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1. Introduction. Let \mathfrak{X} be the complex vector space consisting of all complex-valued functions of a non-negative real variable. For each positive number u , let the *shift operator* I_u be the mapping of \mathfrak{X} into itself defined by the formula

$$I_u x(t) = \begin{cases} 0 & (0 \leq t < u) \\ x(t - u) & (t \geq u) \end{cases}$$

Evidently, $I_{u+v} = I_u I_v$, for any positive numbers u and v .

A linear operator A which maps a subspace \mathfrak{D} of \mathfrak{X} into itself will here be called a *V-operator* (after Volterra) if

(1.1) for each x in \mathfrak{D} , the conjugate function x^* belongs to \mathfrak{D} ,

(1.2) both \mathfrak{D} and $\mathfrak{X} \setminus \mathfrak{D}$ are invariant under the shift operators,

(1.3) every shift operator commutes with A .

Many operators that occur in mathematical physics are of this type. If \mathfrak{D} is any subspace of \mathfrak{X} having the properties (1.1) and (1.2), the restriction to \mathfrak{D} of each shift operator is an example of a *V-operator*. All 'perfect operators' (of which a definition may be found in [5]¹) are *V-operators*, on the space of perfect functions.

In this paper we obtain a representation theorem for *V-operators* which are continuous in a certain sense. This result leads to characterizations of two related classes of perfect operators, one of which has been considered from a different point of view in [5]. The main representation theorem (Theorem 4) is similar to a result obtained by R. E. Edwards [2] for *V-operators* which are continuous in another sense; and it closely resembles a theorem given recently by König and Meixner ([3], Satz 3).

2. Elementary properties of V-operators. An important property of *V-operators* is given by

THEOREM 1. *Let A be a V-operator, and let x_1 and x_2 be two of its operands such that, for some positive number t_0 , $x_1(t) = x_2(t)$ whenever $0 \leq t \leq t_0$. Then $Ax_1(t) = Ax_2(t)$ whenever $0 \leq t \leq t_0$.*

Proof. Let $x = x_1 - x_2$. Then, since $x(t) = 0$ if $0 \leq t \leq t_0$, there is

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¹ And in § 4 below.

a function y such that $x = I_{t_0}y$; and y is an operand of A , by virtue of the property (1.2). Consequently, by virtue of (1.3), $Ax = I_{t_0}Ay$; so that $Ax(t) = 0$ whenever $0 \leq t \leq t_0$. But $Ax = Ax_1 - Ax_2$, since A is linear: hence the conclusion of the theorem.

With products and linear combinations defined in the usual way, the V -operators on a given space \mathfrak{D} constitute a linear algebra $\mathfrak{A}(\mathfrak{D})$. If A belongs to $\mathfrak{A}(\mathfrak{D})$ then so does the operator A^* defined by

$$A^*x = (Ax^*)^* ,$$

where x is any function in \mathfrak{D} . We therefore have the unique decomposition

$$A = B + iC ,$$

where B and C belong to $\mathfrak{A}(\mathfrak{D})$ and are 'real' in the sense that Bx and Cx are real for every real function x in \mathfrak{D} . (The property (1.1) ensures that every function x in \mathfrak{D} can be uniquely expressed as $x_1 + ix_2$, where x_1 and x_2 are real functions in \mathfrak{D} .)

If A is a linear combination of shift operators, we have

$$A = \sum_{j=1}^n \alpha_j I_{u_j} = I_u \sum_{j=1}^n \alpha_j I_{u_j - u} ,$$

where $\alpha_1, \dots, \alpha_n$ are complex numbers, u is the least of the positive numbers u_1, \dots, u_n , and I_0 is the unit operator (to be denoted henceforth by ' I '). From this it is apparent that A has no reciprocal in the algebra $\mathfrak{A}(\mathfrak{X})$; however, $I - A$ has a reciprocal in $\mathfrak{A}(\mathfrak{X})$, as the following result shows.

THEOREM 2. *Let A be a V -operator on a space \mathfrak{D} , and let u be any positive number. Then the formula*

$$Bx(t) = x(t) + \sum_{n=1}^{\infty} I_{nu} A^n x(t) ,$$

where x is any function in \mathfrak{D} , and $t \geq 0$, defines a linear transformation B , of \mathfrak{D} into \mathfrak{X} , which commutes with every shift operator and is such that $B(I - I_u A)x = x$ for every x in \mathfrak{D} and $(I - I_u A)Bx = x$ if Bx is in \mathfrak{D} .

Proof. The series defining B certainly converges (pointwise): in fact, if $t_0 \geq 0$ and m is a positive integer such that $mu \geq t_0$, then, for any x in \mathfrak{D} ,

$$Bx(t) = x(t) + \sum_{n=1}^m I_{nu} A^n x(t)$$

whenever $0 \leq t \leq t_0$. Hence if Bx is in \mathfrak{D} then, by Theorem 1,

$$(I - I_u A)Bx(t) = x(t) - I_{(m+1)u} A^{m+1} x(t) = x(t)$$

whenever $0 \leq t \leq t_0$; so that $(I - I_u A)Bx = x$, since t_0 is arbitrary. Also, if x is in \mathfrak{D} then $(I - I_u A)x$ is in \mathfrak{D} , so that

$$\begin{aligned} B(I - I_u A)x(t) &= (I - I_u A)x(t) + \sum_{n=1}^m I_{nu} A^n (I - I_u A)x(t) \\ &= x(t) - I_{(m+1)u} A^{m+1} x(t) = x(t) \end{aligned}$$

whenever $0 \leq t \leq t_0$. Thus $B(I - I_u A)x = x$. It can be verified in a similar way that B commutes with the shift operators and is linear.

