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1. Introduction. The Banach algebras $\mathfrak{A}$ considered here are over the field of complex numbers, and have isometric involutions $*$. The involution is said to be *hermitian* if for any $x = x^* \in \mathfrak{A}$, the spectrum $Sp_{\mathfrak{A}}(x)$ of x contains only real numbers. The algebra $\mathfrak{A}$ is said to be *symmetric* if for any $y \in \mathfrak{A}$, $Sp_{\mathfrak{A}}(y^*y)$ contains only nonnegative real numbers.

A familiar example of a Banach algebra with an involution is the group algebra over the complex numbers of a locally compact group G. This is obtained by taking the Banach space $L^1(G)$ of all complex valued absolutely integrable functions with respect to the left invariant Haar measure dx on G. Multiplication is defined as convolution, and the involution by the formula $x^*(g) = \overline{x(g^{-1})}\rho(g)$, where $x \in L^1(G)$ and $\rho(\cdot)$ is the modular function relating the given measure to the right invariant measure by $dx^{-1} = \rho(x)dx$. This involution will be called the *natural involution* of the group algebra, and is the only involution on the group algebra we will consider.

It is known that when the group G is either compact or commutative, then its group algebra with respect to the natural involution is symmetric. On the other hand, in 1948 Neumark [6] showed that the natural involution in the group algebra of the homogeneous Lorentz group is not hermitian. (This implies that the algebra is not symmetric. See Theorem $A(a)$.) Later Gelfand and Neumark [3] extended this example to include all complex unimodular groups. Their proofs are quite difficult, entailing a knowledge of the irreducible unitary representations of the groups and considerable computation. Except for finite and commutative groups, the corresponding problems have not been studied for discrete groups. These problems will be our concern.

The main results will be summarized now. In § 2 several facts (some of which are well known) are collected to be used later. § 3 is concerned with the construction of group algebras that are symmetric, or at least have an hermitian involution. It is shown (Corollary 3.4) that the group algebra of the direct product of a commutative group and a group whose group algebra is symmetric, is a symmetric algebra. Theorem 3.7 shows that the natural involution is hermitian in the group algebra of a semidirect product of a commutative group by a finite group.

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In § 4 examples are given of discrete groups for which the natural involution in the group algebra is not hermitian. The examples include free groups on two or more generators, and free groups on three or more generators of order two (Theorem 4.7). It is worth noting that these examples settle the following matrix problem negatively: suppose T is a bounded operator on l^1 (countable absolutely convergent sequences of complex numbers), and with respect to the usual basis, suppose that the matrix (t_{ij}) of T satisfies $t_{ij} = \overline{t_{ji}}$. Then, is the spectrum of the operator T a subset of the real axis? In this connection see Remark 4.8.

Finally in § 5 we show that various connections exist between the above problems and the question of the existence of an invariant mean on the group. The principal results are Theorem 5.6 and Theorem 5.8.

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2. Preliminary theorems.

THEOREM A. *Let $\mathfrak{A}$ be a Banach algebra over the complex numbers with an isometric involution * and identity e. Then:*

(a) *if $\mathfrak{A}$ is symmetric, then the involution is hermitian, and the converse holds whenever $\mathfrak{A}$ is commutative;*

(b) *the involution is hermitian whenever $ie + x$ is regular for any $x = x^*$.*

A proof of this theorem can be found in Rickart [7]. It is not known in general if an algebra with an hermitian involution is symmetric, and it is worth noting that this is exactly the problem in proving that a B^* algebra is a C^* algebra. The essential step in proving this is to show that the B^* algebra (whose involution is hermitian) is symmetric.

Let $\mathfrak{A}$ be a Banach algebra with an identity e of norm one, and let $\mathscr{B}(\mathfrak{A})$ denote the set of all bounded linear operators on $\mathfrak{A}$. For $x \in \mathfrak{A}$, *the left multiplication operator* L_x is defined by the formula $L_x y = xy$.

THEOREM B. (a) *The mapping $x \to L_x$ maps $\mathfrak{A}$ isometrically and isomorphically* into $\mathscr{B}(\mathfrak{A})$.

(b) *Let $\mathscr{L}(\mathfrak{A})$ denote the image of $\mathfrak{A}$ in $\mathscr{B}(\mathfrak{A})$. Then for $x \in \mathfrak{A}$, $Sp_{\mathfrak{A}}(x) = Sp_{\mathscr{L}(\mathfrak{A})}(L_x) = Sp(L_x)$, where $Sp(L_x) = \{\alpha: L_x - \alpha I$ is a singular operator on the Banach space$\}$.*

Proof. (a) and the first identity in (b) are immediate. If y is regular in $\mathfrak{A}$, then L_y is a regular operator on $\mathfrak{A}$, since it has as inverse the operator $L_{y^{-1}}$. This shows that $Sp_{\mathfrak{A}}(x) \supset Sp(L_x)$. Now if L_y is regular on $\mathscr{B}(\mathfrak{A})$, there exists an element $S \in \mathscr{B}(\mathfrak{A})$ such that $L_y S = SL_y = I$. It is then easily computed that $S = L_{Se}$ and Se is the inverse of y in $\mathfrak{A}$.

