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OPERATIONAL CALCULUS OF LINEAR RELATIONS

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1. Introduction. Let X and Y be linear spaces, and T a linear subspace of $X \oplus Y$. We call T a *linear relation* to indicate our interest in those constructions with T which generalize those carried out when T is single-valued [4].

Properly many-valued linear relations arise naturally from operators T when T^{-1} or T^* is contemplated in cases where they are not single-valued. One advantage of not dismissing T^* when it is not single-valued is that $T^{**} = T$ if and only if T is closed (for the details, see 3.34, below.) A more superficial attraction is that linear relations, even self-adjoint linear relations in Hilbert space can exhibit phenomena (unbounded spectrum, domain $\neq X$) in finite-dimensional spaces which linear operators exhibit only in infinite-dimensional spaces.

We present an outline of the paper. In § 2 we define $p(T)$ where p is a polynomial with coefficients in the field ϕ involved in X . We prove that $(pq)(T) = p(T)q(T)$, $(p \circ q)(T) = p(q(T))$, and point out that sometimes $(p + q)(T) \neq p(T) + q(T)$, etc.

In § 3 we turn to relations in dual pairs. In this situation, adjoints can be defined. We build an automorphism $\lambda \rightarrow \bar{\lambda}$ of ϕ into the theory of dual pairs, so as not to *exclude* the Hilbert space situation, which dual pairs are intended to imitate. (Thus the transpose is a special kind of adjoint.) Closedness is defined algebraically, but in a way compatible with the topological concept. Closure of T^* and other algebraic properties of $*$ are established. Finally, it is shown that if T is closed and its resolvent is not void then $p(T)$ is also closed.

Section 4 considers the self-dual case. We give a simple condition (4.3) always true in Hilbert space, that T^*T be self-adjoint, T being closed. In § 5 we give the spectral analysis of self-adjoint linear relations in Hilbert space. In a 1:1 manner these correspond to the unitary *operators*, via the Cayley transform. However, it can be shown directly that X is the direct sum of orthogonal subspaces Y, Z which reduce T ($= T^*$) giving in Z a self-adjoint operator and in Y the inverse of the zero-operator.

2. Linear relations. A *relation* T between members of a set X and members of a set Y is merely a subset of $X \times Y$. For $x \in X$, $T(x) = \{y : (x, y) \in T\}$. The *domain* of T consists of those x such that $T(x)$ is not void. T is called single-valued if $T(x)$ never contains more than one element. The *range* of T is the union of all $T(x)$.

If T is as above and $S \subset Y \times Z$, then $S \circ T = \{(x, z) : (x, y) \in T, (y, z) \in S \text{ for some } y\}$. We shall write this ST . Finally, $T^{-1} = \{(y, x) : (x, y) \in T\}$. The *range* of T is the domain of T^{-1} .

If X and Y are linear spaces over a field Φ then $X \oplus Y$ is $X \times Y$ with the usual linear structure. A *linear relation* T between members of X and members of Y is a linear subspace of $X \oplus Y$. Linearity is characterized by

$$2.01 \quad \alpha T(x_1) + \beta T(x_2) \subset T(\alpha x_1 + \beta x_2), \quad (\alpha, \beta \in \Phi; x_1, x_2 \in X).$$

The null space of T is the class of x such that $(x, 0) \in T$. It is easy to see that

2.02 *if S and T are linear relations with the same null space, and the same range, then $S \subset T$ only if $S = T$.*

Let L be a linear subspace of X , and λ an element of Φ . Then λ_L denotes the single valued operator defined on L by $\lambda_L = \{(x, \lambda x) : x \in L\}$. The unit of Φ we denote by 1. Thus 1_L has a meaning according to the preceeding agreement. For T a linear relation with range L , we define λT as $\lambda_L T$. The zero of Φ we denote by 0. Thus OT is not O_X , but O_L where L is the domain of T .

Addition of linear relations S, T is defined as follows:

$$S + T = \{(x, y) : y = s + t \text{ for some } s, t \text{ such that } (x, s) \in S, (x, t) \in T\}.$$

The linear relations in $X \oplus X$ do not form a linear space, let alone a linear algebra. We list algebraic properties partly for use later, but mainly to call attention, as it were, to those that are lacking.

2.1 THEOREM. *The operations ' \circ ' and ' $+$ ' are associative, ' $+$ ' is commutative. Let R, S, T be linear relations. Then*

$$2.11 \quad \text{domain of } R = X \iff 1_X \subset R^{-1}R;$$

$$2.12 \quad R \text{ is single-valued} \iff RR^{-1} \subset 1_L, \quad L = \text{range of } R;$$

$$2.13 \quad \lambda \in \Phi \Rightarrow \lambda(ST) = (\lambda S)T = S(\lambda T) = ST\lambda_L, \quad L = \text{domain of } T;$$

$$2.14 \quad R \subset S \Rightarrow R + T \subset S + T, \quad RT \subset ST, \quad TR \subset TS, \quad R^{-1} \subset S^{-1};$$

$$2.15 \quad RS + RT \subset R(S + T), \text{ with equality when the domain of } R \text{ coincides with the whole space};$$

$$2.16 \quad (S + T)R \subset SR + TR, \text{ with equality when } R \text{ is single-valued};$$

$$2.17 \quad (ST)^{-1} = T^{-1}S^{-1}.$$

The proof of these may be left to the reader.