If the transformation B of Theorem 2 maps \mathfrak{D} into itself, then $I - I_u A$ has a reciprocal in $\mathfrak{A}(\mathfrak{D})$, namely B . This is certainly the case if \mathfrak{D} consists of all the functions x that have some purely local property (for example, continuity, with $x(0) = 0$, or differentiability, with $x(0) = x'(0) = 0$, or local integrability).² It is also the case with certain other choices of \mathfrak{D} , provided that A is restricted to be a linear combination of shift operators; for example, if \mathfrak{D} consists of the perfect functions, then an operator of the form

$$(2.1) \quad \alpha_0 I + \alpha_1 I_{u_1} + \dots + \alpha_n I_{u_n}$$

has a reciprocal in $\mathfrak{A}(\mathfrak{D})$ if $\alpha_0 \neq 0$ (this can be seen at once on taking Laplace transforms and using Theorem 6 of [5]).

If \mathfrak{D} contains more than the zero function, it is clear that (2.1) represents the zero operator on \mathfrak{D} only if all the coefficients $\alpha_0, \dots, \alpha_n$ are zero; and since the product of two operators of this form is another such operator, the reciprocal of (2.1) cannot be expressed in the same form unless it is a scalar multiple of I . Thus it is usual for $\mathfrak{A}(\mathfrak{D})$ to contain operators other than those of the form (2.1). In general it seems to be difficult to decide whether $\mathfrak{A}(\mathfrak{D})$ is commutative or not; but it is shown in § 4 that \mathfrak{D} can be chosen, of moderate size, so that $\mathfrak{A}(\mathfrak{D})$ is not commutative.

The Laplace transformation is naturally associated with the idea of a V -operator, because it converts the shift operators to exponential factors. A locally integrable function x has an absolutely convergent Laplace integral if x is of exponential order at infinity, in the sense that $x(t) = O(e^{ct})$ as $t \rightarrow \infty$, for some real number c (depending on x). One can consider V -operators on spaces consisting of such functions, and for some of these spaces the following result is available.

THEOREM 3. *Let A be a V -operator on a space \mathfrak{D} consisting of all*

² A property at infinity might be regarded as 'local', but this interpretation is to be excluded here.

the functions in \mathfrak{X} which satisfy some (possibly empty) set of local conditions and are of exponential order at infinity. Then there are positive numbers b , c , and τ such that $|Ax(t)| \leq be^{ct}$ whenever $t \geq \tau$ and $|x(t)| \leq 1$ for all t , with x in \mathfrak{D} .

Proof. Assuming the theorem to be false, we shall construct inductively a sequence $\{x_n\}$ in \mathfrak{D} , and a sequence $\{t_n\}$ of positive numbers, such that, for each positive integer n ,

- (i) $|x_n(t)| \leq 2^{-n}$ for all values of t ,
- (ii) $t_n \geq n$,
- (iii) $x_n(t) = 0$ if $0 \leq t \leq t_{n-1}$, where $t_0 = 0$,
- (iv) $|\sum_{j=1}^n Ax_j(t_n)| \geq e^{nt_n}$.

In the first place, if the theorem is false, we can choose x_1 so that $|x_1(t)| \leq \frac{1}{2}$ for all values of t and $|Ax_1(t)| \geq e^t$ for some value of t , say t_1 , greater than 1. Suppose, then, that the first $m-1$ terms of each sequence have been chosen, where $m > 1$, so that (i)-(iv) hold when $n \leq m-1$. Let

$$y_m = \sum_{j=1}^{m-1} Ax_j.$$

Since y_m belongs to \mathfrak{D} , there is a real number c_m such that $|y_m(t)| \leq e^{c_m t}$ when t is sufficiently large. We can choose x_m so that $|x_m(t)| \leq 2^{-m}$ for all t , $x_m(t) = 0$ if $0 \leq t \leq t_{m-1}$, and

$$|Ax_m(t_m)| \geq 2e^{(c_m+m)t_m},$$

where t_m is chosen so that $t_m \geq m$ and $|y_m(t_m)| \leq e^{c_m t_m}$. Then

$$\left| \sum_{j=1}^m Ax_j(t_m) \right| \geq |Ax_m(t_m)| - |y_m(t_m)| \geq e^{(c_m+m)t_m} \geq e^{m t_m}.$$

Thus (i)-(iv) hold when $n = m$.

Now let $x_0 = \sum_{n=1}^{\infty} x_n$. Then $|x_0(t)| \leq 1$ for all t , by virtue of (i); and x_0 belongs to \mathfrak{D} since, by (iii), it has the appropriate local properties. Hence there is a real number c_0 such that $Ax(t) = O(e^{c_0 t})$ as $t \rightarrow \infty$; so that, by (ii), $Ax(t_n) = O(e^{c_0 t_n})$ as $n \rightarrow \infty$. But, by (iii) and (iv), and Theorem 1, $|Ax(t_n)| \geq e^{n t_n}$ for each n . This contradiction proves the theorem.

3. Strong continuity. If the field of complex numbers is given either the discrete topology or the usual topology, the space \mathfrak{X} can be given the corresponding topology of uniform convergence on finite closed intervals. The first of these topologies for \mathfrak{X} has the property that every V -operator is continuous with respect to it, as Theorem 1 shows; but it does not make \mathfrak{X} a topological vector space (it has the defect that $n^{-1}x \rightarrow 0$ as $n \rightarrow \infty$ only if x is the zero function). The second topology for \mathfrak{X}

is more interesting, and will be referred to as the *strong* topology. In fact we shall consider this only in relation to the closed subspace, \mathfrak{C}_0 , consisting of all the continuous functions x for which $x(0) = 0$. For each x in \mathfrak{C}_0 , and each non-negative number t , we define $\|x\|_t$ to be the least upper bound of $|x(u)|$ with $0 \leq u \leq t$. We can then give \mathfrak{C}_0 a metric, which determines the strong topology, by taking the distance between functions x and y to be

$$\sum_{n=1}^{\infty} 2^{-n} \|x - y\|_n / (1 + \|x - y\|_n) .$$

In this way \mathfrak{C}_0 becomes a Fréchet space.