THEOREM C. *Let $\mathfrak{C}$ be a Banach algebra with an identity over the complex numbers and suppose C is a maximal left ideal* (hence closed) *in $\mathfrak{C}$. Then, with respect to the quotient norm, $\mathfrak{C}/C$ is a Banach space. For $x \in \mathfrak{C}, y + C \in \mathfrak{C}/C$ the mapping defined by $L_x^{\mathfrak{C}/C}(y + C) = xy + C$ gives a bounded algebraically irreducible representation of $\mathfrak{C}$ on $\mathfrak{C}/C$.*

The above representation $x \to L_x^{\mathfrak{C}/C}$ is called the left regular representation of $\mathfrak{C}$ on $\mathfrak{C}/C$. A proof of this theorem can also be found in Rickart [7].

3. Group algebras in which the natural involution is hermitian. It will be seen shortly that the symmetry problem for the group algebra of the direct product of two groups is a special case of a more general problem concerning tensor products of Banach algebras, so the latter will be taken up first. If $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras, then the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{B}$ can be normed with the so called *greatest cross norm* and then completed to give another Banach algebra called the *projective tensor product* $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$. The basic results concerning this can be found in Schatten [9]. We will summarize here only a few pertinent facts.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras over the complex numbers having identities of norm one. It will be convenient for us not to distinguish notationally between the norms or the identities in the two algebras. Let $\mathfrak{A} \otimes \mathfrak{B}$ denote the usual algebraic tensor product of the vector spaces $\mathfrak{A}$ and $\mathfrak{B}$. An element $u \in \mathfrak{A} \otimes \mathfrak{B}$ can be represented in many ways in the form $\sum_{i=1}^n a_i \otimes b_i$ where $a_i \in \mathfrak{A}, b_i \in \mathfrak{B}, i = 1, 2, \cdots, n$. Whenever such a representation occurs, it will be denoted by $u \sim \sum_{i=1}^n a_i \otimes b_i$. The set $\mathfrak{A} \otimes \mathfrak{B}$ becomes an algebra by defining, for $u, v \in \mathfrak{A} \otimes \mathfrak{B}$, a representation of the product uv to be $\sum_{i=1}^n \sum_{j=1}^m a_i c_j \otimes b_i d_j$, where $u \sim \sum_{i=1}^n a_i \otimes b_i$, $v \sim \sum_{j=1}^m c_j \otimes d_j$. It becomes a normed algebra by defining $\| u \| = GLB \sum_{i=1}^n \| a_i \| \cdot \| b_i \|$ where the GLB is extended over all $\sum_{i=1}^n a_i \otimes b_i \sim u$. With this norm, any $u \in \mathfrak{A} \otimes \mathfrak{B}$ that satisfies $u \sim a \otimes b$ has a norm given by $\| u \| = \| a \| \cdot \| b \|$, and the identity $e \sim e \otimes e$ of $\mathfrak{A} \otimes \mathfrak{B}$ has norm one. The completion $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ of $\mathfrak{A} \otimes \mathfrak{B}$ is hence a Banach algebra over the complex numbers with an identity of norm one. Finally we note that if $\mathfrak{A}$ and $\mathfrak{B}$ each have isometric involutions $*$, the definition of $u^* \sim \sum_{i=1}^n a_i^* \otimes b_i^*$ where $u \sim \sum_{i=1}^n a_i \otimes b_i$ gives a well-defined isometric involution on $\mathfrak{A} \otimes \mathfrak{B}$ which can hence be extended to $\mathfrak{A} \hat{\otimes} \mathfrak{B}$.

We now restrict ourselves to commutative $\mathfrak{A}$. Let $\Phi(\mathfrak{A})$ denote the space of maximal ideals of $\mathfrak{A}$, which will we be identify with the corresponding homomorphisms. For $h \in \Phi(\mathfrak{A})$, define $T_h: \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{B}$ by the formula $T_h(u) = \sum_{i=1}^n h(a_i) b_i$. Now, two formal sums represent the same element in $\mathfrak{A} \otimes \mathfrak{B}$ if and only if one can be transformed into the other by successive applications of the distributive law and the commutative law applied to scalars and $\otimes$. It is then clear that T_h is well-defined and a homomorphism. Also if $\mathfrak{A}$ and $\mathfrak{B}$ have involutions, and $\mathfrak{A}$ is symmetric, then for $u \sim \sum_{i=1}^n a_i \otimes b_i$, we have

$$T_h(u^*) = T_h\Big(\sum_{i=1}^n a_i^* \otimes b_i^*\Big) = \sum_{i=1}^n h(a_i^*)b_i^* = \sum_{i=1}^n \overline{h(a_i)}b_i^* = \Big(\sum_{i=1}^n h(a_i)b_i\Big)^*$$
$$= (T_h u)^* ,$$

so that T_h is a *-homomorphism. Finally since

$$\| T_h u \| = \Big\| \sum_{i=1}^n h(a_i)b_i \Big\| \leqq \sum_{i=1}^n \| a_i \| \cdot \| b_i \|$$

holds for any $\sum_{i=1}^n a_i \otimes b_i \sim u$, we have $\| T_h u \| \leqq \| u \|$ for all u, so that T_h can be extended to $\mathfrak{A} \hat{\otimes} \mathfrak{B}$. The extension will also be denoted by T_h.

Except for the notation, the following theorem is essentially the same as that of Bochner and Phillips [1: Theorem 3], which generalizes the Wiener-Gelfand theorem on the existence of an inverse.

