty$$

then  $\int |\log f_{m, n}(x_m, \dots, x_n)| d\nu < \infty$  for  $n = m, m+1, m+2, \dots$  and  $n^{-1} \log f_{m, n}(x_m, \dots, x_n)$  converges as  $n \rightarrow \infty$  with  $\nu$ -probability one to a function  $g$  with  $\int g d\nu = a \geq 0$  where

$$a = \int |\log f_{1, k+1}(x_1, \dots, x_{k+1}) - \log f_{1, k}(x_1, \dots, x_k)| d\nu \geq 0.$$

In particular, if  $\nu$  is ergodic,  $g = a$  with  $\nu$ -probability one.

**COROLLARY 2.** Suppose  $\mu, \nu$  satisfy the hypothesis of Theorem 6 for  $m = 1$ . If  $\nu$  is ergodic and if there is a set  $A \in \mathcal{S}$  such that

$$(20) \quad \nu\{[P_\nu[x_{k+1} \in A | x_1, \dots, x_k] \neq P_\mu[x_{k+1} \in A] | x_1, \dots, x_k]\} > 0$$

Then  $\nu$  is singular with respect to  $\mu$ .

**4. A generalization of McMillan's theorem.** In the setting of this paper, McMillan's Theorem may be stated as the following. Let  $X$  be a finite set of  $K$  points and  $\mathcal{S}$  be the  $\sigma$ -algebra of all subsets of  $X$ . Let  $\nu$  be any stationary probability measure on  $\mathcal{F}$  and  $\mu$  be the measure on  $\mathcal{F}$  such that  $\mu[X_m = a_0, X_{m+1} = a_1, \dots, X_n = a_{n-m}] = K^{-(n-m+1)}$  for any integers  $m, n$  and  $a_0, a_1, \dots, a_{n-m}$  in  $X$ .  $\mu$  may be described as the equally distributed independent measure on  $\mathcal{F}$ . Then  $n^{-1} \log f_{1, n}(x_1, \dots, x_n)$  converges as  $n \rightarrow \infty$  in  $L_1(\nu)$ . In particular, if  $\nu$  is ergodic, the limit function is equal to  $\log K - H$  with  $\nu$ -probability one where  $H$  is the entropy of  $\nu$  measure [4]. We shall generalize this theorem to the case that  $X$  is countable and  $\mu$  is Markovian with stationary transition probabilities.

**THEOREM 7.** *Let the totality of elements of  $X$  be  $a_1, a_2, \dots$  and  $\nu$  be a stationary probability measure on  $\mathcal{F}$  such that  $\int -\log \nu_1(x_1) d\nu < \infty$  where  $\nu_1$  is the function defined on  $X$  by  $\nu_1(a_i) = \nu[x_1 = a_i]$ . Let  $\mu$  be a Markovian measure on  $\mathcal{F}$  with stationary transition probabilities. Let  $p(a_i, a_j)$  be the value of  $P_\mu[x_1 = a_j | x_0 = a_i]$ . Let  $\nu_{1n}$  be absolutely continuous with respect to  $\mu_{1n}$  for  $n = 1, 2, \dots$ . If*

$$\int -\log p(x_1, x_2) d\nu < \infty$$

*and  $\int |\log f_{11}(x_1)| d\nu < \infty$  then  $\int |\log f_{1n}(x_1, \dots, x_n)| d\nu < \infty$  for  $n = 1, 2, \dots$  and  $n^{-1} \log f_{1n}(x_1, \dots, x_n)$  converges as  $n \rightarrow \infty$  in  $L_1(\nu)$ . In particular, if  $\nu$  is ergodic, the limit is equal to a constant with  $\nu$ -probability one.*

*Proof.* Let

$$\nu_n(a_{i_1}, a_{i_2}, \dots, a_{i_n}) = \nu[x_1 = a_{i_1}, x_2 = a_{i_2}, \dots, x_n = a_{i_n}]$$

and

$$\mu_n(a_{i_1}, a_{i_2}, \dots, a_{i_n}) = \mu[x_1 = a_{i_1}, x_2 = a_{i_2}, \dots, x_n = a_{i_n}].$$

