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TWO EXTREMAL PROBLEMS

MARVIN ROSENBLUM AND HAROLD WIDOM

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1. Introduction. Let \mathcal{P}_0 be the class of all complex trigonometric polynomials P of the form $P_0 + P_1 e^{i\phi} + P_2 e^{2i\phi} + \dots$. Let σ and μ be, respectively normalized Lebesgue measure and any finite non-negative Borel measure on the real interval $(-\pi, \pi]$. Suppose $\mu = \mu_A + \mu_S$, with $d\mu_A(\phi) = f(\phi)d\sigma(\phi)$, is the Lebesgue decomposition of μ into absolutely continuous and singular measures. In this note we shall be concerned with two generalizations of the problem Q_0 : Find

$$I_0(\mu) = \inf_{P \in \mathcal{P}_0} \left[\int |1 + e^{i\phi} P(e^{i\phi})|^2 d\mu(\phi) \right]^{\frac{1}{2}}.$$

Q_0 was solved by Szegö for the case $\mu = \mu_A$ and in general by M. G. Krein and Kolmogorov. They showed that $I_0(\mu) = \exp \frac{1}{2} \int \log f d\sigma$ if $\log f$ is integrable and $I_0(\mu) = 0$ otherwise. (See [3], pp. 44, 231.)

We shall consider:

Problem Q_1 : Suppose $\int |g|^2 d\mu < \infty$. Find

$$I_1(g, \mu) = \inf_{P \in \mathcal{P}_0} \left[\int |g + P|^2 d\mu \right]^{\frac{1}{2}},$$

and

Problem Q_2 : Suppose $\int |h| d\sigma < \infty$. Find

$$I_2(h, \mu) = \sup_{P \in \mathcal{P}_0} \left\{ \left| \int Ph d\sigma \right| / \left[\int |P|^2 d\mu \right]^{\frac{1}{2}} \right\}.$$

Clearly $I_1(e^{-i\phi}, \mu) = I_0(\mu)$. Also

$$[I_2(1, \mu)]^{-1} = \inf_{P \in \mathcal{P}_0} \left\{ \left[\int |P|^2 d\mu \right]^{\frac{1}{2}} / \left| \int Pd\sigma \right| \right\} = I_0(\mu),$$

so Q_0 is a particularization of both Q_1 and Q_2 . There are other special cases of Q_1 and Q_2 that can be found in the work of Szegö [5] and Grenander and Szegö [3]. Of particular interest are the following:

(i) Let $g(\phi) = e^{-i(k+1)\phi}$, where k is a positive integer. Then Q_1 is the problem of linear prediction k units ahead of time ([3], p. 184).

(ii) Let $h(\phi) = 1/(1 - \alpha e^{-i\phi})$, $|\alpha| < 1$. Then

$$I_2(h, \mu) = \sup_{P \in \mathcal{P}_0} \left\{ |P(\alpha)| / \left[\int |P|^2 d\mu \right]^{\frac{1}{2}} \right\}.$$

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See [3], p. 48.

Throughout we shall indulge in the following notational conveniences: We shall write $I_1(g, f)$ and $I_2(h, f)$ for $I_1(g, \mu_A)$ and $I_2(h, \mu_A)$ respectively, and, in certain contexts, consider two functions identical that are equal everywhere except for a set of Lebesgue measure zero.

We have divided this note into six sections. First we indicate an interesting duality between $I_1(e^{-i\phi}g(\phi), f)$ and $I_2(g, 1/f)$ that relates the problems Q_1 and Q_2 under certain restrictive hypotheses. In section three we fashion the theory that will handle Q_1 and Q_2 . This is the solution of a Riemann-Hilbert problem (which we call problem Q_3), which is applied in §§ 4, 5 and 6 to Q_1 and Q_2 .