THEOREM 3.1. *An element $u \in \mathfrak{A} \hat{\otimes} \mathfrak{B}$ has a left* (right) *inverse in $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ if and only if $T_h u$ has a left* (right) *inverse in $\mathfrak{B}$ for every $h \in \Phi(\mathfrak{A})$.*

Proof. Only the case of left inverses will be shown. If $u \in \mathfrak{A} \hat{\otimes} \mathfrak{B}$ has a left inverse v, then for any $h \in \Phi(\mathfrak{A})$, $T_h v$ is a left inverse in $\mathfrak{B}$ for $T_h u$, since T_h is a homomorphism taking the identity of $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ to the identity of $\mathfrak{B}$.

Conversely assume that $u_0 \in \mathfrak{A} \hat{\otimes} \mathfrak{B} = \mathfrak{C}$, that $T_h u_0$ has a left inverse in $\mathfrak{B}$ for every $h \in \Phi(\mathfrak{A})$, and that u_0 does not have a left inverse in $\mathfrak{C}$. Then $\mathfrak{C}u_0$ is a proper left ideal containing u_0 and can be extended to a maximal left ideal C. Now consider the left regular representation $u \rightarrow L_u^{\mathfrak{C}/C}$ of $\mathfrak{C}$ on $\mathfrak{C}/C$ (see Theorem C). Since this representation is algebraically irreducible, it follows from Theorem C that the set of all bounded operators on $\mathfrak{C}/C$ commuting with $\{L_u^{\mathfrak{C}/C} : u \in \mathfrak{C}\}$ consits of just scalar multiples of the identity operator. Clearly $L_{a\otimes e}^{\mathfrak{C}/C}$ commutes with all $L_u^{\mathfrak{C}/C}$ so that $L_{a\otimes e}^{\mathfrak{C}/C} = h(a)I$, and since

$$(L_{a_1\otimes e}^{\mathfrak{C}/C})(L_{a_2\otimes e}^{\mathfrak{C}/C}) = L_{a_1 a_2\otimes e}^{\mathfrak{C}/C} ,$$

it follows that h is an element of $\Phi(\mathfrak{A})$. Hence

$$L_{a\otimes b}^{\mathfrak{C}/C} = (L_{a\otimes e}^{\mathfrak{C}/C})(L_{e\otimes b}^{\mathfrak{C}/C}) = h(a)(L_{e\otimes b}^{\mathfrak{C}/C}) = L_{e\otimes h(a)b}^{\mathfrak{C}/C}$$

so that for $u \sim \sum_{i=1}^n a_i \otimes b_i$, we have

$$L_u^{\mathfrak{C}/C} = L_{\sum_{i=1}^n a_i\otimes b_i}^{\mathfrak{C}/C} = \sum_{i=1}^n L_{a_i\otimes b_i}^{\mathfrak{C}/C} = \sum_{i=1}^n L_{e\otimes h(a_i)b_i}^{\mathfrak{C}/C} L_{e\otimes \sum_{i=1}^n h(a_i)b_i}^{\mathfrak{C}/C} = L_{e\otimes T_h u}^{\mathfrak{C}/C} .$$

Moreover, since the representation is continuous we can extend this to $\mathfrak{C}$ so that we have $L_v^{\mathfrak{C}/C} = L_{e\otimes T_h v}^{\mathfrak{C}/C}$ for all $v \in \mathfrak{A} \hat{\otimes} \mathfrak{B}$.

Now by assumption $T_h u_0$ has a left inverse $b_0 \in \mathfrak{B}$. Hence

$$\begin{aligned}((L_{e\otimes b_0}^{\mathfrak{C}/C})(L_{u_0}^{\mathfrak{C}/C}))(e \otimes e + C) &= ((L_{e\otimes b_0}^{\mathfrak{C}/C})(L_{e\otimes T_h u_0}^{\mathfrak{C}/C}))(e \otimes e + C) \\ &= (L_{e\otimes b_0 T_h u_0}^{\mathfrak{C}/C})(e \otimes e + C) = (L_{e\otimes e}^{\mathfrak{C}/C})(e \otimes e + C) = e \otimes e + C \,.\end{aligned}$$

On the other hand, since $u_0 \in C$, we have

$$\begin{aligned}((L_{e\otimes b_0}^{\mathfrak{C}/C})(L_{u_0}^{\mathfrak{C}/C}))(e \otimes e + C) &= (L_{e\otimes b_0}^{\mathfrak{C}/C})(L_{u_0}^{\mathfrak{C}/C}(e \otimes e + C)) \\ &= (L_{e\otimes b_0}^{\mathfrak{C}/C})(u_0 + C) = (L_{e\otimes b_0}^{\mathfrak{C}/C})(C) = C\end{aligned}$$

and we have obtained a contradiction.

Since an element is regular if and only if it has a right inverse and a left inverse we have:

COROLLARY 3.2. *An element $u \in \mathfrak{A} \hat{\otimes} \mathfrak{B}$ is regular if and only if $T_h u$ is regular in $\mathfrak{B}$ for every $h \in \Phi(\mathfrak{A})$. More precisely*:

$$Sp_{\mathfrak{A}\hat{\otimes}\mathfrak{B}}(u) = \bigcup_{h\in\Phi(\mathfrak{A})} Sp_{\mathfrak{B}}(T_h u) \,.$$

COROLLARY 3.3. *If $\mathfrak{A}$ is symmetric and $\mathfrak{B}$ is symmetric* (has an hermitian involution), *then $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ is symmetric* (has an hermitian involution).

Proof. If $\mathfrak{B}$ has an hermitian involution, then for $u = u^* \in \mathfrak{A} \hat{\otimes} \mathfrak{B}$ it follows that $(T_h u)^* = T_h u$ for all $h \in \Phi(\mathfrak{A})$. By the preceding corollary, $Sp_{\mathfrak{A}\hat{\otimes}\mathfrak{B}}(u)$ is a subset of the real axis. The "symmetry argument" is similar.

