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**CONCERNING BOUNDARY VALUE PROBLEMS**

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**1. Introduction.** This paper follows work on integral equations by H. S. Wall [4], [5], J. S. MacNerney [1], [2] and the present author [3]. Some results of these papers are used here to investigate certain boundary value problems.

In §2, results of Wall and MacNerney are used to study a linear boundary value problem which includes problems of the following kind: Suppose that each of  $a_{ij}$ ,  $i, j = 1, \dots, n$  is a continuous function,  $a$  and  $b$  are numbers and each of  $b_{ij}$ ,  $c_{ij}$  and  $d_i$ ,  $i, j = 1, \dots, n$  is a number. Is there a unique function  $n$ -tuple  $f_1, \dots, f_n$  such that

$$f'_i = \sum_{j=1}^n a_{ij} f_j \quad \text{and} \quad \sum_{j=1}^n [b_{ij} f_j(a) + c_{ij} f_j(b)] = d_i, \quad i = 1, \dots, n?$$

Section 3 contains some observations concerning a nonlinear boundary value problem which includes the problem of solving a certain system of nonlinear first order differential equations together with a nonlinear boundary condition. An example is given in the final section.

$S$  denotes a normed, complete, abelian group (norms are denoted by  $\|\cdot\|$ ).  $B$  denotes the normed, complete, abelian group of all bounded endomorphisms from  $S$  to  $S$  (the norm of an element  $T$  of  $B$  is the g.l.b. of the set of all  $M$  such that  $\|Tx\| \leq M\|x\|$  for all  $x$  in  $S$ ).  $B^*$  denotes the set to which  $T$  belongs only if  $T$  is a continuous function from  $S$  to  $S$ . If  $[a, b]$  denotes a number interval, then  $C_{[a, b]}$  denotes the set to which  $f$  belongs only if  $f$  is a continuous function from  $[a, b]$  to  $S$ . The identity function on the numbers is denoted by  $j$ .

The reader is referred to [1] for a definition of the integral of a function from a number interval  $[a, b]$  to  $B$  with respect to a function from  $[a, b]$  to  $B$  and to [3] for a definition of the integral of a function from  $[a, b]$  to  $S$  with respect to a function from  $[a, b]$  to  $B^*$ . [1] and [3] contain existence theorems for these integrals and a discussion of some of their properties.

**2. A linear boundary value problem.** Suppose that  $[a, b]$  is a number interval and  $F$  is a continuous function from  $[a, b]$  to  $B$  which is of bounded variation on  $[a, b]$ . The following are theorems:

(i) There is a unique continuous function  $M$  from  $[a, b] \times [a, b]$  to  $B$  such that  $M(t, u) = I + \int_u^t dF \cdot M(j, u)$  for each of  $t$  and  $u$  in  $[a, b]$ . ( $I$  denotes the identity element in  $B$ )

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(ii)  $M(t, u)M(u, v) = M(t, v)$  if each of  $t, u$  and  $v$  is in  $[a, b]$ .

(iii) If  $h$  is a continuous function from  $[a, b]$  to  $S$  and  $c$  is in  $[a, b]$ , then the only element  $X$  of  $C_{[a,b]}$  such that  $X(t) = h(t) + \int_c^t dF \cdot X$  for each  $t$  in  $[a, b]$  is given by  $X(t) = M(t, c)h(c) + \int_c^t M(t, j)dh$  for each  $t$  in  $[a, b]$ .<sup>2</sup>

**THEOREM A.** *Suppose that  $H$  is a function from  $[a, b]$  to  $B$  which is of bounded variation on  $[a, b]$ . A necessary and sufficient condition that there be a unique element  $Y$  of  $C_{[a,b]}$  such that*

(\*)  $Y(t) = Y(u) + g(t) - g(u) + \int_a^t dF \cdot Y$  and  $\int_a^b dH \cdot Y = C$  for each  $C$  in  $S$  and each  $g$  in  $C_{[a,b]}$  is that  $\int_a^b dH \cdot M(j, a)$  have an inverse which is from  $S$  onto  $S$ .