2. Duality of I_1 and I_2 . This will fall out of the following Banach space lemma:

Let \mathcal{P}_0 be a subspace of a Banach space \mathcal{L} and let \mathcal{P}_0^\perp be the annihilator of \mathcal{P}_0 in the dual space \mathcal{L}^* . If $g \in \mathcal{L}$, then

$$\inf \{ \|g + P\| : P \in \mathcal{P}_0 \} = \sup \{ |l(g)| : l \in \mathcal{P}_0^\perp, \|l\| \leq 1 \}.$$

For a proof see Bonsall [2].

THEOREM 1. Suppose f and $1/f$ are in $L^1(-\pi, \pi)$ and $\int |g|^2 f d\sigma < \infty$.

Then

$$I_1(e^{-i\phi}g(\phi), f) = I_2(g, 1/f).$$

Sketch of proof. By the above lemma

$$I_1(e^{-i\phi}g(\phi), f) = \sup \left\{ \left| \int e^{-i\phi}g(\phi)h(\phi)f(\phi)d\sigma \right| / \left[\int |h|^2 f d\sigma \right]^{\frac{1}{2}} \right\},$$

where the supremum is taken over all h such that $\int e^{in\phi}h(\phi)f(\phi)d\sigma = 0$ for $n = 0, 1, 2, \dots$. Through the substitution $e^{-i\phi}hf = P$ it follows that

$$I_1(e^{-i\phi}g(\phi), f) = \sup \left\{ \left| \int Pf d\sigma \right| / \left[\int |P|^2 \frac{1}{f} d\sigma \right]^{\frac{1}{2}} \right\},$$

where now the supremum is taken over all P such that $\int e^{in\phi}P(\phi)d\sigma = 0$ for $n = 1, 2, \dots$. It can be shown that it is sufficient merely to consider suprema for $P \in \mathcal{P}_0$, which proves the theorem.

The restrictive condition $1/f \in L^1(-\pi, \pi)$ seems essential to the formulation of the preceding duality relation, but at least this relation indicates that there exist close tie-ins between Q_1 and Q_2 . We shall solve a Riemann-Hilbert problem for the unit circle that, when applied to Q_1 and Q_2 , solves both.

3. **The Riemann-Hilbert problem Q_3 .** Let f be a non-negative function in $L^1 = L^1(-\pi, \pi)$, and suppose that \mathcal{S} is the closure of \mathcal{S}_0 in the Hilbert space $L^2(f)$ of functions square integrable with respect to the measure $f d\sigma$. Thus, for example, \mathcal{S} in $L^2(1) = L^2$ can be identified with the Hardy space H^2 . The problem Q_3 is:

Given $k \in L^1$, find functions $P \in \mathcal{S}$ and q satisfying

$$(1) \quad Pf = k + q, \quad \text{and}$$

$$(2) \quad \int q e^{-in\phi} d\sigma = 0, \quad n = 0, 1, \dots$$

(Note that since $\int |P|^2 f d\sigma < \infty$, we have $Pf \in L^1$ and so $q = Pf - k \in L^1$.)

We first list some prefactory material. We associate with any non-negative $f \in L^1$ such that $\log f \in L^1$ the analytic functions

$$(3) \quad \begin{aligned} F^+(z) &= \exp \frac{1}{2} \int \frac{e^{i\phi} + z}{e^{i\phi} - z} \log f(\phi) d\sigma(\phi), \quad |z| < 1, \\ F^-(z) &= \exp \frac{1}{2} \int \frac{z + e^{i\phi}}{z - e^{i\phi}} \log f(\phi) d\sigma(\phi), \quad |z| > 1. \end{aligned}$$

F^+ and F^- belong to H^2 and K^2 respectively, and $\overline{F^-(z)} = F^+(1/\bar{z})$ if $|z| > 1$. (A function $F(z)$ is said to belong to K^p if $F(1/z)$ belongs to H^p .) Since the boundary functions in H^2 and K^2 exist in mean square, we can define

$$(4) \quad \begin{aligned} f^+(\phi) &= \lim_{r \rightarrow 1^-} F^+(re^{i\phi}), \\ f^-(\phi) &= \lim_{r \rightarrow 1^+} F^-(re^{i\phi}). \end{aligned}$$