The following theorem is a special case of a theorem due to Grothendick [4: Théorème 2], and gives the connection between tensor products and group algebras.

THEOREM (Grothendieck). *If G and H are locally compact groups, then after a suitable normalization $L^1(G) \hat{\otimes} L^1(H)$ is isometrically * isomorphic to $L^1(G \times H)$*. ($G \times H$ denotes the direct product of the groups G and H).

The proof of this theorem is not easy. However our concern in the following corollary is with discrete groups, and for this special case the proof is quite direct. In any event, assuming this theorem, Corollary 3.3 gives:

COROLLARY 3.4. *If G is a discrete abelian group and H an arbitrary discrete group whose group algebra is symmetric* (has an hermitian involution), *then the group algebra of* $G \times H$ *is symmetric* (has an hermitian involution).

The case of semi-direct products will now be taken up.

DEFINITION 3.5. Let K and C be groups and suppose that for each $c \in C$ there is an automorphism φ_c of K such that the mapping $c \rightarrow \varphi_c$ is a homomorphism of C onto a group of automorphisms of K. The set of ordered pairs $\{\langle c, k\rangle : c \in C, k \in K\}$ with multiplication defined by $\langle c_1, k_1\rangle\langle c_2, k_2\rangle = \langle c_1c_2, k_1\varphi_{c_1}(k_2)\rangle$ then forms a group G called the *semi-direct product of* K *and* C *by* φ and denoted by $C \times_\varphi K$.

It is immediately verified that the set $\{\langle e, k\rangle : k \in K\}$ forms a normal subgroup of G isomorphic to K, and that $\langle c, k\rangle^{-1} = \langle c^{-1}, \varphi_{c-1}(k^{-1})\rangle$. The generality of semi-direct products is shown by the following theorem.

THEOREM 3.6. *If a group* G *contains subgroups* K *and* C, *where* K *is normal,* $K \cap C = e$, *and* $G = KC$, *then* G *is isomorphic to a semi-direct product of* C *and* K.

Proof. Since K is normal the mapping $c \rightarrow \varphi_c$, where $\varphi_c(k) = ckc^{-1}$, is a homomorphism of C onto a group of automorphisms of K. Since $G = KC$, any $g \in G$ can be written in the form $g = k_gc_g$ where $k_g \in K$, $c_g \in C$, and since $K \cap C = e$ this decomposition is unique. Then

$$gh = k_gc_gk_hc_h = k_gc_gk_hc_g^{-1}c_gc_h = k_g\varphi_{c_g}(k_g)c_gc_h$$

so that $c_{gh} = c_gc_h$, and $k_{gh} = k_g\varphi_{c_g}(k_h)$. It is now obvious that the correspondence $g \leftrightarrow \langle c_g, k_g\rangle$ is an isomorphism between G and the semi-direct product $C \times_\varphi K$.

Before stating the next theorem, it is convenient to establish some special conventions. The group algebra of a discrete group G will be denoted by $l^1(G)$, and elements of $l^1(G)$ will be written as sums rather than functions, i.e. if $x \in l^1(G)$, then $x = \sum_{g\in G} x(g)g$, where the $x(g)$ are complex numbers satisfying $\sum_{g\in G} |x(g)| < \infty$. Convolution in $l^1(G)$ is then the usual multiplication of these formal sums, and the involution is given by $x^* = \sum_{g\in G} \overline{x(g^{-1})}g$.

Let $G = C \times_\varphi K$ be a semi-direct product of C and K. We will abuse notation and consider C and K as subgroups of G. This is justified by Theorem 3.6. The elements of G can then be uniquely written in the form $g = kc$ and $gg' = kck'c' = k\varphi_c(k')cc'$ where $k, k' \in K$ and $c, c' \in C$. Finally, for $x \in l^1(G)$, we have

$$x = \sum_{k\in K}\sum_{c\in C} x(kc)kc = \sum_{c\in C}\left(\sum_{k\in K} x(kc)k\right)c = \sum_{c\in C} x'(c)c$$

where $x'(c) \in l^1(K)$. Dropping the primes, we will now write any element $x \in l^1(G)$ in the form $x = \sum_{c\in C} x(c)c$, $x(c) \in l^1(K)$.

THEOREM 3.7. *If C is a finite group and K is a discrete abelian group, then the natural involution is hermitian in the group algebra of any semi-direct product $G = C \times_\varphi K$.*

Proof. Let $G = C \times_\varphi K$ be a semi-direct product of C and K, and let $x = x^* \in l^1(G)$, $x = \sum_{c\in C} x(c)c$. Then

$$x^* = \sum_{c\in C} (x(c)c)^* = \sum_{c\in C} c^{-1}(x(c)^*) = \sum_{c\in C} c^{-1}(x(c)^*)cc^{-1}$$

$$= \sum_{c\in C} \Phi_{c^{-1}}(x(c)^*)c^{-1} = \sum_{c\in C} \Phi_c(x(c^{-1})^*)c \ .$$

(Φ_c denotes the extension of φ_c to $l^1(K)$ defined by $\Phi_c(\sum_{k\in K} x(k)k) = \sum_{k\in K} x(k)\varphi_c(k)$.) Since $x = x^*$ and the decomposition is unique we have $\Phi_c((x(c^{-1}))^*) = x(c)$ for all $c \in C$.