Then

$$f_{1n}(x_1, \dots, x_n) = \frac{\nu_n(x_1, \dots, x_n)}{\mu_n(x_1, \dots, x_n)}$$

with  $\mu$ -probability one and

$$f_{1n-1}(x_1, \dots, x_{n-1}) = \frac{\nu_n(x_1, \dots, x_{n-1}, a_i)}{\nu_{n-1}(x_1, \dots, x_{n-1})}$$

with  $\nu$ -probability one and

$$P_\mu[x_n = a_i | x_{n-1}] = \frac{\mu_n(x_1, \dots, x_{n-1}, a_i)}{\mu_n(x_1, \dots, x_{n-1})}$$

with  $\mu$ -probability one. Hence

$$\frac{f_{1n}(x_1, \dots, x_n)}{f_{1n-1}(x_1, \dots, x_{n-1})} = \sum_{i=1}^{\infty} \frac{P_\nu[x_n = a_i | x_{n-1}, \dots, x_1]}{P_\mu[x_n = a_i | x_{n-1}]} I_{x_n = a_i}$$

with  $\nu$ -probability one and

$$\begin{aligned} (21) \quad \log \frac{f_{1n-1}(x_1, \dots, x_n)}{f_{1n-1}(x_1, \dots, x_{n-1})} &= \sum_{i=1}^{\infty} \log P_\nu[x_n = a_i | x_{n-1}, \dots, x_1] I_{x_n = a_i} \\ &\quad - \log p(x_{n-1}, x_n) \\ &= T^n g_n \end{aligned}$$

with  $\nu$ -probability one where

$$(22) \quad g_n = \sum_{i=1}^{\infty} \log P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}] I_{x_0 = a_i} \\ - \log p(x_{-1}, x_0).$$

We know that  $P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}]$  converges with  $\nu$ -probability one as  $n \rightarrow \infty$  to  $P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}, \dots]$  by Doob's Martingale Convergence Theorem. Hence  $L\{P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}]\}$  converges with  $\nu$ -probability one to  $L\{P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}, \dots]\}$ . But  $L(\xi)$  is a bounded function for  $0 \leq \xi \leq 1$ , hence  $L\{P_{\nu}[x_0 = a_i | x_{-1}, x_{-(n-1)}]\}$  are uniformly bounded with  $\nu$ -probability one. Hence  $L\{P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}]\}$  also converges in  $L_1(\nu)$  to  $L\{P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}, \dots]\}$  as  $n \rightarrow \infty$ . Now by Jensen's inequality  $\int -L\{P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}]\} d\nu \leq -L\{P_{\nu}[x_0 = a_i]\}$ . Since

$$\sum_{i=1}^{\infty} -L\{P_{\nu}[x_0 = a_i]\} = \int -\log \nu_1(x_0) d\nu < \infty \\ \sum_{i=1}^m -L\{P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}]\}$$

converges in  $L_1(\nu)$ , as  $m \rightarrow \infty$ , to

$$\sum_{i=1}^{\infty} -L\{P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}]\}$$

uniformly in  $n$ . Hence

$$\sum_{i=1}^{\infty} -L\{P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}]\}$$

converges in  $L_1(\nu)$  to

$$\sum_{i=1}^{\infty} -L\{P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}, \dots]\} \text{ as } n \rightarrow \infty. \text{ Now}$$

$$\int -\sum_{i=1}^{\infty} \log P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}] I_{x_0 = a_i} d\nu \\ = \int -\sum_{i=1}^{\infty} L\{P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}]\} d\nu \text{ and} \\ \int -\sum_{i=1}^{\infty} \log P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}, \dots] I_{x_0 = a_i} d\nu \\ = \int -\sum_{i=1}^{\infty} L\{P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}, \dots]\} d\nu, \text{ hence}$$

$$(23) \quad \lim_{n \rightarrow \infty} \int -\sum_{i=1}^{\infty} \log P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}] I_{x_0 = a_i} d\nu \\ = \int -\sum_{i=1}^{\infty} \log P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}, \dots] I_{x_0 = a_i} d\nu.$$