These functions satisfy

$$(5) \quad f(\phi) = f^-(\phi)f^+(\phi) = |f^+(\phi)|^2 = |f^-(\phi)|^2.$$

For any non-negative $f \in L^1$ and $\varepsilon > 0$ we define $F_\varepsilon^\pm(z), f_\varepsilon^\pm(\phi)$ by (3) and (4) with f replaced by $f_\varepsilon = f + \varepsilon$. Here we need not assume that $\log f \in L^1$. Note that since $f + \varepsilon \geq \varepsilon > 0$, we have $1/F_\varepsilon^+ \in H^\infty$ and $1/F_\varepsilon^- \in K^\infty$. Moreover $|f_\varepsilon^+(\phi)|^2 = f(\phi) + \varepsilon$, so $|f_\varepsilon^-(\phi)| = |f_\varepsilon^+(\phi)| \geq [f(\phi)]^{1/2}$.

Next we define an operator $(\)_+$ as follows. Its domain D consists of all L^1 functions k with Fourier series $\sum_{-\infty}^\infty c_n e^{in\phi}$ such that $\sum_0^\infty |c_n|^2 < \infty$, and k_+ is the function with Fourier series $\sum_0^\infty c_n e^{in\phi}$. We define the operator $(\)_-$ by $k_- = k - k_+$. Notice that $k_+ \in H^2$ and $k_- \in K^1$ with $\int k_- d\sigma = 0$.

Our discussion of Q_3 proceeds in the following order. First we prove uniqueness. Then we solve Q_3 in certain special cases (these being sufficient, it will turn out, to handle Q_1), and finally find the solution in

the general case.

We are indebted to the referee for the proof of the next lemma.

LEMMA 2. Q_3 has at most one solution.

Proof. Suppose $Pf = q$ where $P \in \mathcal{S}$ and q satisfies (2). Then P is orthogonal, in $L^2(f)$, to all exponentials $e^{in\phi}$ ($n \geq 0$). Since P belongs to the closed manifold \mathcal{S} spanned by these exponentials we conclude $P = 0$.

One can formally solve Q_3 by means of the usual factorization methods (see [4], for example). Write $f = f^+f^-$, so $Pf = k + q$ implies

$$Pf^+ = \frac{k}{f^-} + \frac{q}{f^-}.$$

Applying $(\)_+$ to both sides we obtain $Pf^+ = (k/f^-)_+$, $P = (1/f^+)(k/f^-)_+$. The following theorem justifies this procedure in certain cases.

THEOREM 3. (i) Suppose $\log f \in L^1$ and $k/f^- \in D$. Then Q_3 has the solution

$$(6) \quad P = \frac{1}{f^+} \left(\frac{k}{f^-} \right)_+ \quad q = -f^- \left(\frac{k}{f^-} \right)_-.$$

(ii) Suppose $\log f \notin L^1$ and $k^2/f \in L^1$. Then Q_3 has the solution

$$P = \frac{k}{f} \quad q = 0.$$

Proof. (i) Let $\varepsilon > 0$. Since the function f^+ is outer, it follows from a theorem of Beurling [1] that there exists a $P_0 \in \mathcal{S}_0$ such that

$$\int \left| \left(\frac{k}{f^-} \right)_+ - P_0 f^+ \right|^2 d\sigma < \varepsilon.$$

Therefore by (5)

$$\int \left| \frac{1}{f^+} \left(\frac{k}{f^-} \right)_+ - P_0 \right|^2 f d\sigma < \varepsilon,$$

so P as defined in (6) belongs to \mathcal{S} . Furthermore, with q as defined in (6),

$$Pf - q = f^- \left[\left(\frac{k}{f^-} \right)_+ + \left(\frac{k}{f^-} \right)_- \right] = k.$$

It remains to show that $q \in K^1$. Certainly q belongs to $K^{1/2}$ since it is the product of the two K^1 functions $-f^-$ and $(k/f^-)_-$. But since also

$q = Pf - k$, it belongs to L^1 . Therefore ([6], p. 163) $q \in K^1$.