In the case of \mathfrak{C}_0 , which is an example of a space \mathfrak{D} satisfying (1.1) and (1.2), a large class of V -operators, including those of the form (2.1), can be defined in terms of Riemann-Stieltjes convolution integrals. If ν is a function which belongs to \mathfrak{X} and has bounded variation in every finite interval $[0, t]$, then the formula

$$(3.1) \quad Ax(t) = \int_0^t x(t - u) d\nu(u)$$

where x is any function in \mathfrak{C}_0 , defines a V -operator A on \mathfrak{C}_0 (cf. [5], Theorem 3). Moreover, if $0 \leq v \leq t$ then

$$|Ax(v)| \leq \int_0^v |x(v - u)| d\nu(u) \leq \int_0^t \|x\|_t |d\nu(u)|, \quad (t \geq 0),$$

so that

$$\|Ax\|_t \leq \|x\|_t \left| \int_0^t d\nu(u) \right|;$$

whence it follows that A is strongly continuous (continuous with respect to the strong topology). The theorem we are about to prove shows that every strongly continuous V -operator on a sufficiently large space \mathfrak{D} of continuous functions can be represented in this way (and can therefore be extended from \mathfrak{D} to the whole of \mathfrak{C}_0).

If A is a linear operator on a subspace \mathfrak{D} of \mathfrak{C}_0 , and if $t \geq 0$, we denote by ‘ $\|A\|_t$ ’ the least upper bound of $\|Ax\|_t$ with x in \mathfrak{D} and $\|x\|_t \leq 1$. It is clear that A is strongly continuous if and only if $\|A\|_t$ is finite for all values of t (or, equivalently, for all sufficiently large values of t).

THEOREM 4. *Let A be a strongly continuous V -operator on a strongly dense subspace \mathfrak{D} of \mathfrak{C}_0 , and let t be any positive number. Then there is a function ν in \mathfrak{X} , with $\nu(0) = 0$ and $\nu(u-) = \nu(u)$ whenever $0 < u \leq t$, such that $Ax(t)$ is given by (3.1) for every x in \mathfrak{D} . This function ν is uniquely determined by A , and is independent of t ; its total variation*

in the interval $[0, t]$ is $\|A\|_t$.

Proof. For each function x in \mathfrak{D} , and for each positive number t , let x_t be the restriction of x to the closed interval $[0, t]$. Then, for a fixed value of t , the mapping $x \rightarrow x_t$ is a linear transformation of \mathfrak{D} on to a subspace \mathfrak{D}_t of the complex Banach space $C[0, t]$, consisting of all continuous functions on the interval $[0, t]$; moreover, $\|x_t\| = \|x\|_t$. If $x_t = 0$ then $Ax(t) = 0$, by Theorem 1; we can therefore define a linear functional φ on \mathfrak{D}_t by the formula

$$\varphi(x_t) = Ax(t).$$

This functional is continuous, with $\|\varphi\| = \|A\|_t$.

An integral representation of φ can be found by adapting a construction used by Banach ([1], 59-60). By a well-known theorem³, φ can be extended without change of norm to the complex Banach space $M[0, t]$, which contains the characteristic functions of all the subintervals of $[0, t]$. A function ν_t can then be defined on $[0, t]$ so that $\nu_t(0) = 0$ and

$$(i) \quad \int_0^t |d\nu_t(u)| \leq \|\varphi\|,$$

$$(ii) \quad \varphi(f) = \int_0^t f(t-u)d\nu_t(u)$$

for every function f in $C[0, t]$.

Without affecting the validity of (i) or (ii), we can adjust ν_t so that it is continuous on the left at each interior point of the interval $[0, t]$. Moreover, if f is a continuous function such that $f(0) = 0$, then the jump of ν_t at the point t makes no contribution to the integral in (ii); therefore, as far as such functions f are concerned, we may suppose ν_t chosen so that $\nu_t(t-) = \nu_t(t)$, giving left-hand continuity throughout the interval $(0, t]$, and retaining (i). Under these conditions, ν_t is uniquely determined by A . For, if $0 < v \leq t$ and $0 < \delta < v$, there is a function f_δ in $C[0, t]$ such that $\|f_\delta\| = 1$ and

$$f_\delta(u) = \begin{cases} 0 & (0 \leq u \leq t-v) \\ 1 & (t-v+\delta \leq u \leq t). \end{cases}$$

Thus

$$\varphi(f_\delta) = \int_0^{v-\delta} d\nu_t(u) + \int_{v-\delta}^v f_\delta(t-u)d\nu_t(u),$$

and therefore

$$|\varphi(f_\delta) - \nu_t(v-\delta)| \leq \int_{v-\delta}^v |d\nu_t(u)|,$$

³ The Hahn-Banach-Bohnenblust-Sobczyk extension theorem: see, for example, [8], 113.

so that $\varphi(f_\delta) \rightarrow \nu_t(v)$ as $\delta \rightarrow 0$.⁴ But since \mathfrak{D} is strongly dense in \mathfrak{C}_0 , f_δ belongs to the closure of \mathfrak{D}_t , in $C[0, t]$; so that, φ being continuous, $\varphi(f_\delta)$ is uniquely determined by A , for each value of δ . This establishes the uniqueness of ν_t .