*Proof.* Consider first the following lemma. If  $Y$  is in  $C_{[a,b]}$  and satisfies (\*) for each of  $u$  and  $t$  in  $[a, b]$ , then

$$\left[ \int_a^b dH \cdot M(j, a) \right] Y(a) = C - \int_a^b \left[ \int_a^b dH(s) \cdot M(s, j) \right] dg .$$

Suppose  $Y$  is in  $C_{[a,b]}$  and satisfies (\*) for each of  $u$  and  $t$  in  $[a, b]$ . By (iii),  $Y(t) = M(t, a)Y(a) + \int_a^t M(t, j)dg$  for each  $t$  in  $[a, b]$  and thus

$$\begin{aligned} C &= \int_a^b dH \cdot Y = \left[ \int_a^b dH \cdot M(j, a) \right] Y(a) + \int_a^b dH(s) \cdot \left[ \int_a^s M(s, j)dg \right] \\ &= \left[ \int_a^b dH \cdot M(j, a) \right] Y(a) + \int_a^b \left[ \int_a^b dH(s) \cdot M(s, j) \right] dg .^3 \end{aligned}$$

Hence,

$$\left[ \int_a^b dH \cdot M(j, a) \right] Y(a) = C - \int_a^b \left[ \int_a^b dH(s) \cdot M(s, j) \right] dg .$$

Denote  $\int_a^b dH \cdot M(j, a)$  by  $Q$ . Suppose that (\*) has a unique solution for each  $g$  in  $C_{[a,b]}$  and each  $C$  in  $S$ .

Denote by  $W$  a point of  $S$ , by  $g$  an element of  $C_{[a,b]}$ ,

<sup>2</sup> Certain essential ideas for Theorems (i) and (ii) were given by Wall in [4]. In [5], Wall gave these theorems for  $S$  an  $n$ -dimensional Euclidean space or suitable infinite dimensional space. In [1], MacNerney extended Wall's theory in proving these theorems for any normed, linear and complete space. Modifications of MacNerney's proofs to the case of  $S$  a normed, complete, abelian group are so slight that the proofs are omitted. Discussion concerning the properties and computation of  $M$  can be found in each paper listed as reference to this paper.

<sup>3</sup> A proof that  $\int_a^b dH(s) \cdot \left[ \int_a^s M(s, j)dg \right] = \int_a^b \left[ \int_a^b dH(s) \cdot M(s, j) \right] dg$  which follows closely a similar argument for ordinary integrals, is omitted.

$$W + \int_a^b \left[ \int_j^b dH(s) \cdot M(s, j) \right] dg$$

by  $C$  and by  $X$  the unique element of  $C_{[a,b]}$  satisfying (\*) for this  $g$  and  $C$ . By the above lemma,  $QX(a) = C - \int_a^b \left[ \int_j^b dH(s) \cdot M(s, j) \right] dg = W$ . Thus each point of  $S$  is the image of some point of  $S$  under  $Q$ , that is,  $Q$  takes  $S$  onto  $S$ .

Suppose that  $Q$  is not reversible and denote by each of  $W, U$  and  $V$  a point in  $S$  such that  $QU = W, QV = W$  and  $U \neq V$ . Denote by  $Y$  and  $Z$  two elements of  $C_{[a,b]}$  such that  $Y(t) = U + g(t) - g(a) + \int_a^t dF \cdot Y$  and  $Z(t) = V + g(t) - g(a) + \int_a^t dF \cdot Z$  for each  $t$  in  $[a, b]$ . Thus,  $Y(t) = Y(u) + g(t) - g(u) + \int_u^t dF \cdot Y$  and  $Z(t) = Z(u) + g(t) - g(u) + \int_u^t dF \cdot Z$ , for each of  $u$  and  $t$  in  $[a, b]$ . Since  $Y(a) = U$  and  $Z(a) = V$ , it follows that  $Y \neq Z$ . As in the proof of the lemma,

$$\int_a^b dH \cdot Y = QU + \int_a^b \left[ \int_j^b dH(s) \cdot M(s, j) \right] dg$$

and

$$\int_a^b dH \cdot Z = QV + \int_a^b \left[ \int_j^b dH(s) \cdot M(s, j) \right] dg$$

and so

$$\int_a^b dH \cdot Y = \int_a^b dH \cdot Z,$$

which means that there is a boundary value problem of the type (\*) which has two solutions, which contradicts the above assumption. Thus if (\*) has a unique solution for each  $g$  in  $C_{[a,b]}$  and each  $C$  in  $S$ ,  $Q$  takes  $S$  onto  $S$  reversibly.