(ii) If $\log f \notin L^1$, the space \mathcal{P} is identical with $L^2(f)$ ([3], § 33) and so $k/f \in \mathcal{P}$.

We now give the complete solution of Q_3 .

THEOREM 4. (i) *The limit*

$$\lim_{\varepsilon \rightarrow 0^+} \int \left| (k/f_\varepsilon^-)_+ \right|^2 d\sigma$$

exists either finitely or infinitely.

(ii) *A necessary and sufficient condition that Q_3 have a solution P, q is that the limit be finite.*

(iii) *If the limit is finite then*

$$P = \lim (1/f_\varepsilon^+)(k/f_\varepsilon^-)_+$$

in the space $L^2(f)$, and

$$\int |P|^2 f d\sigma = \lim_{\varepsilon \rightarrow 0^+} \int |(k/f_\varepsilon^-)_+|^2 d\sigma.$$

Proof. Assume first that Q_3 has a solution P, q and divide both sides of (1) by f_ε^- . Since $q/f_\varepsilon^- \in K^1$ and $\int q/f_\varepsilon^- d\sigma = 0$ we have $q/f_\varepsilon^- \in D$ and $(q/f_\varepsilon^-)_+ = 0$; also $Pf/f_\varepsilon^- \in L^2 \subset D$. Therefore we can apply ()₊ to both sides, obtaining

$$(Pf/f_\varepsilon^-)_+ = (k/f_\varepsilon^-)_+.$$

Consequently

$$(7) \quad \int |(k/f_\varepsilon^-)_+|^2 d\sigma \leq \int |Pf/f_\varepsilon^-|^2 d\sigma \leq \int |P|^2 f d\sigma,$$

and so

$$(8) \quad \limsup_{\varepsilon \rightarrow 0^+} \int |(k/f_\varepsilon^-)_+|^2 d\sigma < \infty.$$

Conversely suppose that $\{\varepsilon_n\}$ is a sequence of ε 's such that $\varepsilon_n \rightarrow 0^+$ and

$$(9) \quad \int |(k/f_\varepsilon^-)_+|^2 d\sigma = O(1) \text{ for } \varepsilon = \varepsilon_n.$$

By Theorem 3(i) there corresponds to each $\varepsilon = \varepsilon_n$ a solution $P_\varepsilon, q_\varepsilon$ of $(f + \varepsilon)P_\varepsilon = k + q_\varepsilon$. We have

$$(10) \quad \int |P_\varepsilon|^2 f d\sigma \leq \int |P_\varepsilon|^2 f_\varepsilon d\sigma = \int |(k/f_\varepsilon^-)_+|^2 d\sigma = O(1).$$

Thus there exists a subsequence of $\{\varepsilon_n\}$ such that $\{P_\varepsilon\}$ converges weakly

in $L^2(f)$ to an element $P \in \mathcal{P}$. It will follow that $P, Pf - k$ satisfies Q_3 if the L^1 function $q = Pf - k$ satisfies (2). We have for $n = 0, 1, 2, \dots$

$$\begin{aligned} \int q(\phi)e^{-in\phi}d\sigma &= \int \{P_\varepsilon(\phi)[f(\phi) + \varepsilon] - k(\phi)\}e^{-in\phi}d\sigma \\ &+ \int [P(\phi) - P_\varepsilon(\phi)]f(\phi)e^{-in\phi}d\sigma - \varepsilon \int P_\varepsilon(\phi)e^{-in\phi}d\sigma \\ &= J_1 + J_2 + J_3 . \end{aligned}$$