Now suppose that $t' > t$. By what has been proved, we have, for any x in \mathfrak{D} ,

$$Ax(t) = \int_0^t x(t - u) d\nu_t(u) .$$

But $Ax(t) = I_{t'-t}Ax(t')$, and $I_{t'-t}A = AI_{t'-t}$; hence

$$Ax(t) = \int_0^{t'} I_{t'-t}x(t' - u) d\nu_t(u) = \int_0^t x(t - u) d\nu_t(u) .$$

It follows that $\nu_t(u) = \nu_{t'}(u)$ whenever $0 \leq u \leq t$; in particular, $\nu_t(t) = \nu_{t'}(t)$. Hence if we define the function ν by

$$\nu(t) = \nu_t(t) \tag{t \geq 0} ,$$

we obtain the required representation of A .

Finally, (i) shows that

$$\int_0^t |d\nu(u)| \leq \|A\|_t ,$$

and we have previously noted that, for any x in \mathfrak{D} ,

$$\|Ax\|_t \leq \|x\|_t \int_0^t |d\nu(u)| .$$

Thus $\int_0^t |d\nu(u)| = \|A\|_t$, and the proof is complete.⁵

As a corollary, we have

THEOREM 5. *Suppose that the formula*

$$Ax(t) = \int_0^t K(t, u)x(u)du \tag{t \geq 0}$$

defines a V-operator A on \mathfrak{C}_0 , the kernel K being such that $\int_0^t |K(t, u)| du$ exists as a Lebesgue integral which is locally bounded with respect to t. Then there is a function k in \mathfrak{X} such that, for each t, $K(t, u) = k(t - u)$ for almost all values of u.

⁴ Here we use the fact that if a function of bounded variation is continuous on the left, then so is its total variation.

⁵ In this proof we have not fully used the fact that A maps \mathfrak{D} into itself: it is enough that A maps \mathfrak{D} into \mathfrak{C}_0 .

Proof. For each t , let $\|K\|_t$ be the least upper bound of $\int_0^v |K(v, u)| du$ with $0 \leq v \leq t$; this is finite, by hypothesis. Then, for each x in \mathfrak{C}_0 ,

$$\|Ax\|_t \leq \|K\|_t \|x\|_t,$$

so that A is strongly continuous. But

$$Ax(t) = \int_0^t K(t, t-u)x(t-u)du,$$

so that if

$$L_t(u) = \int_0^u K(t, t-v)dv$$

then

$$Ax(t) = \int_0^t x(t-u)dL_t(u).$$

Hence, by Theorem 4, $L_t = \nu$, a function which is independent of t . Since ν has bounded variation, there is a function k such that

$$k(u) = \frac{d}{du} \nu(u)$$

except when u is in a set E whose Lebesgue measure is 0. However, for each value of t ,

$$\frac{d}{du} \nu(u) = \frac{d}{du} L_t(u) = K(t, t-u)$$

except when u is in a set E_t of measure 0. Thus

$$K(t, u) = k(t-u)$$

except when u is in the set $t - (E_t \cup E)$, which has measure 0.

The functions in \mathfrak{C}_0 which are of exponential order at infinity form a subspace \mathfrak{C}_0 . The perfect functions form a smaller subspace, \mathfrak{D}_0 (in fact \mathfrak{D}_0 is the largest subspace of \mathfrak{C}_0 which is invariant under the differential operator, D).

THEOREM 6. \mathfrak{D}_0 is strongly dense in \mathfrak{C}_0 .

Proof. It is easily seen that \mathfrak{C}_0 is strongly dense in \mathfrak{C}_0 : in fact, if x is in \mathfrak{C}_0 and x_n is defined by

$$x_n(t) = \begin{cases} x(t) & (0 \leq t \leq n) \\ x(n) & (t \geq n), \end{cases}$$

then x_n belongs to \mathfrak{C}_0 , for each n , and $x_n \rightarrow x$ strongly as $n \rightarrow \infty$. To show that \mathfrak{D}_0 is dense in \mathfrak{C}_0 , let x be any function in \mathfrak{C}_0 and, for each positive number δ , let $g_{(\delta)}$ be a positive perfect function such that if $t \geq \delta$ then $g_{(\delta)}(t) = 0$ and $\int_0^t g_{(\delta)}(u)du = 1$ (for example, we could take $g_{(\delta)}$ to be $Dh_{(\delta)}$, where $h_{(\delta)}$ is given by Lemma 1 of [5]). Let $x_{(\delta)} = x * g_{(\delta)}$. Then $x_{(\delta)}$ belongs to \mathfrak{D}_0 (' $x*$ ' is a perfect operator), and, if $v \geq \delta$,

$$\begin{aligned} x_{(\delta)}(v) - x(v) &= \int_0^v x(v-u)g_{(\delta)}(u)du - x(v) \\ &= \int_0^\delta \{x(v-u) - x(v)\}g_{(\delta)}(u)du . \end{aligned}$$

Now let t and ε be any positive numbers. Since x is uniformly continuous in the interval $[0, t]$, with $x(0) = 0$, we can choose δ so that

$$|x(v-u) - x(v)| < \varepsilon$$

whenever $\delta \leq v \leq t$, and $|x(v)| < \frac{1}{2}\varepsilon$ whenever $0 \leq v \leq \delta$; then

$$|x_{(\delta)}(v) - x(v)| < \varepsilon \int_0^\delta g_{(\delta)}(u)du = \varepsilon$$

if $\delta \leq v \leq t$, and if $0 \leq v \leq \delta$,

$$\begin{aligned} |x_{(\delta)}(v) - x(v)| &\leq \int_0^\delta |x(v-u)|g_{(\delta)}(u)du + |x(v)| \\ &\leq \frac{1}{2}\varepsilon \int_0^\delta g_{(\delta)}(u)du + \frac{1}{2}\varepsilon = \varepsilon . \end{aligned}$$

Thus $\|x_{(\delta)} - x\|_t < \varepsilon$. It follows that \mathfrak{D}_0 is strongly dense in \mathfrak{C}_0 .