(23) together with the facts that the sequence

$$\left\{ - \sum_{i=1}^{\infty} \log P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}] I_{x_0=a_i} \right\}$$

is also convergent with  $\nu$ -probability one and that the functions

$$- \sum_{i=1}^{\infty} \log P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}] I_{x_0=a_i}$$

are non negative with  $\nu$ -probability one imply that

$$\sum_{i=1}^{\infty} P_{\nu}[x_0 = a_i | x_{-1}, \dots, x_{-(n-1)}] I_{x_0=a_i}$$

converges as  $n \rightarrow \infty$  in  $L_1(\nu)$  to

$$\sum_{i=1}^{\infty} P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}, \dots] I_{x_0=a_i} .$$

Thus we have  $\{g_n\}$  to be an  $L_1(\nu)$  convergent sequence. Let the limit of the sequence be  $h$ . Let  $\bar{h}$  be the  $L_1(\nu)$  limit of  $1/n(h + Th + \dots + T^n h)$  as  $n \rightarrow \infty$ . Now by (21)

$$\log f_{1_2}(x_1, \dots, x_n) = \log f_{1_1}(x_1) + \sum_{i=2}^n T^i g_i. \quad \text{Thus}$$

$$\begin{aligned} & \int \left| \frac{1}{n} \log f_{1_n}(x_1, \dots, x_n) - \bar{h} \right| d\nu \\ & \leq \frac{1}{n} \int |\log f_{1_1}(x_1)| d\nu + \int \left| \frac{1}{n} \left( \sum_{i=2}^n T^i g_i - \sum_{i=2}^n T^i h \right) \right| d\nu \\ & \quad + \int \left| \frac{1}{n} \sum_{i=2}^n T^i h - \bar{h} \right| d\nu \\ & = \frac{1}{n} \int |\log f_{1_1}(x_1)| d\nu + \frac{1}{n} \sum_{i=2}^n \int |g_i - h| d\nu \\ & \quad + \int \left| \frac{1}{n} \sum_{i=2}^n T^i h - \bar{h} \right| d\nu \rightarrow 0 \text{ as } n \rightarrow \infty . \end{aligned}$$

**COROLLARY 3.** *Under the hypothesis of Theorem 7, if  $\nu$  is ergodic and not Markovian then  $\nu$  is singular to  $\mu$ .*

*Proof.* If  $\nu$  is ergodic then the  $L_1(\nu)$  limit,  $\bar{h}$ , of  $\{1/n \log f_{1_n}(x_1, \dots, x_n)\}$  is equal with  $\nu$  probability one to

$$\int \sum_{i=1}^{\infty} L\{P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}, \dots]\} d\nu - \int \log p(x_{-1}, x_0) d\nu$$

which is greater or equal to

$$\int \sum_{i=1}^{\infty} L\{P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}]\} d\nu - \int \log p(x_{-1}, x_0) d\nu .$$

Hence by (21)

$$\begin{aligned} \bar{h} &\geq \int \sum_{i=1}^{\infty} \log P_{\nu}[x_0 = a_i | x_{-1}, x_{-2}] I_{x_0=a_i} d\nu - \int \log p(x_{-1}, x_0) d\nu \\ &= \int \log f_{13}(x_1, x_2, x_3) d\nu - \int \log f_{12}(x_1, x_2) d\nu . \end{aligned}$$

However  $\int \log f_{13}(x_1, x_2, x_3) d\nu - \int \log f_{12}(x_1, x_2) d\nu = 0$  if and only if

$$(24) \quad \mu[f_{12}(x_1, x_2) \neq f_{13}(x_1, x_2, x_3)] = 0 .$$

(24) implies that

$$P_{\nu}[x_3 \in A | x_1, x_2] = P_{\mu}[x_3 \in A | x_1, x_2]$$

with  $\nu$ -probability one for any  $A \in \mathcal{S}$ . This is impossible since  $\mu$  is Markovian and  $\nu$  is not. Hence  $\bar{h} > 0$  with  $\nu$ -probability one. Hence  $f_{1n}(x_1, \dots, x_n) \rightarrow \infty$  with  $\nu$  probability one and  $\nu$  is singular to  $\mu$  by the same argument used in the proof in Corollary 1.

The extensions of Theorem 7 and Corollary 3 to  $k$ -Markovian  $\mu$  is obvious.

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