Theorem 3(i) implies that $J_1 = 0$. By the weak convergence of the P_ε we can make J_2 as small as desired by taking ε_n sufficiently small. Finally (10) implies that $\int |\varepsilon^{1/2}P_\varepsilon|^2 d\sigma = O(1)$, so by the Schwarz inequality $|J_3| \leq \varepsilon^{1/2} \int |\varepsilon^{1/2}P_\varepsilon|^2 d\sigma = O(\varepsilon^{1/2})$ as $\varepsilon_n \rightarrow 0$. Thus P, q satisfy Q_3 , so (8), holds and (9) is true for *any* sequence $\{\varepsilon_m\}$ of ε 's that converge to $0+$. By what we have shown there corresponds to any such sequence $\{\varepsilon_m\}$ a subsequence such that P_ε converges weakly to the unique (Lemma 2) element P . Thus we can consider ε to be a real variable and conclude that P_ε converges weakly in $L^2(f)$ to $P \in \mathcal{P}$ as $\varepsilon \rightarrow 0+$ provided that

$$\liminf_{\varepsilon \rightarrow 0+} \int |k/f_\varepsilon^-|_+^2 d\sigma < \infty .$$

We next prove that in fact P_ε converges strongly to P in $L^2(f)$. It suffices to show that $\int |P_\varepsilon|^2 fd\sigma \rightarrow \int |P|^2 fd\sigma$. Weak convergence gives

$$\liminf_{\varepsilon \rightarrow 0+} \int |P_\varepsilon|^2 fd\sigma \geq \int |P|^2 fd\sigma .$$

On the other hand, as in (7),

$$\int |P_\varepsilon|^2 fd\sigma \leq \int |P_\varepsilon|^2 f_\varepsilon d\sigma = \int |(k/f_\varepsilon^-)_+|^2 d\sigma \leq \int |P|^2 fd\sigma .$$

so

$$\limsup_{\varepsilon \rightarrow 0+} \int |P_\varepsilon|^2 fd\sigma \leq \int |P|^2 fd\sigma .$$

Thus

$$\lim_{\varepsilon \rightarrow 0+} \int |P_\varepsilon|^2 fd\sigma$$

exists, and equals

$$\lim_{\varepsilon \rightarrow 0+} \int |(k/f_\varepsilon^-)_+|^2 d\sigma = \int |P|^2 fd\sigma .$$

Thus the proof is complete.

4. **Solution of Q_1 .** In Q_1 we wish to find

$$I_1(g, \mu) = \inf_{P \in \mathcal{S}_0'} \left[\int |g + P|^2 d\mu \right]^{\frac{1}{2}},$$

where g is a given function in $L^2(\mu)$. Since $I_1(g, \mu)$ represents the distance from g to the manifold \mathcal{S}_0 in $L^2(\mu)$, there exists a (unique) function P belonging to the closure \mathcal{S}_0' of \mathcal{S}_0 in $L^2(\mu)$ such that

$$I_1(g, \mu) = \left[\int |g + P|^2 d\mu \right]^{\frac{1}{2}}.$$

This function P is such that $g + P$ is orthogonal to \mathcal{S}_0 , so

$$\int [g(\phi) + P(\phi)] e^{-in\phi} d\mu(\phi) = 0 \quad n = 0, 1, 2, \dots$$

It follows from a theorem of the brothers Riesz ([6], p. 158) that the measure ν given by

$$\nu(E) = \int_E [g(\phi) + P(\phi)] d\mu(\phi)$$

is absolutely continuous with respect to Lebesgue measure. Let F be a Borel set of Lebesgue measure zero such that $\mu_s((-\pi, \pi] - F) = 0$. Then $g + P$ vanishes on F almost everywhere with respect to μ_s , so

$$\int_F |g + P|^2 d\mu_s = 0$$

and

$$\int |g + P|^2 d\mu = \int_{\mathcal{E}} |g + P|^2 d\mu_A = \int |g + P|^2 f d\sigma.$$

Since $\mu \geq \mu_A$ it follows that $I_1(g, \mu) = I_1(g, f)$, and this common value is attained by the same extremizing function $P \in \mathcal{S}_0' \subset \mathcal{S}$.