We say S and T commute is $ST = TS$. Suppose $SR = RS, TR = RT$. Then $(S + T)R \subset R(S + T)$. The equality may not hold, as the example $S = -T = 1_X$, domain of $R \neq X$, will show.

T^n is defined as $T^{n-1}T$, as usual. If T^n appears in a formula where $n = 0$ is allowed, then T^0 stands for 1_X .

These things can all be extended to the case of moduls over a ring Φ . However, we now turn to a lemma whose proof requires that Φ be a

field.

For the remainder of § 2, T will denote a linear relation in $X \oplus X$, and for $\lambda \in \Phi$, we write just ' λ ' for ' λ_x '.

It is clear that $\alpha_0 + \alpha_1 T + \dots + \alpha_n T^n$ has for its domain, just the domain of T^n . *This is true even if $\alpha_n = 0$!* If a polynomial p has coefficients $\alpha_0, \alpha_1, \dots, \alpha_n$, then by $p(T)$ we mean $\alpha_0 + \alpha_1 T + \dots + \alpha_n T^n$ provided $\alpha_n \neq 0$. Otherwise we omit α_n and consider whether $\alpha_{n-1} \neq 0$, etc. If $\alpha_n \neq 0$ and $\alpha_i = 0$ for some $i < n$, then it does not matter whether α_i is omitted or not (but we have already agreed to retain it) because, for example $T^3 + 0T = T^3$.

The next lemma settles a little difficulty that arises in the 'multiple-valued' situation. It enables us to include the multiple valued case in the succeeding theorem, whose substance is that the usual laws of algebra apply to the multiplication of linear polynomials in T . The importance of this theorem is based on the natural fear that even in the single valued case (see 2.15, 2.16), factoring might produce a proper extension of the "multiplied-out" polynomial.

2.2 LEMMA. *Let $(x, y) \in \alpha_0 + \alpha_1 T + \dots + \alpha_n T^n$, where $\alpha_n \neq 0$. Then there exist y_0, y_1, \dots, y_n such that*

$$2.21 \quad y_0 = x, \sum_{i=0}^n \alpha_i y_i = y$$

and

$$2.22 \quad (y_{i-1}, y_i) \in T \quad (i = 1, \dots, n).$$

Proof. Assume that for some j , we have y_0, y_1, \dots, y_n such that 2.21 holds, and (instead of 2.22)

$$(j) \quad (y_{i-1}, y_i) \in T \quad (1 \leq i \leq j)$$

and

$$(x, y_i) \in T^i \quad (1 \leq i \leq n).$$

Let k be the next integer greater than j such that $\alpha_k \neq 0$. We shall establish (k). This will suffice to prove the lemma.

Because $\alpha_k \neq 0$ we can find $\lambda_1, \dots, \lambda_j$ such that, for $1 \leq h \leq j$,

$$\sum_{m=k-j+h}^k \alpha_m \lambda_{j-k+m+1-h} = \alpha_k.$$

We can find z_1, z_2, \dots, z_k where $z_k = y_k$ and $(x, z_1), (z_1, z_2), \dots, (z_{k-1}, z_k) \in T$. This implies that $(0, y_1 - z_1) \in T$, and $(y_{i-1} - z_{i-1}, y_i - z_i) \in T$ for $i \leq j$.

Now we define w_0, w_1, \dots, w_n as follows. $w_0 = x, w_1 = z_1$, for $1 \leq m \leq k$,

$$2.23 \quad w_m = z_m + \sum_{i=1}^{j-k+m} \lambda_i (y_{j-k+m+1-i} - z_{j-k+m+1-i})$$

while $w_{k+1} = y_{k+1}, \dots, w_n = y_n$. It is clear that $(w_{i-1}, w_i) \in T$ for $i \leq k$, and $(x, w_i) \in T^i$ for all i . There remains only the question, does $\sum \alpha_i w_i = y$, or, equivalently, does

$$2.24 \quad \sum_{m=1}^k \alpha_m (w_m - y_m) = 0?$$

The sum in 2.24 has the value

$$\sum_{m=1}^{k-1} \alpha_m (z_m - y_m) + \sum_{m=1}^k \sum_{i=1}^{j-k+m} \alpha_m \lambda_i (y_{j-k+m+1-i} - z_{j-k+m+1+i}).$$

It is not hard to verify that for $0 \leq h < k$ the coefficient of $y_h - z_h$ in this sum is

$$2.25 \quad -\alpha_h + \sum_{m=k-j+h}^k \alpha_m \lambda_{j-k+m+1-h},$$

where the \sum -term is understood to be absent when $k - j + h > k$. These λ were chosen in order to make this vanish for $0 \leq h \leq j$. For $j < h < k$, $\alpha_h = 0$; since $k < k - j + h$, the \sum term is absent. Thus the sum in 2.24 is 0, and this concludes the proof of the Lemma (2.2).

N.B. This lemma does not imply that T could be cut down to a linear operator U whose domain contains c, Ux, \dots , and $U^{n-1}x$, where

$$\sum_{m=0}^n \alpha_m U^m(x) = y,$$

for x could be 0 and y be not 0.