In [5] it is shown that any positive perfect operator has the representation (3.1), with ν a non-decreasing function (in fact this holds for any positive V -operator on a space \mathfrak{D} such that $\mathfrak{D}_0 \subseteq \mathfrak{D} \subseteq \mathfrak{C}_0$). It follows that the linear combinations of positive perfect operators, which form a linear algebra $\mathfrak{M}(\mathfrak{D}_0)^6$, are strongly continuous. On the other hand, there are strongly continuous perfect operators which do not belong to $\mathfrak{M}(\mathfrak{D}_0)$: for example, if $\nu(t) = \sin(e^{t^2} - 1)$, and A is defined on \mathfrak{D}_0 according to (3.1), then, as is shown in [5], A is a perfect operator which is not in $\mathfrak{M}(\mathfrak{D}_0)$; but of course A is strongly continuous. However, it is possible to characterize $\mathfrak{M}(\mathfrak{D}_0)$ in terms of seminorms, as follows.

THEOREM 7. *A V -operator A on \mathfrak{D}_0 is an element of $\mathfrak{M}(\mathfrak{D}_0)$ if and only if there is a real number c such that $\|A\|_t = O(e^{ct})$ as $t \rightarrow \infty$.*

Proof. By Theorem 1 of [5], an operator A on \mathfrak{D}_0 is in $\mathfrak{M}(\mathfrak{D}_0)$ if

⁶ $\mathfrak{M}(\mathfrak{D}_0)$ is denoted in [5] by ' \mathfrak{M} '.

and only if it admits the representation (3.1) with ν a linear combination of positive non-decreasing functions which are of exponential order at infinity. This condition on ν is equivalent to the existence of a real number c such that $\int_0^t |d\nu(u)| = O(e^{ct})$ as $t \rightarrow \infty$. Therefore, by Theorems 4 and 6 above, A is in $\mathfrak{M}(\mathfrak{D}_0)$ if and only if $\|A\|_t = O(e^{ct})$ as $t \rightarrow \infty$.

Each function y in \mathfrak{C}_0 determines a strongly continuous V -operator A on \mathfrak{C}_0 according to the formula $Ax = x*y$; for, integration by parts shows that this formula is equivalent to (3.1), with

$$\nu(t) = D^{-1}y(t) = \int_0^t y(u)du \quad (t \geq 0).$$

An important property of convolution in \mathfrak{C}_0 is the fact that it obeys the associative law (as well as the commutative law); more generally, we have

THEOREM 8. *Let A and B be strongly continuous V -operators, on \mathfrak{C}_0 and on a subspace \mathfrak{D} of \mathfrak{C}_0 respectively. If x is any function in \mathfrak{D} then Ax belongs to the strong closure of \mathfrak{D} ; if Ax is in \mathfrak{D} itself, then $ABx = BAx$. In particular, if y is a function in \mathfrak{C}_0 such that $x*y$ is in \mathfrak{D} , then $B(x*y) = (Bx)*y$.*

Proof. Let A be represented by a function ν in accordance with Theorem 4. Then for any x in \mathfrak{D} , each value $Ax(t)$ can be arbitrarily approximated by sums of the form

$$\sum_{j=1}^n \{\nu(u_j) - \nu(u_{j-1})\}x(t - u_j),$$

where $0 \leq u_1 \leq \dots \leq u_n \leq t$; and this approximation is locally uniform with respect to t . Now the above sum is the value at t of the function

(i)
$$\sum_{j=1}^n \alpha_j I_{u_j} x,$$

where $\alpha_j = \nu(u_j) - \nu(u_{j-1})$. This function belongs to \mathfrak{D} , since \mathfrak{D} satisfies (1.2). Thus Ax belongs to the strong closure of \mathfrak{D} . Further, the points u_j can be chosen in such a way that, while Ax is strongly approximated by (i), ABx is simultaneously approximated, in the same sense, by

(ii)
$$\sum_{j=1}^n \alpha_j I_{u_j} Bx.$$

But, since B is a V -operator, (ii) is the same as

$$B \sum_{j=1}^n \alpha_j I_{u_j} x.$$

Since B is strongly continuous, it follows that $ABx = BAx$ if Ax is an operand of B .

We can now prove a partial converse of Theorem 1, namely.

THEOREM 9. *Let A be a non-zero strongly continuous V -operator on \mathfrak{C}_0 . Then there is a non-negative number τ such that (i) for any function x in C_0 , $Ax(t) = 0$ whenever $0 \leq t \leq \tau$, and (ii) if $Ax(t) = 0$ whenever $0 \leq t \leq t_0$, where x belongs to \mathfrak{C}_0 and $t_0 \geq \tau$, then $x(t) = 0$ whenever $0 \leq t \leq t_0 - \tau$. In particular, $x = 0$ if $Ax = 0$.*