Suppose that  $Q$  takes  $S$  onto  $S$  reversibly. Denote by  $g$  an element of  $C_{[a,b]}$  and by  $C$  a point in  $S$ . Denote

$$\left[ \int_a^b dH \cdot M(j, a) \right]^{-1} \left\{ C - \int_a^b \left[ \int_j^b dH(s) \cdot M(s, j) \right] dg \right\}$$

by  $U$  and denote by  $X$  the element of  $C_{[a,b]}$  such that  $X(t) = U + g(t) - g(a) + \int_a^t dH \cdot X$  for each  $t$  in  $[a, b]$ . Noting that  $X(t) = X(u) + g(t) - g(u) + \int_u^t dH \cdot X$  and that  $X(t) = M(t, a)U + \int_a^t M(t, j)dg$  for each of  $u$  and  $t$  in  $[a, b]$  and substituting for  $X$  in  $\int_a^b dH \cdot X$ , it is seen that  $\int_a^b dH \cdot X = C$ . Thus  $X$  satisfies (\*) for this  $g$  and  $C$ . Suppose  $Y$  is in  $C_{[a,b]}$  and satisfies (\*). Then, by the above lemma,

$$QY(a) = C - \int_a^b \left[ \int_j^b dH(s) \cdot M(s, j) \right] dg$$

and so  $Y(a) = U$  which means that  $Y(t) = U + g(t) - g(u) + \int_a^t dF \cdot Y$  and hence by (iii),  $X = Y$ . Thus if  $Q$  takes  $S$  onto  $S$  reversibly, there is a unique solution to (\*) for each  $g$  in  $C_{[a,b]}$  and  $C$  in  $S$ .

**THEOREM B.** *If  $\int_a^b dH \cdot M(j, a)$  has a bounded inverse which takes  $S$  onto  $S$ , that is, if  $\left[\int_a^b dH \cdot M(j, a)\right]^{-1}$  is in  $B$ , then there is a function  $R$  from  $[a, b]$  to  $B$  and a function  $K$  from  $[a, b] \times [a, b]$  to  $B$  such that if  $g$  is in  $C_{[a,b]}$  and  $C$  is in  $S$ , then the only element  $Y$  of  $C_{[a,b]}$  satisfying (\*) for each of  $t$  and  $u$  in  $[a, b]$  is given by  $Y(t) = R(t)C + \int_a^b K(t, j)dg$  for each  $t$  in  $[a, b]$ . Moreover, such a pair of functions  $R$  and  $K$  is given by  $R(t) = \left[\int_a^b dH \cdot M(j, t)\right]^{-1}$  and*

$$K(t, u) = \begin{cases} -\left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_u^b dH \cdot M(j, u) + M(t, u) & \text{if } a \leq u \leq t \\ -\left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_u^b dH \cdot M(j, u) & \text{if } t \leq u \leq b. \end{cases}$$

*Proof.* Suppose that  $g$  is in  $C_{[a,b]}$  and  $C$  is in  $S$ . From Theorem A, (\*) has a unique solution  $Y$  for this  $C$  and  $g$ , and from the lemma in the proof of Theorem A,

$$\left[\int_a^b dH \cdot M(j, a)\right]X(a) = C - \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right]dg$$

and so

$$X(a) = \left[\int_a^b dH \cdot M(j, a)\right]^{-1} \left\{ C - \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right]dg \right\}.$$

Using (iii) and the fact that

$$M(t, a) \left[\int_a^b dH \cdot M(j, a)\right]^{-1} = \left[\int_a^b dH \cdot M(j, t)\right]^{-1},$$

$$\begin{aligned} X(t) &= \left[\int_a^b dH \cdot M(j, t)\right]^{-1} C - \int_a^b \left\{ \left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_j^b dH(s) \cdot M(s, j) \right\} dg \\ &\quad + \int_a^t M(t, j) dg \\ &= R(t)C + \int_a^b K(t, j) dg \end{aligned}$$

where  $R$  and  $K$  are defined as in the statement of the theorem.