2.3 THEOREM. *Let p and q be two polynomials with coefficients in Φ . Then*

$$2.31 \quad (qp)(T) = q(T)p(T).$$

Proof. Suppose the degrees of p and q are m and n respectively. Let $p(\xi) = \alpha_0 + \alpha_1 \xi + \dots + \alpha_m \xi^m$. *Mutatis mutandis*, let the coefficients of q and qp be β_j and γ_k .

Now suppose $(x, y) \in (pq)(T)$. By 2.2 there exist x_1, \dots, x_{m+n} such that $(x_{k-1}, x_k) \in T$ for $k = 1, \dots, m+n$ where $x_0 = x$, and $\sum \gamma_k x_k = y$. Let $y_j = \sum_{i=1}^m d_i x_{i+j}$ for $j = 0, \dots, n$. Then $(x, y_0) \in p(T)$ and $(y_{j-1}, y_j) \in T$. Let $z = \sum_{j=0}^n \beta_j y_j$, so that $(y_0, z) \in q(T)$. Then $(x, z) \in q(T)p(T)$. But obviously $z = \sum \gamma_k x_k = y$. This shows that $(qp)(T) \subset q(T)p(T)$.

Now suppose $(x, z) \in q(T)p(T)$. Then there must exist y such that $(x, y) \in p(T)$ and $(y, z) \in q(T)$. By 2.2 we can find x_0, \dots, x_m and y_0, \dots, y_n (where $x_0 = x$, and $y_0 = y$) such that $\sum \alpha_i x_i = y$ and $\sum \beta_j y_j = z$. We now turn to the free linear space E (over Φ) generated by elements $\xi_0, \dots, \xi_m, \eta_1, \dots, \eta_n$. In E we define a linear operator S , whose domain is spanned by ξ_0, \dots, η_{n-1} , as follows:

$S(\xi_{i-1}) = \xi_i$ ($i = 1, \dots, m$), $S(\eta_0) = \eta_1$, where $\eta_0 = \sum \alpha_i \xi_i$, and $S(\eta_j) = \eta_{j+1}$

($j = 1, \dots, n - 1$). We can map \mathcal{E} linearly into X by a mapping f which sends ξ_i into X_i , and η_j into y_j . This mapping has the property that for ξ in the domain of S , $(f(\xi), f(S\xi)) \in T$. Derivable from this is that if r is a polynomial and $r(S)\xi$ is defined some ξ in \mathcal{E} then $(f(\xi), f(r(S)\xi)) \in r(T)$. We apply this to $\xi = \xi_0$ and $r = qp$. It is easy to see that $p(S)(\xi_0) = \eta_0$, whence $f(qp(S))(\xi_0) = f(\sum \beta_j \eta_j) = \sum \beta_j y_j = z$, and $(x, z) \in (qp)(T)$.

This completes the proof of 2.3.

[*Further remarks on polynomials of relations.* Inspection of the first argument in the proof of 2.3 yields the following result.

2.32 THEOREM. *Let p and q be as in 2.3. Then*

$$2.33 \quad (p + q)(T) \subset p(T) + q(T) .$$

The '=' does not always hold. While

$$2.34 \quad (\sum \alpha_i)T = \sum (\alpha_i T)$$

hold when $\sum \alpha_i \neq 0$, it does not hold when $\sum \alpha_i = 0$, some $\alpha_i \neq 0$, and T is not single-valued.

As the assertion connected with 2.34 implies, the reason that 2.33 cannot be strengthened to an inequality, is that $T - T$ is not 0 times some relation, if T is not single-valued. We close this little discourse on the peculiarities of many-valued relations by showing that the difficulty arises only with the terms of highest order.

2.35 THEOREM. *Let p, q be as above, and suppose the sum of their leading coefficients is not 0. Then $(p + q)(T) = p(T) + q(T)$.*

Proof. We combine the monomials of like degree on the right, and use 2.34 in each case. Eventually one may have to apply the following

$$2.36 \text{ LEMMA. } \textit{If } n \geq k \text{ then } T^n = T^n + \lambda(T^k - T).$$

Proof. Let (x, y) belong to the right side. Then $y = u + v$ where $(x, u) \in T^n + \lambda T^k$ and $(x, v) \in \lambda T^k$. From 2.2 we obtain u_0, \dots, u_n which are successively T -related, $u_0 = x, u_n + \lambda_{u_k} = u$. Therefore $\lambda_{u_k} + v \in T^k(0)$, whence $u_n + \lambda_{u_k} + v \in T^k(u_{n-k}) \subset T^n(x)$. Thus $(x, y) \in T^n$.

2.37 THEOREM. *Let q and p be polynomials. Then $(q \circ p)(T) = q(p(T))$.*

Proof. The polynomial $q \circ p$ is the result of substituting p into q , by definition. The leading coefficients may be taken as not zero. We can multiply out the terms $\beta_j p(T)^j$ on the right side, without affecting

that sum, by 2.3. (The associative law holds for addition.) We can arrange the sum as a polynomial, by virtue of 2.35 there being in fact at all times a unique term $\alpha_m \beta_n T^{m+n}$ of highest degree. The resulting polynomial is of course $(q \circ p)(T)$, for formal reasons.]