By Theorem A (b) the involution in $l^1(G)$ is hermitian if $ie + x$ is regular for all $x = x^*$. We will now construct a right inverse for $ie + x$. Indeed $ie + x$ will have a right inverse if and only if elements $y(c) \in l^1(K)$ can be found for each $c \in C$ such that $y = \sum_{c\in C} y(c)c$ satisfies $(ie + x)y = e$. Expressing this condition in terms of the coefficients we have:

$$e = \Big(ie + \sum_{c\in C} x(c)c\Big)\Big(\sum_{d\in C} y(d)d\Big) = i\sum_{d\in C} y(d)d + \sum_{c,d\in C} x(c)cy(d)d$$

$$= i\sum_{d\in C} y(d)d + \sum_{c,d\in C} x(c)\Phi_c(y(d))cd = i\sum_{b\in C} y(b)b + \sum_{b\in C}\Big(\sum_{c\in C} x(c)\Phi_c(y(c^{-1}b))\Big)b$$

$$= \sum_{b\in C} (iy(b) + \sum_{c\in C} x(c)\Phi_c(y(c^{-1})))b \ .$$

Hence our problem is to find $y(c)$'s satisfying the simultaneous set of equations:

$$iy(b) + \sum_{c\in C} x(c)\Phi_c(y(c^{-1}b)) = \begin{cases} e & \text{for} \quad b = e \\ 0 & \text{for} \quad b \neq e \ . \end{cases}$$

Write the elements for the finite group C as $\{e = c_0, c_1, \cdots, c_n\}$ so that we have:

$$iy(c_k) + \sum_{i=0}^{n} x(c_i)\Phi_{c_i}(y(c_i^{-1}c_k)) = e\delta_{k0} \ ,$$

or

$$iy(c_k) + \sum_{i=0}^{n} x(c_k c_r^{-1})\Phi_{c_k c_r^{-1}}(y(c_r)) = e\delta_{k0} \ . \tag{1}$$

Since $\Phi_c(0) = 0$ and $\Phi_c(e) = e$ for any $c \in C$ the application of $\Phi_{c_k^{-1}}$ to the kth equation gives:

$$i\Phi_{c_k^{-1}}(y(c_k)) + \sum_{r=0}^{n} \Phi_{c_r^{-1}}(x(c_k c_r^{-1}))\Phi_{c_r^{-1}}(y(c_r)) = e\delta_{k0} \tag{2}$$

for $k = 0, 1, \cdots, n$. The matrix of coefficients of these equations is:

$$\begin{bmatrix} ie + x(c_0) & x(c_1^{-1}) & \cdots & x(c_n^{-1}) \\ \Phi_{c_1^{-1}}(c_1c_0^{-1})) & ie + \Phi_{c_1^{-1}}(x(c_1c_1^{-1})) & \cdots & \Phi_{c_1^{-1}}(x(c_1c_n^{-1})) \\ & \cdots & \cdots & \\ \Phi_{c_n^{-1}}(x(c_nc_0^{-1})) & \Phi_{c_n^{-1}}(x(c_nc_1^{-1})) & \cdots & ie + \Phi_{c_n^{-1}}(x(c_nc_n^{-1})) \end{bmatrix}.$$

Now the elements of this matrix are elements of the commutative algebra $l^1(K)$, and hence the determinant Δ of this matrix is a well defined element of $l^1(K)$. Moreover the usual "Cramer's rule" formula will furnish a solution of the set of equations (2) if it can be shown that Δ is a nonsingular element of $l^1(K)$. Let α_{ij} denote the element in the ith row and jth column of the above matrix so that $\Delta = \det(\alpha_{ij})$. Now Δ is nonsingular if and only if $h(\Delta)$ is non-zero for any $h \in \Phi(l^1(K))$. Since h is a homomorphism $h(\Delta) = \det(h(\alpha_{ij}))$, and $h(\Delta)$ will be non-zero if it can be shown that $\overline{h(\alpha_{ij})} = h(\alpha_{ji})$ for $i \neq j$ and $h(\alpha_{jj}) = i + \beta_{jj}$ where $\beta_{jj} = \overline{\beta_{jj}}$. Indeed in this case the matrix $(h(\alpha_{ij}))$ is the matrix corresponding to an operator on a finite dimensional Hilbert space of the form $iI + H$ where H is an hermitian operator. Hence the operator $iI + H$ is nonsingular so that the determinant of any matrix representations of it must be non-zero.

It remains to verify the above equations. Since $\Phi_c(x^*) = (\Phi_c(x))^*$ for any $c \in C$ we have

$$(\Phi_{c_k^{-1}}(x(c_0)))^* = \Phi_{c_k^{-1}}(x(c_0)^*) = \Phi_{c_k^{-1}}(x(c_0)) ,$$
$$(\Phi_{c_k^{-1}}(x(c_k)))^* = \Phi_{c_k^{-1}}(x(c_k)^*) = x(c_k^{-1}) ,$$

and

$$\begin{aligned}(\Phi_{c_i^{-1}}(x(c_ic_k^{-1})))^* &= \Phi_{c_i^{-1}}(x(c_ic_k^{-1})^*) = \Phi_{c_k^{-1}}(\Phi_{c_kc_i^{-1}}(x(c_ic_k^{-1})^*)) \\ &= \Phi_{c_k^{-1}}(\Phi_{c_kc_i^{-1}}(x((c_kc_i^{-1})^{-1})^*)) = \Phi_{c_k^{-1}}(x(c_kc_i^{-1})) .\end{aligned}$$

What we have shown is that the elements α_{ii} are of the form $\alpha_{ii} = ie + \delta_{ii}$ where $\delta_{ii}^* = \delta_{ii}$ and $(\alpha_{ij})^* = \alpha_{ji}$ for $i \neq j$. Finally any $h \in \Phi(l^2(K))$ satisfies $h(a^*) = \overline{h(a)}$ so that the matrix $(h(\alpha_{ij}))$ is in the desired form, and $h(\Delta)$ is non-zero. Hence we have a solution to the equations (2), and the application of Φ_{c_k} to the kth equation of (2) gives the solution to the equations (1) and therefore the desired right inverse y. A left inverse for $ie + x$ can be constructed in a similar way.