**3. A nonlinear boundary value problem.** Here a problem is considered which includes the one in the preceding section. Essentially, the requirements of § 2 that each of  $F(t)$  and  $H(t)$  be an element of  $B$  for every  $t$  in  $[a, b]$  and that  $F$  and  $H$  be of bounded variation are

replaced by considerably weaker conditions. Theorem *D* gives a necessary and sufficient condition for the nonlinear problem considered to have a unique solution. First a fundamental theorem for a certain type of integral equation is given.

**THEOREM C.** *Suppose that  $[a, b]$  is a number interval and  $F$  is a function from  $[a, b]$  to  $B^*$  such that if  $A$  is in  $S$  and  $r > 0$ , there is a variation function  $U$  on  $[a, b]$  and a variation function  $V$  on  $[a, b]$  such that*

$$\| [F(p) - F(q)]x \| \leq U(p, q)$$

and

$$\| [F(p) - F(q)]x - [F(p) - F(q)]y \| \leq V(p, q) \| x - y \|$$

if each of  $p$  and  $q$  is in  $[a, b]$ ,  $\| A - x \| \leq r$  and  $\| A - y \| \leq r$ . Then, if  $c$  is in  $[a, b]$ , there is a segment  $Q'$  containing  $c$  such that if  $Q$  is the common part of  $Q'$  and  $[a, b]$ , there is only one continuous function  $Y$  from  $Q$  to  $S$  such that  $Y(t) = A + \int_c^t dF \cdot Y$  if  $t$  is in  $Q$ .

This follows from Theorem *F* of [3].

**DEFINITION.** Suppose  $F$  is a function from  $[a, b]$  to  $B^*$  and  $c$  is in  $[a, b]$ . If there is a point  $A$  in  $S$  and an element  $Y$  of  $C_{[a,b]}$  such that  $Y(t) = A + \int_c^t dF \cdot Y$  for each  $t$  in  $[a, b]$ , then the set which contains only each such point  $A$  is denoted by  $F_{c,[a,b]}$ .

**LEMMA 4.1.** *Suppose that  $F$  satisfies the hypothesis of Theorem C and for some number  $c$  in  $[a, b]$  and that there is a segment  $Q'$  as in the theorem which has  $[a, b]$  as subset. Then, for each number  $u$  in  $[a, b]$ , there is a set  $F_{u,[a,b]}$ .*

*Proof.* Given such a number  $c$  and segment  $Q'$ , then  $Q = [a, b]$  and there is a point  $A$  in  $S$  and an element  $Y$  of  $C_{[a,b]}$  such that  $Y(t) = A + \int_c^t dF \cdot Y$  for each  $t$  in  $[a, b]$ . Thus if  $u$  is in  $[a, b]$ ,  $Y(u) = A + \int_c^u dF \cdot Y$  and  $Y(t) = Y(u) + \int_u^t dF \cdot Y$  for each  $t$  in  $[a, b]$ . Thus there is a set  $F_{u,[a,b]}$ .

**DEFINITION.** Suppose the hypothesis of Lemma 4.1 holds.  $M$  denotes a function from  $[a, b] \times [a, b]$  such that if each of  $t$  and  $u$  is in  $[a, b]$ ,  $M(t, u)$  is the function from  $F_{u,[a,b]}$  to  $F_{t,[a,b]}$  such that if  $A$  is in  $F_{u,[a,b]}$ ,  $M(t, u)A$  is  $Y(t)$  where  $Y$  is the element of  $C_{[a,b]}$  satisfying  $Y(s) = A + \int_u^s dF \cdot Y$  for each  $s$  in  $[a, b]$ .