We now make some definitions which coincide with the usual ones for closed operators in F -spaces. We call a linear relation T *resolvable* if T^{-1} is single-valued with domain X (that is, by 2.11, if $T^{-1}T \subset 1_X \subset TT^{-1}$. If $T^{-1}T = 1 = TT^{-1}$ we call T *regular*.)

2.4 PROPOSITION. *The product of (finitely many pairwise) commuting linear relations is resolvable only if, and if, each factor is resolvable.*

Proof. It is inevitable and sufficient to consider the case of two factors. If these are resolvable, so is their product. The criterion $T^{-1}T \subset 1 \subset TT^{-1}$ can be used here.

If on the other hand, a linear relation S is not resolvable, then either $(x, 0) \in S$ for some $x \neq 0$, or the range $\neq X$. Accordingly, TS or ST shares the defect. (This suffices for 2.4).

The *resolvent set* of a linear relation T is the class of λ in Φ for which $T - \lambda$ (by which we mean $T - \lambda 1_X$) is resolvable; and its complement is the spectrum $\sigma(T)$ of T .

2.5 (Spectral polynomial theorem). *Let Φ be algebraically closed, and let p be a polynomial over Φ . Then $\sigma(p(T)) = p(\sigma(T))$, where by the latter is meant the class of $p(\lambda)$, $\lambda \in \sigma(T)$.*

Proof. For $\mu \in \Phi$ we can write

$$p(T) - \mu = \alpha(T - \lambda_1) \cdots (T - \lambda_n), \mu = p(\lambda_1) = \cdots = p(\lambda_n)$$

where $T - \lambda_1, \dots, T - \lambda_n$ commute.

If $\mu \in \sigma(p(T))$ then $p(T) - \mu$ is not resolvable, whence (by 2.4) some $\lambda_i \in \sigma(T)$, or $\mu \in p(T)$. If $\mu \in p(T)$ then $\mu = p(\lambda)$, $\lambda \in \sigma(T)$, and so $\lambda = \lambda_i$ for some i . Then $p(T) - \mu$ has a non-resolvable factor, and so is not resolvable. Therefore $\mu \in \sigma(p(T))$. This proves 2.5.

We have defined the sum (and difference) of two linear subspaces U and V (say) of $X \oplus Y$, but occasionally one is concerned with the linear subspace of $X \oplus Y$ which they span. We will have to use some other symbol for this, and we choose

$$2.6 \quad U \neq V.$$

Our purpose is to prove the following

2.61 THEOREM. *The range of $1 - V^{-1}U$ is the null-space of $U \neq V$, and the null-space of $1 - V^{-1}U$ is the domain of $U \cap V$.*

Proof. Let $(x, z) \in 1 - V^{-1}U$. Then $(x, z - x)\varepsilon - V^{-1}U$ whence $(x, y) \in U$ and $(y, x - z)\varepsilon - V^{-1}$, for some y . Therefore $(z - x, -y) \in V$ and so $(z, 0) \in U \neq V$. If moreover, $z = 0$ (so that x is in the null-space) then $(-x, -y)$ and thus (x, y) belongs to V and thus $x \in \text{dom } U \cap V$. The reverse inclusions can be established by reversing the steps of this argument.

3. Adjoints. For the formalism of adjoints, it is good to suppose that the field Φ has an involutory automorphism

$$\lambda \rightarrow \bar{\lambda},$$

and we shall do so. Whether Φ admits a non-trivial involution or not, one *can* base the discussion on the identity. Thus the discussion includes the *transpose*.

Let X, A be two linear spaces over Φ . We shall say X, A are a $(\Phi, -)$ *dual pair* (or, more briefly, a dual pair) if there is a non-degenerate bi-additive, Φ -valued form \langle, \rangle defined on $X \oplus A$, linear in first argument, and semi-linear in the second:

$$\langle x, \lambda a \rangle = \bar{\lambda} \langle x, a \rangle.$$

Let Y, B be another $(\Phi, -)$ dual pair. Let T be a linear relation between elements of X and elements of Y , i.e., let T be a linear subspace of $X \oplus Y$. $X \oplus Y, A \oplus B$ form a $(\Phi, -)$ dual pair, in a natural way:

$$\langle (x, y), (a, b) \rangle = \langle x, a \rangle + \langle y, b \rangle.$$

The *adjoint* T^* is defined as follows:

$$3.11 \quad T^* = \{(b, a) : \langle x, a \rangle = \langle y, b \rangle \text{ for all } (x, y) \in T\}.$$

T^* is (evidently) a linear subspace of $B \oplus A$.