Proof. Let ν be the function representing A according to Theorem 4, and let τ be the greatest lower bound of the numbers t for which $\nu(t) \neq 0$. Obviously, τ has the property (i) required by the theorem. Suppose that x is a function in \mathfrak{C}_0 such that $Ax(t) = 0$ whenever $0 \leq t \leq t_0$, where $t_0 \geq \tau$. Let $g_{(\delta)}$ be defined as in the proof of Theorem 6, and let $x_{(\delta)} = x * g_{(\delta)}$. Then, for each value of δ , $x_{(\delta)}$ has a derivative $x'_{(\delta)}$ in \mathfrak{C}_0 ; in fact $x'_{(\delta)} = x * g'_{(\delta)}$. Also, if $0 \leq t \leq t_0$,

$$\begin{aligned} \int_0^t x'_{(\delta)}(t-u)\nu(u)du &= Ax_{(\delta)}(t) = (Ax) * g_{(\delta)}(t) \\ &= \int_0^t Ax(t-u)g_{(\delta)}(u)du = 0. \end{aligned}$$

Therefore, by a theorem of Titchmarsh [4, 327], $x'_{(\delta)}(t) = 0$ whenever $0 \leq t \leq t_0 - \tau$ (we cannot have $\nu(t) = 0$ for almost all t in a neighbourhood of τ , since ν is continuous on the left). Hence $x_{(\delta)}(t) = 0$ whenever $0 \leq t \leq t_0 - \tau$. Since $x_{(\delta)}(t) \rightarrow x(t)$ as $\delta \rightarrow 0$, the theorem follows.

It is a consequence of Theorem 8 that every strongly continuous V -operator on \mathfrak{D}_0 is a perfect operator (the converse is false; in fact it is easy to see that the differential operator D is not strongly continuous). Thus an operator A represented by (3.1) is a perfect operator if and only if it maps \mathfrak{D}_0 into itself. An equivalent condition is given by

THEOREM 10. *The formula (3.1), with x in \mathfrak{D}_0 , represents a perfect operator A if and only if there is a positive integer n such that $D^{-n}\nu$ belongs to \mathfrak{C}_0 , where*

$$D^{-n}\nu(t) = \int_0^t \cdots \int_0^{u_2} \nu(u_1)du_1 \cdots du_n \quad (t = u_{n+1} \geq 0).$$

Proof. For any perfect function x and any positive integer n , we have from (3.1), after integration by parts,

$$Ax(t) = \int_0^t x^{(n+1)}(t - u)D^{-n}\nu(u)du \quad (t \geq 0).$$

Thus if $D^{-n}\nu$ belongs to \mathfrak{E}_0 for some value of n , then A is a perfect operator. On the other hand, suppose that A , given by (3.1), is a perfect operator (when restricted to \mathfrak{D}_0). By a general representation theorem for perfect operators [6], there is a function y in \mathfrak{E}_0 such that, for some positive integer n , and every perfect function x ,

$$Ax(t) = \int_0^t x^{(n+1)}(t - u)y(u)du \quad (t \geq 0).$$

Hence $x^{(n+1)} * (y - D^{-n}\nu) = 0$, so that, by Theorem 9, $y = D^{-n}\nu$.

If $\nu(t) = e^{t^2}$, the V -operator A given by (3.1) does not map \mathfrak{D}_0 into itself, since ν does not satisfy the condition of Theorem 10.

Every perfect operator A has a Laplace transform, \bar{A} : if A is given by (3.1), \bar{A} may or may not be given by

$$(3.2) \quad \bar{A}(z) = \int_0^\infty e^{-zt}d\nu(t),$$

the integral being convergent when $\Re z$ is sufficiently large. This representation of \bar{A} is certainly valid if A belongs to $\mathfrak{M}(\mathfrak{D}_0)$ (cf. [5], Theorem 4); and also if $\nu(t) = \sin(e^{t^2} - 1)$, for example. But if $D^{-1}\nu(t) = \sin(e^{t^2} - 1)$ the integral in (3.2) does not converge for any value of z (as can be seen on integrating twice by parts). However, (3.2) holds whenever the integral is convergent, as the following result shows.

THEOREM 11. *Let A be any strongly continuous perfect operator, and let ν be a function such that A is represented by (3.1). Then the Laplace transform \bar{A} is represented by (3.2), with $\Re z$ sufficiently large, if the infinite integral is interpreted in the sense of summability (C, n) , where n is any non-negative integer such that $D^{-n}\nu$ belongs to \mathfrak{E}_0 .*

Proof. Let B be the perfect operator obtained on replacing ν by $D^{-1}\nu$ in (3.1). Then, if x is any perfect function, and $t \geq 0$,

$$DBx(t) = Bx'(t) = \int_0^t x'(t - u)\nu(u)du = \nu(0)x(t) + \int_0^t x(t - u)d\nu(u).$$

Thus $DB = \nu(0)I + A$. If ν belongs to \mathfrak{E}_0 then, since B is determined by the function ν in the sense that $Bx = x * \nu$, B has the same Laplace transform as ν ; that is to say, when $\Re z$ is sufficiently large,

$$\bar{B}(z) = \int_0^\infty e^{-zt}\nu(t)dt.$$

Therefore, in this case,

$$\bar{A}(z) = z\bar{B}(z) - \nu(0) = \int_0^\infty ze^{-zt}\{\nu(t) - \nu(0)\}dt = \int_0^\infty e^{-zt}d\nu(t),$$

so that (3.2) holds, the integral being convergent.