**LEMMA 4.2.** *Under the hypothesis of Lemma 4.1,  $M(s, t)M(t, u) = M(s, u)$  for each of  $s, t$  and  $u$  in  $[a, b]$ .*

*Proof.* Suppose that each of  $s, t$  and  $u$  is in  $[a, b]$  and  $A$  is in  $F_{u;[a,b]}$ . Then,  $Y(s) = A + \int_s^s dF \cdot Y$  and  $Y(t) = A + \int_t^t dF \cdot Y$  so that  $Y(s) = Y(t) + \int_t^s dF \cdot Y$ ,  $Y(t) = M(t, u)A$  and  $Y(s) = M(s, u)A$ . Therefore,  $Y(s) = M(t, u)A + \int_t^s dF \cdot Y$  and  $Y(s) = M(s, t)[M(t, u)A] = [M(s, t)M(t, u)]A$ . Thus,  $M(s, u) = M(s, t)M(t, u)$ .

**THEOREM D.** *Suppose that in addition to the hypothesis of Theorem C, it is true that for some  $c$  in  $[a, b]$ , there is a set  $F_{c;[a,b]}$ . Suppose furthermore that  $T$  is a function from  $C_{[a,b]}$  to  $S$  and that  $C$  is in  $S$ . The following two statements are equivalent:*

(i) *There is only one element  $Y$  of  $C_{[a,b]}$  such that*

(\*\*)  *$TY = C$  and  $Y(t) = Y(u) + \int_u^t dF \cdot Y$  for each of  $t$  and  $u$  in  $[a, b]$ .*

(ii) *For some  $u$  in  $[a, b]$ , the function  $R$  from  $F_{u;[a,b]}$ , defined by  $RA = T[M(j, u)A]$  for each  $A$  in  $F_{u;[a,b]}$  takes only one element of  $F_{u;[a,b]}$  into  $C$ .*

*Proof.* Suppose that for some  $u$  in  $[a, b]$ , the function  $R$  as defined in Theorem D takes only the point  $U$  of  $F_{u;[a,b]}$  into  $C$ . Denote by  $Y$  the element of  $C_{[a,b]}$  such that  $Y(t) = U + \int_u^t dF \cdot Y$  for each  $t$  in  $[a, b]$ . Thus,  $Y(t) = Y(s) + \int_s^t dF \cdot Y$  and  $Y(t) = M(t, u)U$  for each of  $t$  and  $s$  in  $[a, b]$  and  $TY = T[M(j, u)Y(u)] = C$ . Suppose  $X$  is in  $C_{[a,b]}$  and satisfies (\*\*). Then,  $X(t) = M(t, s)X(s)$  for each of  $t$  and  $s$  in  $[a, b]$  and so  $TX = T[M(j, u)X(u)]$  which means that  $R[X(u)] = C$  which in turn implies that  $X(u) = U$  and so  $X(t) = U + \int_u^t dF \cdot X$  for each  $t$  in  $[a, b]$ . By Theorem B,  $X = Y$ . Thus the existence of such a  $u$  in  $[a, b]$  and such a function  $R$  implies that (\*\*) has a unique solution.

Suppose that (\*\*) has a unique solution  $Y$  which is in  $C_{[a,b]}$ . Denote by  $u$  a number in  $[a, b]$ . Thus  $Y(t) = Y(u) + \int_u^t dF \cdot Y$  and  $Y(t) = M(t, u)Y(u)$  for each  $t$  in  $[a, b]$  and so  $TY = T[M(j, u)Y(u)]$ . Denote by  $R$  the function from  $F_{u;[a,b]}$  to  $S$  so that  $RA = T[M(j, u)A]$  for each  $A$  in  $F_{u;[a,b]}$ . Thus  $R[Y(u)] = C$ . Suppose that  $V \neq Y(u)$  and  $RV = C$ . Denote by  $X$  the element of  $C_{[a,b]}$  so that  $X(t) = V + \int_u^t dF \cdot X$  for each  $t$  in  $[a, b]$ .  $X \neq Y$  as  $X(u) \neq Y(u)$ . But  $X(t) = X(s) + \int_s^t dF \cdot X$  for each of  $t$  and  $s$  in  $[a, b]$  and  $TX = [M(j, u)X(u)] = T[M(j, u)V] = RV = C$ , a contradiction. Thus there is not such a point  $V$  in  $F_{u;[a,b]}$  and so the existence of a unique element of  $C_{[a,b]}$  satisfying (\*\*) implies the existence of the required function  $R$ .