For a linear subspace U of $B \oplus A$ we define

$$3.12 \quad U^* = \{(x, y) : \langle x, a \rangle = \langle y, b \rangle \text{ for all } (b, a) \in U\}.$$

It is usually supposed that 3.12 need hardly be written down, once 3.11 is presented. We mention three obvious properties of this process (or, rather, these processes. See § 4)

$$3.2 \quad T \subset T^{**}, S \subset T \Rightarrow T^* \subset S^*$$

$$3.21 \quad (\lambda T)^* = \bar{\lambda} T^*$$

$$3.22 \quad (T^{-1})^* = (T^*)^{-1}.$$

For a subset M of X , let

$$3.23 \quad M^\perp = \{a : \langle x, a \rangle = 0 \text{ for all } x \in M\}$$

while if $M \subset A$ then

$$3.24 \quad M^\perp = \{x : \langle x, a \rangle = 0 \text{ for all } a \in M\}.$$

In this sense (cf. [4])

$$3.25 \quad T^* = (-T^{-1})^\perp.$$

In 3.3 we have in mind the natural pairing of $Y \oplus X$ and $B \oplus A$, of course.

Again, considering X, A as a typical pair, and M a linear subspace of X , we define $M^{\perp\perp}$ as the *closure* of M . This requires no topology in X, A , or Φ , and resembles the Stone topology [1, p. 466] in this respect—and in fact admits a natural, joint generalization.

M is *closed* if $M = M^{\perp\perp}$, and *dense* if $M^{\perp\perp} = X$.

PROPOSITION.

3.31 *The null-space of $T^* = (\text{range of } T)^\perp$*

3.32 *T^* is single-valued only if and if the domain of T is dense*

3.33 *T^* is closed*

3.34 *T^{**} is the smallest closed linear relation containing T .*

Here 3.31 is easily established on the definitions, and 3.32 follows from it by considering the null space of T^{*-1} . 3.33 is obvious, because any M^\perp is closed, while 3.34 follows from 3.33.

Turning to the adjoint of a sum, let S and T be two linear subspaces of $X \oplus Y$. It is quite elementary that

$$3.4 \quad S^* + T^* \subset (S + T)^*.$$

The following gives an unsymmetric condition which insures the equality.

3.41 THEOREM. *If the domain of $S^* = B$, and the domain of S includes that of T , then*

$$(S + T) = S^* + T^*.$$

Proof. Let $(b, a) \in (S + T)^*$. Then there is an element a_1 such that $(b, a_1) \in S^*$. Let us show that $(b, a - a_1) \in T^*$. To this end, suppose $(x, t) \in T$. Then $(x, s) \in S$ for $s = S(x)$, and $(x, s + t) \in S + T$. Now

$$\begin{aligned} \langle x, a - a_1 \rangle - \langle t, b \rangle &= \langle x, a \rangle - \langle x, a_1 \rangle - \langle t, b \rangle \\ &= \langle x, a \rangle - \langle s, b \rangle - \langle t, b \rangle = \langle x, a \rangle - \langle s + t, b \rangle = 0. \end{aligned}$$

Thus $(b, a - a_1) \in T^*$, which, with $(b, a_1) \in S^*$ gives $(b, a) \in S^* + T^*$ as was to be shown.

Although our T is not a *function*, we may adapt a symbolism usually used in a functional context, and write

$$X \text{---}_T Y, \text{ or } Y_T \text{---} X,$$

to convey that T is a linear subspace of $X \oplus Y$.

If we introduce S

$$Y \text{---}_S Z$$

where Z, C is another $(\phi, -)$ dual pair, then

$$S \text{---}_{ST} Z, \text{ and } C \text{---}_{(ST)^*} A.$$

Since $A_{T^*} \text{---} B_{S^*} \text{---} C$ we also have $C \text{---}_{T^*S^*} A$ and there arises the question of the relation of $(ST)^*$ and T^*S^* . In fact, it is quite elementary that $(ST)^* \supset T^*S^*$, but we wish to examine also the reverse inclusion, which is initiated by the following lemma. Here f_a (for example) is the linear functional on X defined by $f_a(x) = \langle x, a \rangle$, etc.

3.5 LEMMA. *Let $c \in C$, $a \in A$. Consider these linear functionals defined in Y*

$$3.51 \quad f_c \circ S, f_a \circ T^{-1}.$$

Then $(c, a) \in (S \circ T)^$ if and only if these functionals are single-valued and agree on the intersection of their domains; and $(c, a) \in T^* \circ S^*$ if and only if they have a common extension to some f_b , $b \in B$.*

Proof. The second assertion is the easier to show. If $(c, a) \in T^* \circ S^*$ then $(c, b) \in S^*$, $(b, a) \in T^*$ for some $b \in B$. Let $y \in D(S) \cap D(T^{-1})$ (' D ' means 'domain'). I say these functionals (3.51) agree with f_b for such y . Indeed, if $(y, z) \in S$ and $(y, x) \in T^{-1}$ then $f_c(z) = \langle z, c \rangle = \langle y, b \rangle = \langle x, a \rangle = f_a(x)$.

Conversely, if b having this property exists, then $(c, b) \in S^*$ and $(b, a) \in T^*$ or $(c, a) \in T^* \circ S^*$.

Now let $(c, a) \in (S \circ T)^*$, and let $y \in D(S) \cap D(T^{-1})$. Let $(y, z) \in S$, $(x, y) \in T$. Then $(x, z) \in S \circ T$ and $\langle x, a \rangle = \langle z, c \rangle$, and these are generic elements of $(f_a \circ T^{-1})(y)$, $(f_c \circ S^{-1})(y)$ respectively. Thus 3.51 are single-valued, and agree on $D(S) \cap D(T^{-1})$. The converse is obvious.