REMARK 3.8. We do not know in general if the group algebra of a semi-direct product of a finite group and a discrete abelian group is symmetric with respect to the natural involution, in spite of the fact that the above theorem shows the hermitianess of the involution. The following theorem describes a special case where this is true.

THEOREM 3.9. *If $C = \{e, a : a^2 = e\}$ and K is abelian, then any semi-direct product $G = C \times_\varphi K$ has a symmetric group algebra.*

Proof. $x \in l^1(G)$ has the form $x = x_1 + ax_2$, $x_1, x_2 \in l^1(K)$, and hence $e + x^*x = e + x_1^*x_1 + x_2^*x_2 + x_2^*ax_1 + x_1^*ax_2$. Let $\Phi_a = \Phi$, $z_1 = e + x_1^*x_1 + x_2^*x_2$, and $z_2 = \Phi(x_2^*)x_1 + \Phi(x_1^*)x_2$. Then $e + x^*x = z_1 + az_2$. Thus $e = (z_1 + az_2)(y_1 + ay_2)$ if and only if $z_1y_1 + \Phi(z_2)y_2 = e$ and $z_2y_1 + \Phi(z_1)y_2 = 0$ so that $y_2 = -\Phi(z_1^{-1})z_2y_1$ and $(z_1 - \Phi(z_2)\Phi(z_1^{-1})z_2)y_1 = e$. Assume $z_1 - \Phi(z_2)\Phi(z_1^{-1})z_2$ and hence $\Phi(z_1)z_1 - \Phi(z_2)z_2$ is singular. Then there is a homomorphism h such that $h(\Phi(z_1))h(z_1) = h(\Phi(z_2))h(z_2)$. But

$$\begin{aligned}
h(\Phi(z_2))h(z_2) &= (h(x_2^*)h(\Phi(x_1)) + h(x_1^*)h(\Phi(x_2)))(h(\Phi(x_2^*))h(x_1) + h(\Phi(x_1^*))h(x_2)) \\
&= \overline{h(x_2)}h(\Phi(x_1))\overline{h(\Phi(x_2))}h(x_1) + \overline{h(x_1)}h(\Phi(x_2))\overline{h(\Phi(x_2))}h(x_1) \\
&\quad + \overline{h(x_2)}h(\Phi(x_1))\overline{h(\Phi(x_1))}h(x_2) + \overline{h(x_1)}h(\Phi(x_2))\overline{h(\Phi(x_1))}h(x_2) \\
&= |h(x_1)h(\Phi(x_2))|^2 + |h(x_2)h(\Phi(x_1))|^2 + 2h(x_1)h(\Phi(x_1))\overline{h(x_2)}\overline{h(\Phi(x_2))} \\
&\leqq |h(x_1)h(\Phi(x_2))|^2 + |h(x_2)h(\Phi(x_1))|^2 + |h(x_1)h(\Phi(x_2))|^2 \\
&\quad + |h(x_2)h(\Phi(x_1))|^2 = 2|h(x_1)h(\Phi(x_2))|^2 + 2|h(x_2)h(\Phi(x_1))|^2 \\
&\leqq |h(x_1)|^2 + |h(\Phi(x_2)|^2 + |h(x_2)|^2 + |h(\Phi(x_1)|^2 \\
&< (1 + |h(x_1)|^2 + |h(x_2)|^2)(1 + |h(\Phi(x_1))|^2 + |h(\Phi(x_2))|^2) \\
&= h(z_1)h(\Phi(z_1))
\end{aligned}$$

and we have obtained a contradiction.

It is known (see Rickart [7]) that the symmetry and hermitianess properties are preserved in passing from a Banach algebra to a norm closed * closed subalgebra. In the case of the group algebra of a discrete group, and the group algebra of a subgroup, an elementary proof of a more general result can be given. Specifically:

THEOREM 3.10. *Let G be a discrete group, and H a subgroup of G. Then the natural imbedding of H in G induces an isometric * isomorphic imbedding of $l^1(H)$ into $l^1(G)$. With respect to this imbedding, for $x \in l^1(H)$*

$$Sp_{l^1(H)}(x) = Sp_{l^1(G)}(x) .$$

In particular, if $l^1(G)$ is symmetric (has an hermitian involution), *then $l^1(H)$ is symmetric* (has an hermitian involution).

Proof. The only non-trivial part of the proof consists in showing that if $x \in l^1(H)$, and x is regular in $l^1(G)$, then x is already regular in $l^1(H)$.