4. **An example.** Suppose that  $[a, b]$  is a number interval,  $S$  the number plane, each of  $p$  and  $q$  a continuous function from  $[a, b]$  to a number set such that  $p(t) > 0$  for each  $t$  in  $[a, b]$  and each of  $a_{ij}$ ,  $b_{ij}$  and  $c_i$ ,  $i, j = 1, 2$ , a number. The problem of solving

$$(4) \quad \begin{aligned} (py')' qy &= G \\ a_{11}y(a) + a_{12}p(a)y'(a) + b_{11}y(b) + b_{12}p(b)y'(b) &= c_1 \\ a_{12}y(a) + a_{22}p(a)y'(a) + b_{21}y(b) + b_{22}p(b)y'(b) &= c_2 \end{aligned}$$

for each continuous function  $G$  from  $[a, b]$  to a number set and each ordered number pair  $(c_1, c_2)$  is equivalent to the problem of finding a function pair  $f_1, f_2$  each of which is from  $[a, b]$  to a number set such that

$$\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} 0 & 1/q \\ q & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} f_1(a) \\ f_2(a) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} f_1(b) \\ f_2(b) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

i.e., the problem of finding a continuous function  $f$  from  $[a, b]$  to  $S$  such that

$$(8) \quad \begin{aligned} f &= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \\ f(t) &= f(u) + g(t) - g(u) + \int_u^t dF \cdot f \end{aligned}$$

and

$$\int_a^b dH \cdot f = A_1 f(a) + A_2 f(b) = C$$

for each of  $u$  and  $t$  in  $[a, b]$  where  $g(t) = \begin{bmatrix} 0 \\ G(t) \end{bmatrix}$ ,  $F(t)$  is the linear transformation from  $S$  to  $S$  associated with

$$\begin{bmatrix} 0 & \int_a^t (1/p) dj \\ \int_a^t q dj & 0 \end{bmatrix}$$

for each  $t$  in  $[a, b]$ , each of  $A_1$  and  $A_2$  is a linear transformation from  $S$  to  $S$  with  $A_1$  associated with  $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  and  $A_2$  associated with  $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$  and  $H$  is defined in the following way:  $H(a) = N_b$ , the transformation which takes each point of  $S$  into  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $H(u) = A_1$  if  $a < u < b$  and  $H(b) = A_1 + A_2$ . Suppose that  $M$  satisfies  $M(t, u) = I + \int_u^t dF \cdot M(j, u)$  for each of  $t$  and  $u$  in  $[a, b]$ . From § 2, for (8) to have a unique con-



tinuous solution for each  $g$  and each  $C$  it is necessary and sufficient that  $\int_a^b dH \cdot M(j, a) = A_1 + M(b, a)A_2$  have an inverse which is from  $S$  onto  $S_r$ . Here is  $\int_a^b dH \cdot M(j, a)$  has an inverse, it is from  $S$  to  $S$  and is bounded

Suppose that  $\int_a^b dH \cdot M(j, a)$  has an inverse,  $G$  is a continuous function from  $[a, b]$  to a number set,  $C$  is in  $S$  and  $g = \begin{bmatrix} 0 \\ G \end{bmatrix}$ . By Theorem B, there is a function  $K$  from  $[a, b] \times [a, b]$  to  $B$  and a function  $R$  from  $[a, b]$  to  $B$  such that  $f(t) = R(t)C + \int_a^b K(t, j)dg$  for each  $t$  in  $[a, b]$ . Denote by each of  $R_{ij}$ ,  $K_{ij}$ ,  $i, j = 1, 2$  a function from  $[a, b]$  to a number set such that if each of  $t$  and  $u$  is in  $[a, b]$ ,  $R(t)$  is associated with

$$\begin{bmatrix} R_{11}(t) & R_{12}(t) \\ R_{21}(t) & R_{22}(t) \end{bmatrix}$$

and  $K(t, u)$  is associated with

$$\begin{bmatrix} K_{11}(t, u) & K_{12}(t, u) \\ K_{21}(t, u) & K_{22}(t, u) \end{bmatrix}.$$

Thus,  $f_1(t) = R_{11}(t)c_1 + R_{12}(t)c_2 + \int_a^b K_{12}(t, j)dG$  for each  $t$  in  $[a, b]$  and  $f_1$  is the unique solution to (A).

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