This establishes 3.5.

From this, a useful conclusion may be drawn.

3.52 PROPOSITION. *Suppose either that the domain of S^* is C , or that the range of T^* is A . Then*

$$(S \circ T)^* = T^* \circ S^*.$$

Proof. Let $(c, a) \in (S \circ T)^*$. Consider the case in which the domain of S^* is c . Then $(c, b) \in S^*$ for some b . Let $(y, z) \in S$. Then $(f_c \circ S)(y) = \langle z, c \rangle = \langle y, b \rangle$, i.e., f_b is an extension of $f_c \circ S$. Hence it is also an ex-

tension of $f_a \circ T^{-1}$ (the latter confined, if need be, to the domain of $S + T^{-1}$.) We apply 3.5, and obtain $(c, a) \in T^* \circ S^*$.

If the range of T^* is A , the proof is similar. But it may be reduced to the case treated, by using 3.22, and the general fact $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$.

We may now drop the 'o' again, which was reintroduced to make 3.5 easier to present.

3.6 PROPOSITION. *Let U be a linear subspace of $X \oplus Y$, and V , of $Y \oplus Z$. If either the domain of U^{**} is X , or the range of V^{**} is Z , then $(VU)^{**} \subset V^{**}U^{**}$.*

Proof. In any case $U^*V^* \subset (VU)^*$ and $(VU)^{**} \subset (U^*V^*)^*$. We think of U^* as S and V^* as T and apply 3.52, *mutatis mutandis*.

We recall (3.34) that T is closed precisely when $T \supset T^{**}$. The merit of our "many-valued" approach is that this criterion is available whether T^* is single-valued or not.

3.7 THEOREM. *Let S and T be linear relations as above. Suppose they are closed, and that either the domain of T is X or the range of S is Z . Then ST is closed.*

Proof. By 3.6, we obtain $(ST)^{**} \subset S^{**}T^{**} = ST$ provided the domain of T is X or the range of S is Z , which suffices.

The relevance of the existence of resolvent values, to the question of closedness of polynomials in a (closed) operator, was noticed by Taylor [3] (see also [2, p. 56]).

3.8 THEOREM. *Let T be a closed linear subspace of $X \oplus X$, for which there is at least one $\lambda \in \Phi$ such that $T - \lambda$ has range X . Then $p(T)$, for any polynomial p over Φ , is closed.*

Proof. By the algebraic Theorem 2.3 we have

$$[p - p(\lambda)](T) = (T - \lambda)q(T)$$

where q is a polynomial of degree less than that of p . By 3.7 and an obvious inductive approach, we see that $[p - p(\lambda)](T)$ is closed. Now $[p - p(\lambda)](T) = p(T) - p(\lambda)$ by 2.35, so the latter is closed. Note that $p(T) = U + V$ where $U = p(T) - p(\lambda)$, $V = p(\lambda)$.

Now $(U + V)^* \supset U^* + V^*$ and so $(U + V)^{**} \subset (U^* + V^*)^*$. Let V^* be the S of 3.41. Then its domain is the whole space, while $S^* = V$ and its domain is also the whole space. Thus $(U + V)^{**} \subset U^{**} + V^{**} = U + V$, so that $p(T)$ is closed. Of course, we also know that

$$p(T) = p(\lambda) + (T - \lambda)p(T)$$

which does not emerge from the proof given in [2].

4. Self-duality. When X, A is a $(\phi, -)$ dual pair and $A = X$, we speak of a self-dual pair. This situation presents two definitions of M^\perp , that given by 3.23, and another, which we might call ${}^\perp M$, given by 3.24. These coincide if and only if

$$4.1 \quad \langle x, y \rangle = 0 \text{ if and only if } \langle y, x \rangle = 0$$

which, in turn, is equivalent to

$$4.11 \quad \text{There exists a } p \in \phi \text{ such that } p\bar{p} = 1 \text{ and}$$

$$\langle y, x \rangle = p\overline{\langle x, y \rangle} \text{ for all } x, y \in X.$$

(We leave the proof of this equivalence to the reader. One should note that 4.1 for X is transmitted, *via* 4.11, to $X \oplus X$, so that when $T \subset X \oplus X$, $T^\perp = {}^\perp T$ when 4.1 holds.)

The situation $M^\perp \neq {}^\perp M$ would not be awkward if one had ${}^\perp(M^\perp) = ({}^\perp M)^\perp$, but for all we know this condition might be equivalent to 4.1. In any case, it does not hold in general (see 5.41).

We therefore *assume* 4.1 in this section.

Let T be a linear subspace of $X \oplus X$. Then $W = T \mp T^\perp$ (see 2.6) is of interest, because for closed relations in Hilbert space, $W = X \oplus X$.

In general, the following relations hold:

$$4.2 \quad \begin{array}{ccc} W = X \oplus X & & \\ \Downarrow & \Downarrow & \\ \text{null-space of } W = X & & W \text{ is dense} \\ \Downarrow & \Downarrow & \Downarrow \\ \text{null-space of } W \text{ is dense} & & T^\perp \cap T(0, 0). \end{array}$$

We proceed to generalize a proposition of von Neumann's [5].