We now proceed by induction. Suppose that, for some non-negative integer n , (3.2) holds in the sense of summability (C, n) provided that $D^{-n}\nu$ belongs to \mathfrak{C}_0 and $\Re z$ is sufficiently large. If $D^{-n-1}\nu$ belongs to \mathfrak{C}_0 , and $t > 0$, then

$$\begin{aligned} \int_0^t \left(1 - \frac{u}{t}\right)^{n+1} e^{-zu} d\nu(u) &= -\nu(0) + z \int_0^t \left(1 - \frac{u}{t}\right)^{n+1} e^{-zu} dD^{-1}\nu(u) \\ &\quad + \frac{n+1}{t} \int_0^t \left(1 - \frac{u}{t}\right)^n e^{-zu} dD^{-1}\nu(u). \end{aligned}$$

But, by the induction hypothesis (with $D^{-1}\nu$ in place of ν),

$$\bar{B}(z) = \lim_{t \rightarrow \infty} \int_0^t \left(1 - \frac{u}{t}\right)^{n+1} e^{-zu} dD^{-1}\nu(u) = \lim_{t \rightarrow \infty} \int_0^t \left(1 - \frac{u}{t}\right)^n e^{-zu} dD^{-1}\nu(u)$$

when $\Re z$ is sufficiently large; so that

$$\lim_{t \rightarrow \infty} \int_0^t \left(1 - \frac{u}{t}\right)^{n+1} e^{-zu} d\nu(u) = -\nu(0) + z\bar{B}(z) = \bar{A}(z).$$

Thus

$$\bar{A}(z) = \int_0^\infty e^{-zt} d\nu(t) \quad (C, n+1),$$

and the theorem follows.

If \mathfrak{D} is any subspace of \mathfrak{C}_0 satisfying (1.1) and (1.2), the strongly continuous V -operators on \mathfrak{D} form a subalgebra of $\mathfrak{A}(\mathfrak{D})$, say $\mathfrak{N}(\mathfrak{D})$. If \mathfrak{D} is strongly dense in \mathfrak{C}_0 , it follows from Theorem 4 that $\mathfrak{N}(\mathfrak{D})$ effectively consists of those operators in $\mathfrak{N}(\mathfrak{C}_0)$ which leave \mathfrak{D} invariant. In this case, Theorems 8 and 9 show that $\mathfrak{N}(\mathfrak{D})$ is an integral domain (it is commutative, and has no divisors of zero). The full algebra $\mathfrak{N}(\mathfrak{C}_0)^7$ has the further property that any operator which is inverse to an operator in $\mathfrak{N}(\mathfrak{C}_0)$ is itself in $\mathfrak{N}(\mathfrak{C}_0)$: this is special case of

THEOREM 12. *Let A and B be strongly continuous V -operators on a strongly closed subspace \mathfrak{D} of \mathfrak{C}_0 , and suppose that there is an operator C on \mathfrak{D} such that $A = BC$. Suppose also that $Bx = 0$ only if $x = 0$. Then C is a strongly continuous V -operator.*

⁷ $\mathfrak{N}(\mathfrak{C}_0) = \mathfrak{M}(\mathfrak{C}_0)$, consisting of the linear combinations of positive V -operators on \mathfrak{C}_0 .

Proof. If $u > 0$ and x is any function in \mathfrak{D} then, since A and B are V -operators,

$$B(I_u Cx - CI_u x) = I_u Ax - AI_u x = 0;$$

so that, by the hypothesis concerning B , $I_u Cx = CI_u x$. In a similar way it can be verified that C is linear, and is therefore a V -operator. To show that C is strongly continuous, let $\{x_n\}$ be a strongly convergent sequence in \mathfrak{D} such that the sequence $\{Cx_n\}$ is also strongly convergent. Since A and B are strongly continuous,

$$B(\lim_{n \rightarrow \infty} Cx_n - C \lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} Ax_n - A \lim_{n \rightarrow \infty} x_n = 0,$$

so that $\lim_{n \rightarrow \infty} Cx_n = C \lim_{n \rightarrow \infty} x_n$; thus the graph of C is closed. Now \mathfrak{D} , being strongly closed, is a Fréchet space relative to the strong topology; hence, by Banach's closed-graph theorem [1, 41], C is strongly continuous.

4. Operators that commute with convolution. It is a consequence of Theorem 8 that a subspace \mathfrak{D} of \mathfrak{C}_0 , satisfying (1.1) and (1.2), is closed under convolution if it is strongly closed. On the other hand, \mathfrak{D}_0 is closed under convolution though it is not strongly closed. If \mathfrak{D} is any subspace of \mathfrak{C}_0 which is closed under convolution (so forming an integral domain with no unit element), an operator A on \mathfrak{D} will be said to *commute with convolution* if

$$A(x*y) = (Ax)*y$$

for all x and y in \mathfrak{D} . Such operators are necessarily linear (cf. [5], § 4), and, for a given choice of \mathfrak{D} , they form an integral domain \mathfrak{D}^* in which \mathfrak{D} is isomorphically embedded (by the correspondence $x \rightarrow x*$).

A shift operator belongs to \mathfrak{D}^* if it maps \mathfrak{D} into itself. Hence if \mathfrak{D} satisfies (1.1) and (1.2), in addition to being closed under convolution, then all the operators in \mathfrak{D}^* are V -operators; in fact \mathfrak{D}^* is then a maximal commutative subalgebra of $\mathfrak{A}(\mathfrak{D})$. In this case, Theorem 8 shows that every strongly continuous V -operator commutes with convolution; so that

$$\mathfrak{N}(\mathfrak{D}) \subseteq \mathfrak{D}^* \subseteq \mathfrak{A}(\mathfrak{D}).$$

If, further, \mathfrak{D} is strongly closed, then $\mathfrak{N}(\mathfrak{D}) = \mathfrak{D}^*$: for, if B is defined by $Bx = x*y$, with y in \mathfrak{D} , and $A = BC$, where C is any operator in \mathfrak{D}^* , then, for any x in \mathfrak{D} ,

$$Ax = (Cx)*y = C(x*y) = C(y*x) = (Cy)*x;$$

thus the conditions of Theorem 12 are satisfied, so that C belongs to $\mathfrak{N}(\mathfrak{D})$. In particular, the operators on \mathfrak{C}_0 that commute with convolution

are precisely the strongly continuous V -operators on \mathfrak{C}_0 (and can therefore be represented according to Theorem 4).