Now,

$$\int [g(\phi) + P(\phi)] e^{-in\phi} f(\phi) d\sigma = 0 \quad n = 0, 1, \dots,$$

so if we set $q = (g + P)f$ we have $Pf = -gf + q$, where $P \in \mathcal{S}$ and q satisfies (2). Since $(gf)^2/f = g^2f \in L^1$, we can apply Theorem 3 to this situation. The extremizing function

$$P = \begin{cases} -(1/f_+)(gf_+) & \text{if } \log f \in L^1 \\ -g & \text{if } \log f \notin L^1, \end{cases}$$

and since

$$I_1(g, f) = \left[\int |g + P|^2 f d\sigma \right]^{\frac{1}{2}} = \left[\int |q|^2 f d\sigma \right]^{\frac{1}{2}}$$

we have

$$I_1(g, \mu) = I_1(g, f) = \begin{cases} \left[\int |(gf^+)_-|^2 d\sigma \right]^{\frac{1}{2}} & \text{if } \log f \in L^1 \\ 0 & \text{if } \log f \notin L^1. \end{cases}$$

5. **Solution of Q_2 .** Given $h \in L^1$, we will evaluate

$$I_2(h, \mu) = \sup_{P \in \mathcal{P}_0} \left\{ \left| \int Ph d\sigma \right| / \left[\int |P|^2 d\mu \right]^{\frac{1}{2}} \right\}.$$

Since $\mu \geq \mu_A$ it is clear that if $I_2(h, f)$ is finite so is $I_2(h, \mu)$. We shall show that, conversely, if $I_2(h, \mu)$ is finite then so is $I_2(h, f)$ and in fact $I_2(h, f) = I_2(h, \mu)$. So now suppose $I_2(h, \mu) < \infty$. Then the linear functional L on \mathcal{P}_0 given by

$$L(P) = \int Ph d\sigma$$

is bounded on $L^2(\mu)$. Therefore if \mathcal{S}' denotes the closure of \mathcal{P}_0 in $L^2(\mu)$, there is a uniquely determined $Q \in \mathcal{S}'$ such that $L(P) = \int P\bar{Q} d\mu$. Then we have

$$\int e^{-in\phi} [Q(\phi) d\mu(\phi) - \bar{h}(\phi) d\sigma(\phi)] = 0 \quad n = 0, 1, \dots.$$

We again apply the F. and M. Riesz theorem, and deduce that the measure ν given by

$$\nu(E) = \int_E Q d\mu - \int_E h d\sigma$$

is absolutely continuous with respect to Lebesgue measure. Letting F be a Borel set of Lebesgue measure zero such that $\mu_S((-\pi, \pi] - F) = 0$, we see that Q vanishes on F almost everywhere with respect to μ_S . Consequently

$$\int e^{-in\phi} [Q(\phi) f(\phi) - \bar{h}(\phi)] d\sigma(\phi) = 0 \quad n = 0, 1, \dots,$$

so $Qf = \bar{h} + q$, where $Q \in \mathcal{S}' \subset \mathcal{P}$ and q satisfies (2). Thus the linear functional

$$L(P) = \int Ph d\sigma = \int P\bar{Q}f d\sigma,$$

$P \in \mathcal{P}_0$, is bounded on $L^2(f)$, so $I_2(h, f)$ is finite and in fact equals $I_2(h, \mu)$. We deduce from Theorem 4 that

$$I_2(h, \mu) = I_2(h, f) = \lim_{\varepsilon \rightarrow 0+} \left[\int |(\bar{h}/f_\varepsilon^-)_+|^2 d\sigma \right]^{\frac{1}{2}},$$

and Q may be exhibited as an $L^2(f)$ limit in the mean.