4.3 THEOREM. *Let T be closed. Let $W = T \mp T^\perp$ and suppose that the null-space of W is all of X . Then the null-space of $1 + T^*T$ is (0) , the range is X , and $(T^*T)^* = T^*T$ (i.e., T^*T is self-adjoint.)*

Proof. Let U (in 2.61) $= T$, and $V = T^\perp$. Then $-V^{-1} = T^*$. Therefore the range of $1 + T^*T$ is the null-space of W , that is, X . Moreover, the null-space of $(1 + T^*T)^*$ is (by 3.31) $(\text{range of } 1 + T^*T)^\perp$, which is (0) .

We know that $T^*S^* \subset (ST)^*$ in general, so if we set $S = T^*$, $S^* = T^{**} = T$, we get $T^*T \subset (T^*T)^*$, or $1 + T^*T \subset (1 + T^*T)^*$. Here we have used 3.41.

Considering 2.02, and what we know about the null-spaces and ranges, we conclude that $1 + T^*T = (1 + T^*T)^*$, $T^*T = (T^*T)^*$.

We have already defined T to be self-adjoint if $T = T^*$. We call

T unitary if $T^* = T^{-1}$. We say nothing about single-valuedness. In the Hilbert-space-situation, there are no unitary linear relations except those single-valued relations which are usually called unitary, as the following shows.

4.4 PROPOSITION. $T^{-1} \subset T^*$ if and only if $\langle x, x \rangle = \langle y, y \rangle$ for all $(x, y) \in T$. If $T^* = T^{-1}$ and $T \mp T^\perp = X \oplus X$ then the domain and range of T both equal X .

Proof. The statement about $\langle x, x \rangle$ and $\langle y, y \rangle$ is obviously true.

Now assume $T \mp T^\perp = X \oplus X$ and $T^* = T^{-1}$. Let $y \in X$. Then $(0, y) = (x, t) + (-x, y - t)$ where $(x, t) \in T$ and $(-x, y - t) \in T^\perp = (-T^*)^{-1} = -T$, or $(x, y - t) \in T$. Then $(2x, y) \in T$, or the given y is in the range of T . Now the things assumed about T are inherited by T^{-1} so that the range of T^{-1} is also X .

Returning briefly to the Hilbert-space-situation, if $T^* = T^{-1}$ then T is closed and so $T \mp T^\perp$ does equal $X \oplus X$, whence T is unitary in the usual sense.

To generalize the formal aspects of the Cayley transform [4] we assume now that Φ contains an element i such that $i^2 = -1$ and $\bar{i} = -i$.

Cayley's map sends $X \oplus X$ onto $X \oplus X$ thus

$$C(x, y) = (x - iy, x + iy).$$

Its third iterate is scalar, and it preserves orthogonality, etc. If $T \subset X \oplus X$ then

$$C(T) = \{(s - it, s + it) : (s, t) \in T\}$$

is the Cayley transform of T .

We list several elementary properties.

- | | |
|------|--------------------------------------|
| 4.51 | $S \subset T \iff C(S) \subset C(T)$ |
| 4.52 | $C(-T) = C(T)^{-1}$ |
| 4.53 | $C(T^{-1}) = -C(T)^{-1}$ |
| 4.54 | $C(T^\perp) = C(T)^\perp$ |
| 4.55 | $C(T^*) = C(T)^{*^{-1}}$. |

4.6 THEOREM. $T \subset T^*$ if and only if $C(T)^{-1} \subset C(T)^*$, $T = T^*$ if and only if $C(T)$ is unitary.

If $C^2(T)$ were unitary, and we were in Hilbert space, then T would have a spectral resolution, but $C^2(T)$ is unitary if and only if $T^* = -T$.

The spectral mapping theorem holds for this Cayley transform:

$$4.7 \quad \sigma(C(T)) = \{(1 + i\tau)(1 - i\tau)^{-1} : \tau \in (T)\}$$

with the following understanding: $\infty \in \sigma(S)$ means $0 \in \sigma(S^{-1})$, $2/0 = \infty$, $(1 + i\infty)(1 - i\infty)^{-1} = -1$. Moreover, *eigenvalues* correspond to *eigenvalues*.

The set consisting of the spectrum of T , plus the symbol ∞ if $0 \in \sigma(T^{-1})$ we call, following Taylor, the *augmented spectrum*. The *augmented spectrum* thus contains ∞ whenever T is not single-valued.

5. Hilbert space. In Hilbert space X , (ϕ = complex numbers), self-adjoint linear relations T may be analyzed in just the same way as the single-valued ones are, by von Neumann, in [4]. The general theory is perfect in a way that the usual theory is not: every unitary operator is the Cayley transform of a unique self-adjoint linear relation, and conversely (4.6).

However, rather than repeat the application of the Cayley transform method, we prefer to analyze the general self-adjoint linear relation in term of self-adjoint operators.

If T is a closed linear subspace of $X \oplus X$, X being a Hilbert space (as shall be assumed in all of this section) then

$$5.1 \quad T = T_{\infty} \pm T_1$$

where T_{∞}, T_1 are *orthogonal* closed linear subspaces (so we write ' \pm ' instead of ' \mp ') and $T_{\infty} = T \cap (\{0\} \oplus X)$. Thus T_{∞} has only 0 in its domain, while its range is $T(0)$ (see § 2). $T(0)$ is closed, since $T_{\infty} = \{0\} \oplus T(0)$. The domain of T_1 is the domain of T , and T_1 is single-valued.