An operator A on \mathfrak{C}_0 which commutes with convolution can be extended to the whole of \mathfrak{C}_0 so as to preserve this property. For, if x is any function in \mathfrak{C}_0 , let x_n be defined, for each positive integer n , as in the proof of Theorem 6: then x_n belongs to \mathfrak{C}_0 , and Theorem 1 shows that $Ax_n(t)$ is independent of n provided that $n \geq t$; therefore, if $t \geq 0$, we can define $Ax(t)$ to be $Ax_n(t)$, where $n \geq t$, without ambiguity. Since convolution is defined locally this extension of A is an operator on \mathfrak{C}_0 which commutes with convolution. It follows that A is strongly continuous, and that its extension to \mathfrak{C}_0 is unique (since \mathfrak{C}_0 is strongly dense in \mathfrak{C}_0).

The integration operator, D^{-1} , is an example of an operator on \mathfrak{C}_0 which commutes with convolution. Since \mathfrak{D}_0 can be expressed as $\bigcap_{n=1}^{\infty} D^{-n}\mathfrak{C}_0$, any operator on \mathfrak{C}_0 which commutes with convolution and leaves \mathfrak{C}_0 invariant must leave \mathfrak{D}_0 invariant. The converse of this is false: for, if A is defined by (3.1), ν being such that $D^{-2}\nu$ belongs to \mathfrak{C}_0 but $D^{-1}\nu$ does not, and $\nu(0) = 0$, then A maps \mathfrak{D}_0 into itself, by Theorem 10; however, if $x(t) = t$ then

$$Ax(t) = \int_0^t (t - u)d\nu(u) = D^{-1}\nu(t),$$

so that x is in \mathfrak{C}_0 but Ax is not.

The operators on \mathfrak{D}_0 that commute with convolution are the perfect operators. These can be characterized as those V -operators on \mathfrak{D}_0 which are continuous in a sense defined in terms of Laplace transforms [7]⁸. The strongly continuous perfect operators are the strongly continuous V -operators on \mathfrak{D}_0 , constituting the algebra $\mathfrak{M}(\mathfrak{D}_0)$; this algebra, and also its subalgebra $\mathfrak{M}(\mathfrak{D}_0)$, can be characterized in terms of convolution, as follows.

THEOREM 13. *A perfect operator belongs to $\mathfrak{M}(\mathfrak{D}_0)$ if and only if it can be extended to the whole of \mathfrak{C}_0 so as to commute with convolution; it belongs to $\mathfrak{M}(\mathfrak{D}_0)$ if and only if this extension (necessary unique) leaves \mathfrak{C}_0 invariant.*

Proof. If an operator A on \mathfrak{D}_0 can be extended to \mathfrak{C}_0 so as to commute with convolution, then its extension belongs to $\mathfrak{M}(\mathfrak{C}_0)$, so that A itself belongs to $\mathfrak{M}(\mathfrak{D}_0)$. On the other hand, any operator A in $\mathfrak{M}(\mathfrak{D}_0)$ admits the representation (3.1), which provides an extension of A to \mathfrak{C}_0 : this extension, being strongly continuous, commutes with convolution;

⁸ It is not at present known whether there are any V -operators on \mathfrak{D}_0 which are not perfect; that is to say, it is not known whether $\mathfrak{M}(\mathfrak{D}_0)$ is commutative or not (but there are linear operators on \mathfrak{D}_0 which commute with D and are not perfect [6]).

it is also unique, since \mathfrak{D}_0 is strongly dense in \mathfrak{C}_0 .

If a perfect operator A has a strongly continuous extension to \mathfrak{C}_0 which leaves \mathfrak{C}_0 invariant, we can regard A as a V -operator on \mathfrak{C}_0 ; then, by Theorem 3, there is a real number c such that $\|A\|_t = O(e^{ct})$ as $t \rightarrow \infty$, and this implies, by Theorem 7, that A belongs to $\mathfrak{M}(\mathfrak{D}_0)$. On the other hand, if A belongs to $\mathfrak{M}(\mathfrak{D}_0)$ then the extension of A to \mathfrak{C}_0 given by (3.1) leaves \mathfrak{C}_0 invariant, by Theorem 3 of [5].

Finally, we give an example of a V -operator, on a strongly dense subspace of \mathfrak{C}_0 , which does not commute with convolution. Let h be the Heaviside unit function ($h(t) = 1$ if $t \geq 0$), and let \mathfrak{D}_1 be the class of all functions x given by

$$(4.1) \quad x = D^{-1}(y + Bh),$$

where y belongs to \mathfrak{C}_0 and B is an operator of the type (2.1). Then $\mathfrak{D}_0 \subseteq \mathfrak{D}_1 \subseteq \mathfrak{C}_0$, and \mathfrak{D}_1 satisfies (1.1) and (1.2); moreover, \mathfrak{D}_1 is closed under convolution. It is clear that y and B in (4.1) are uniquely determined by x , and that the mapping $x \rightarrow y$ is a V -operator, say A , on \mathfrak{D}_1 . The operator D^{-1} maps \mathfrak{D}_1 into itself and commutes with convolution. However, $AD^{-1}x = x$ and $D^{-1}Ax = y$, so that $AD^{-1} \neq D^{-1}A$. Hence A does not commute with convolution. It follows that the algebra $\mathfrak{A}(\mathfrak{D}_1)$, of all V -operators on \mathfrak{D}_1 , is not commutative.

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