6. Some formulae for $I_2(h, \mu)$. We can obtain a simpler formula for $I_2(h, \mu)$ if we assume that $h^2/f \in L^1$ and apply Theorem 3. Then

$$I_2(h, \mu) = \begin{cases} \left[\int |(\bar{h}/f^-)_+|^2 d\sigma \right]^{\frac{1}{2}} = \left[\int |(e^{-i\phi}h(\phi)/f^+(\phi))_-|^2 d\sigma(\phi) \right]^{\frac{1}{2}} & \text{if } \log f \in L^1, \\ \left[\int |h|^2/f d\sigma \right]^{\frac{1}{2}} & \text{if } \log f \notin L^1. \end{cases}$$

This, in conjunction with our solution of Q_1 , gives the duality discussed in Theorem 1. Note that the hypothesis $1/f \in L^1$ of Theorem 1 implies that $\log f \in L^1$.

Another simple formula for $I_2(h, \mu)$ is available if we know that the Fourier series $\sum_{-\infty}^{\infty} h_n e^{in\phi}$ of h is such that $h_{-n} = O(R_0^{-n})$ as $n \rightarrow +\infty$ for some $R_0 > 1$. Then the function $H(z) = \sum_{-\infty}^{\infty} h_{-n} z^{-n}$ is analytic in $|z| > 1/R_0$. We have

$$\int |(\bar{h}/f_\varepsilon^-)_+|^2 d\sigma = \int |(e^{-i\phi}h(\phi)/f_\varepsilon^+(\phi))_-|^2 d\sigma,$$

which by the Parseval relation equals

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \int e^{in\phi} h(\phi) f_\varepsilon^+(\phi) d\sigma \right|^2 &= \sum_{n=0}^{\infty} \left| \frac{1}{2\pi} \int_{|z|=1} z^{n+1} H(z) / F_\varepsilon^+(z) dz \right|^2 \\ &= \sum_{n=0}^{\infty} \left| \frac{1}{2\pi} \int_{|z|=R} z^{n+1} H(z) / F_\varepsilon^+(z) dz \right|^2, \end{aligned}$$

where $1/R_0 < R < 1$. Let us also assume that $\log f \in L^1$, so F^+ is well-defined and

$$H(Re^{i\phi})/F_\varepsilon^+(Re^{i\phi}) \longrightarrow H(Re^{i\phi})/F^+(Re^{i\phi})$$

in L^2 as $\varepsilon \rightarrow 0+$. It follows that

$$I_2(h, \mu)^2 = \sum_{n=0}^{\infty} \left| \frac{1}{2\pi} \int_{|z|=R} z^{n+1} H(z) / F^+(z) dz \right|^2.$$

Now, if we write

$$\frac{1}{F^+(z)} = \sum_{n=0}^{\infty} f_n z^n,$$

then

$$I_2(h, \mu)^2 = \sum_{n=0}^{\infty} \left| \sum_{m=0}^{\infty} h_{-n-m} f_m \right|^2.$$

Thus if H is the Hankel matrix $[h_{-n-m}]_{n,m=0}^{\infty}$, and Φ the column vector with components f_0, f_1, \dots , then

$$I_2(h, \mu) = \| H\Phi \|,$$

where the norm is that of l^2 .

For example, let α be such that $|\alpha| < 1$ and consider

$$\sup_{P \in \mathcal{P}} \left\{ |P(\alpha)| / \left(\int |P|^2 d\mu \right)^{\frac{1}{2}} \right\}.$$

Thus we wish to evaluate $I_2(1/(1 - \alpha e^{-i\phi}), \mu)$. Here $h_{-n} = \alpha^n$, $n = 0, 1, \dots$, so

$$I_2(h, \mu)^2 = \sum_{n=0}^{\infty} \left| \sum_{m=0}^{\infty} \alpha^{n+m} f_m \right|^2 = 1/[(1 - |\alpha|^2) |F^+(\alpha)|^2],$$

as in [2], p. 48.