5.2 LEMMA. $T(0) = (\text{dom } T^*)^{\perp}$, $\text{dom } T_1$ is dense in $T^*(0)^{\perp}$, and the range of T_1 lies in $T(0)^{\perp}$.

Proof. 3.31 tells us that $T^{*-1}(0) = (\text{dom } T^{-1})^{\perp}$. We can replace T here by T^{-1} , and then replace T^* by T since T is closed. Thus $T(0) = (\text{dom } T^*)^{\perp}$. From $T^*(0) = (\text{dom } T)^{\perp}$ we obtain $(\text{dom } T)^{-1} = T^*(0)^{\perp}$, and thus the second assertion. Finally, if $(x, y) \in T_1$, and $(0, z) \in T_{\infty}$ then $(x, y) \perp (0, z)$, because T_1 is the orthogonal complement of T_{∞} relative to T . Hence $\langle y, z \rangle = 0$.

5.3 THEOREM. Let T be a self-adjoint linear subspace of $X \oplus X$. Let $T = T_{\infty} \pm T_1$ as above. Then

$$X = Y \pm Z$$

and T_{∞} consists of all pairs $(0, y)$, $y \in Y$ while T_1 is a closed linear operator whose domain is dense in Z , and whose range is in Z . T_1 , restricted to Z , coincides with a self-adjoint linear operator in Z .

Proof. Let $Y = T(0)$, $Z = T(0)^\perp$. Then the domain of T_1 is dense in $T^*(0)^\perp = Y^\perp = Z$ and the range lines in $T(0)^\perp = Z$, all by 5.2.

Suppose that $(z, w) \in S^*$ where S is T_1 restricted to Z . Then $\langle x, w \rangle = \langle v, z \rangle$ for all $(x, v) \in T_1$. Each $(x, u) \in T$ is of the form $(x, y + v)$ where $y \in T(0)$ and $(x, v) \in T_1$. Now $\langle y, z \rangle = 0$, so $\langle x, w \rangle = \langle y + v, z \rangle$ for all $(x, y + v) \in T$. It follows that $(z, w) \in T^* = T$. But since $z, w \in Z$ we have $(z, w) \in T_1$. This proves 5.3.

We return here to the question raised in second paragraph of § 4, because a counterexample in a Hilbert space context is more desirable than any other. Let $X = L_2[0, 2]$, in which the inner product will be denoted by \langle, \rangle , and orthogonality, by \perp . Select a bounded operator T , domain X , range dense, with single-valued inverse, and define a self-dual pairing by means of the formula

$$5.4 \quad [f, g] = \langle Tf, g \rangle = \langle f, T^*g \rangle.$$

The associated orthogonality will be denoted by ' \circ ' to prevent confusion with ' \perp ' already present.

5.41 PROPOSITION. *It is possible to select T and M (a linear subspace of X) such that*

$$5.42 \quad {}^\circ(M^\circ) = M \text{ but } ({}^\circ M)^\circ \neq M.$$

Before deciding on a specific T we shall establish

5.43 LEMMA. *${}^\circ(M^\circ)$ is the closure of M in the norm $\|x\|_T = \|Tx\|$ [4, 298], and $({}^\circ M)^\circ$ is the closure of M in $\|\cdot\|_{T^*}$.*

Proof. $M^\circ = \{a : [M, a] = 0\} = {}^\perp(TM)$, and ${}^\circ M = {}^\perp(T^*M)$. Consequently ${}^\circ(M^\circ) = {}^\perp[T^*{}^\perp(TM)]$, and so $g \in {}^\circ(M^\circ)$ precisely when $g \perp T^*{}^\perp(TM)$ or $Tg \perp {}^\perp(TM)$, i.e.,

$$5.44 \quad Tg \in (TM)^{\perp\perp} = \overline{TM}.$$

But this characterizes the closure of M in $\|\cdot\|_T$, and this observation suffices to establish 5.43.

Now we select $T = J$ where

$$(Jf)(t) = \int_0^t f(\tau) d\tau.$$

This J meets our requirement for T . We have

$$(J^*f)(t) = \int_t^2 f(\tau) d\tau,$$

whence $J^* = E - J$ where E is the projection on the constant functions

in X .

Let N be the linear subspace of those functions that vanish on $[1, 2]$. Let

$$h(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 0 \leq t \leq 2. \end{cases}$$

Then $h \in N$ and $M = N \cap \{h\}^\perp \neq N$. Thus $EM = (0)$. It is easy to establish, in the order given, the following: $JM \subset N$, $J^*N \subset N$, $\overline{JM} = N$, $\overline{J^*M} = N$.

Then one observes that $Jf \in N$ implies $f \in M$ while $J^*f \in N$ implies $f \in N$, (and each converse holds, because $JM \subset N$, $J^*N \subset N$.) Using 5.44 as a criterion for $Jg \in {}^\circ(M^\circ)$ we obtain ${}^\circ(M^\circ) = M$, $({}^\circ M)^\circ = N$.

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