Let $\{Hg_\alpha : 0, \alpha \in A, g_0 = e\}$ be a left coset decomposition of G with respect to H. Write $x = \sum_{k\in H} x_0(k)kg_0$, and its inverse $y \in l^1(G)$ as $y = \sum_{a\in A}(\sum_{h\in H} y_a(h)hg_a)$. Then $xy = e$ means

$$e = \sum_{a\in A}\Big(\sum_{k,h\in H} x_0(k)y_a(h)khg_a\Big) = \sum_{a\in A}\Big(\sum_{e\in H}\Big(\sum_{h\in H} x_0(lh^{-1})y_a(h)lg_a\Big)\Big) .$$

For any fixed $a \neq 0$ and $l \in H$ we then have $\sum_{h\in H} x_0(1h^{-1})y_a(h) = 0$. Define $y'_a \in l^1(H)$ by $y'_a = \sum_{h\in H} y_a(h)h$. Then the above equation gives that $xy'_a = 0$, so that x is a divisor of zero in $l^1(H)$. But then x is a divisor of zero in $l^1(G)$. Since x is assumed to be regular in $l^1(G)$ we must have that $y'_a = 0$, and hence that $y_a(h) = 0$, for all $h \in H$. As this is true for all $a \neq 0$ we have that the inverse $y = \sum_{h\in H} y_0(h)h$ is an element of $l^1(H)$.

REMARK 3.11. It is easily seen that if the group algebra of G is symmetric or hermitian, then so is the group algebra of any quotient group. However we do not know if the symmetry or hermitianess of the group algebras of both H and G/H imply that of G.

4. Group algebras where the natural involution is not hermitian. In this section G will be a countable discrete group. The notations following Theorem 3.5 will be used. The conjugate space of $l^1(G)$ will be denoted by $\mathscr{C}(G)$ (all bounded sequences of complex numbers). Let L_x be the left multiplication on $l^1(G)$ defined by x, i.e. $L_x y = xy$, the multiplication being convolution. For a given ordering $\{g_1, g_2, \cdots\}$ of all the elements of G, the matrix of L_x, $\operatorname{mat}(L_x)$, is then defined as (a_{ij}) where $x = \sum_{k=1}^{\infty} x(g_k)g_k$ and $a_{ij} = x(g_j g_i^{-1})$. Since

$$L_x g_n = \sum_{k=1}^{\infty} x(g_k)g_k g_n = \sum_{k=1}^{\infty} x(g_k g_n^{-1})g_k = \sum_{k=1}^{\infty} a_{nk} g_k ,$$

we may speak of the nth row of $\operatorname{mat}(L_x)$ as the image of g_n under L_x. An element $\varphi = (\varphi_1, \varphi_2, \cdots) \in \mathscr{C}(G)$ will be said to be *orthogonal* to a row $R_i = (a_{i1}, a_{i2}, \cdots)$ of $\operatorname{mat}(L_x)$ if $\sum_{j=1}^{\infty} a_{ij}\varphi_j = 0$.

LEMMA 4.1. (i) *If* $\operatorname{mat}(L_x) = (a_{ij})$, *then* $\operatorname{mat}(L_{x^*}) = (b_{ij})$ *where* $b_{ij} = \overline{a_{ji}}$. *In particular if* $x = x^*$, *then* $a_{ij} = \overline{a_{ji}}$.

(ii) *If there is a non-zero element* $\varphi \in \mathscr{C}(G)$ *orthogonal to all the rows of* $\operatorname{mat}(L_x)$, *then* L_x *is a singular operator.*

Proof. (i) Since $x^* = \sum_{g\in G} y(g)g$ where $y(g) = \overline{x(g^{-1})}$ we have that $\operatorname{mat}(L_{x^*}) = (b_{ij})$ with $b_{ij} = y(g_j g_i^{-1}) = \overline{x(g_i g_j^{-1})} = \overline{a_{ji}}$.

(ii) If such a φ exists, then $\varphi(L_x g_n) = 0$ for all n. Hence all finite linear combinations of the $L_x g_n$'s are in the nullspace of φ. From the continuity of φ and L_x, and the fact that linear combinations of the g_n's are dense in $l^1(G)$, it follows that L_x maps $l^1(G)$ into the nullspace of φ.

Since φ is non-zero, L_x is singular because it is not onto. Note that from Theorem B(b) we have that x is singular in $l^1(G)$.

LEMMA 4.2. *Let $a_1, a_2, \cdots, a_n$, $n \geqq 3$ be complex numbers of absolute value one, and $x_1, x_2, \cdots, x_r$ complex numbers of absolute value one or zero. Then for $2r \leqq n$ there are complex numbers $x_{r+1}, x_{r+2}, \cdots, x_n$ of absolute value one or zero, not all of which are zero, such that $a_1x_1 + a_2x_2 + \cdots + a_rx_r = a_{r+1}x_{r+1} + a_{r+2}x_{r+2} + \cdots + a_nx_n$. Moreover if $n \geqq 4$ and $2r < n$, there are at least two linearly independent solutions.*

Proof. Suppose first that not all the x_i's are zero. Let $x_{r+k} = a_kx_k/a_{r+k}$ for $1 \leqq k \leqq r$ and $x_{r+k} = 0$ for $k > r$. This gives a non-zero solution. If $2r < n$ then $x_n = 0$ and some $x_{r+i_0} \neq 0$. Let $x'_n = x_{r+i_0}a_{r+i_0}/a_n$, $x'_{i_0+r} = 0$, and $x'_j = x_j$ for $j \neq n, j \neq r + i_0$. Then the primed sequence is also a solution and is clearly not a scalar multiple of the unprimed sequence. (In the above case $n \geqq 3$ is all that is required).