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M. Altman, <i>An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space</i>	1107
Nesmith Cornett Ankeny, <i>Criterion for rth power residuacity</i>	1115
Julius Rubin Blum and David Lee Hanson, <i>On invariant probability measures I</i>	1125
Frank Featherstone Bonsall, <i>Positive operators compact in an auxiliary topology</i>	1131
Billy Joe Boyer, <i>Summability of derived conjugate series</i>	1139
Delmar L. Boyer, <i>A note on a problem of Fuchs</i>	1147
Hans-Joachim Bremermann, <i>The envelopes of holomorphy of tube domains in infinite dimensional Banach spaces</i>	1149
Andrew Michael Bruckner, <i>Minimal superadditive extensions of superadditive functions</i>	1155
Billy Finney Bryant, <i>On expansive homeomorphisms</i>	1163
Jean W. Butler, <i>On complete and independent sets of operations in finite algebras</i>	1169
Lucien Le Cam, <i>An approximation theorem for the Poisson binomial distribution</i>	1181
Paul Civin, <i>Involutions on locally compact rings</i>	1199
Earl A. Coddington, <i>Normal extensions of formally normal operators</i>	1203
Jacob Feldman, <i>Some classes of equivalent Gaussian processes on an interval</i>	1211
Shaul Foguel, <i>Weak and strong convergence for Markov processes</i>	1221
Martin Fox, <i>Some zero sum two-person games with moves in the unit interval</i>	1235
Robert Pertsch Gilbert, <i>Singularities of three-dimensional harmonic functions</i>	1243
Branko Grünbaum, <i>Partitions of mass-distributions and of convex bodies by hyperplanes</i>	1257
Sidney Morris Harmon, <i>Regular covering surfaces of Riemann surfaces</i>	1263
Edwin Hewitt and Herbert S. Zuckerman, <i>The multiplicative semigroup of integers modulo m</i>	1291
Paul Daniel Hill, <i>Relation of a direct limit group to associated vector groups</i>	1309
Calvin Virgil Holmes, <i>Commutator groups of monomial groups</i>	1313
James Fredrik Jakobsen and W. R. Utz, <i>The non-existence of expansive homeomorphisms on a closed 2-cell</i>	1319
John William Jewett, <i>Multiplication on classes of pseudo-analytic functions</i>	1323
Helmut Klingen, <i>Analytic automorphisms of bounded symmetric complex domains</i>	1327
Robert Jacob Koch, <i>Ordered semigroups in partially ordered semigroups</i>	1333
Marvin David Marcus and N. A. Khan, <i>On a commutator result of Taussky and Zassenhaus</i>	1337
John Glen Marica and Steve Jerome Bryant, <i>Unary algebras</i>	1347
Edward Peter Merkes and W. T. Scott, <i>On univalence of a continued fraction</i>	1361
Shu-Teh Chen Moy, <i>Asymptotic properties of derivatives of stationary measures</i>	1371
John William Neuberger, <i>Concerning boundary value problems</i>	1385
Edward C. Posner, <i>Integral closure of differential rings</i>	1393
Marian Reichaw-Reichbach, <i>Some theorems on mappings onto</i>	1397
Marvin Rosenblum and Harold Widom, <i>Two extremal problems</i>	1409
Morton Lincoln Slater and Herbert S. Wilf, <i>A class of linear differential-difference equations</i>	1419
Charles Robson Storey, Jr., <i>The structure of threads</i>	1429
J. François Treves, <i>An estimate for differential polynomials in $\partial/\partial z_1, \dots, \partial/\partial z_n$</i>	1447
J. D. Weston, <i>On the representation of operators by convolutions integrals</i>	1453
James Victor Whittaker, <i>Normal subgroups of some homeomorphism groups</i>	1469