Now assume that $x_1 = x_2 = \cdots = x_r = 0$. Since $n \geqq 3$ and $2r \leqq n$, a_{r+1} and a_{r+2} exist. Letting $x_{r+1} = 1$, $x_{r+2} = -a_{r+1}/a_{r+2}$, and the remaining x_i's zero we have a solution. Finally if $2r < n$ and $n \geqq 4$, a_{r+1}, a_{r+2}, and a_{r+3} exist. In this case pick $x'_{r+1} = 0$, $x'_{r+2} = 1$, $x'_{r+3} = -a_{r+2}/a_{r+3}$, and the remaining x'_i's zero. Again the primed sequence is a solution and clearly not a scalar multiple of the unprimed sequence.

Let $\{g_1, g_2, \cdots\}$ be an ordering $\mathscr{O}$ of all the elements of G. For a subset A of G, let $|A|$ denote the number of elements of A, and $[A]$ the subgroup generated by A. The following definition is pertinent to both the symmetry of $l^1(G)$, and the existence of an invariant mean on G.

DEFINITION 4.3. A finite set S of G will be said to be *singular with respect to the ordering* $\mathscr{O}$ if:

(i) $|S| \geqq 3$;

(ii) $[S]| = \infty$;

(iii) There is an integer n_0 such that

$$2\,|Sg_n \cap (Sg_1 \cup Sg_2 \cup \cdots \cup Sg_{n-1})| \leqq |S| \quad \text{for all} \quad n > n_0 .$$

In the following theorem an element $\varphi = (\varphi_1, \varphi_2, \cdots) \in \mathscr{C}(G)$ is going to be constructed with respect to a given matrix. We will start out with the sequence consisting of all zeros, and then begin replacing the zeros by other entries. At any given stage in the construction, the kth column of the matrix will be termed an *old* column if φ_k has already replaced a zero (the φ_k may itself be zero), and a *new* column otherwise.

THEOREM 4.4. *Let S be a singular set in G with respect to the* *ordering* $\{g_1, g_2, \cdots\}$. *Then the element* $x = \sum_{s_i \in S} \alpha_i s_i$, $|\alpha_i| = 1$, *is singular in* $l^1(G)$.

Proof. By Theorem B (b) it is enough to show that L_x is a singular operator, and by Lemma 4.1 (ii) it suffices to find an element $\varphi \in \mathscr{C}(G)$ orthogonal to all the rows of mat (L_x).

Take mat (L_x). In the columns that contain a non-zero entry from one of the first n_0 rows of mat (L_x), replace the zeros in φ by zeros. In other words, these columns will now be called old columns. We have that $2\,|Sg_n \cap (Sg_1 \cup Sg_2 \cup \cdots \cup Sg_{n-1})| \leqq r$ for $n > n_0$ where $|S| = r$. The $(n_0 + 1)$ row of mat (L_x) contains non-zero entries $a_{c_1}, a_{c_2}, \cdots, a_{c_r}$ in columns $c_1, c_2, \cdots, c_r$ respectively, corresponding to the elements in the set Sg_{n_0+1}. Since S is singular, at least half of these columns are new. Denote the new columns by $c'_1, c'_2, \cdots, c'_s$ where $2s \geqq r$ and select, using Lemma 4.2, $\varphi_{c'_1}, \varphi_{c'_2}, \cdots, \varphi_{c'_s}$ of absolute value one or zero (but not all zero) such that $\sum_{i=1}^{s} \varphi_{c'_i} a_{c'_i} = 0$. At this stage the $\varphi \in \mathscr{C}(G)$ is orthogonal to the first $n_0 + 1$ rows of mat (L_x). Now take the $(n_0 + 2)$ row of mat (L_x). The non-zero entries $a_{p_1}, a_{p_2}, \cdots, a_{p_r}$ now occurs in columns $p_1, p_2, \cdots p_r$ respectively, and since S is singular at least half of the p_i's are new. Denote the old columns by $p'_1, p'_2, \cdots, p'_t$, and the new ones by $p''_1, p''_2, \cdots, p''_{r-t}$. Again by Lemma 4.2 there are complex numbers $\varphi_{p''_1}, \varphi_{p''_2}, \cdots, \varphi_{p''_{r-t}}$ of absolute value one or zero such that

$$\sum_{i=1}^{t} \varphi_{p'_i} a_{p'_i} = \sum_{j=1}^{r-t} \varphi_{p''_j} a_{p''_j} \ .$$

Replacing the zeros by these new φ_i's then gives an element of $\mathscr{C}(G)$ orthogonal to the first $n_0 + 2$ rows of mat (L_x).

The proof is completed by induction. For any $m \geqq n_0 + 2$ assume that scalars of absolute value one or zero have been selected in columns where a non-zero entry occurs in one of the first m rows of mat (L_x), and that the sequence constructed is orthogonal to these rows. By again using the definition of singularity and Lemma 4.2, new φ_i's of absolute value one or zero, in new columns corresponding to the non-zero entries of the $(m + 1)$ rows of mat (L_x) can be constructed so that the resulting sequence is orthogonal to the first $m + 1$ rows of mat (L_x).

COROLLARY 4.5. *If* $|S| \geqq 4$ *and* $2\,|Sg_n \cap (Sg_1 \cup Sg_2 \cup \cdots \cup Sg_{n-1})| < |S|$ *for* $n > n_0$, *the range of* L_x *is not of finite deficiency.*

Proof. The second part of Lemma 4.2 assures us that at each stage in the above construction, starting with the $(n_0 + 1)$ row, there are two linearly independent sets of new φ_i's to choose from. Therefore we can construct infinitely many linearly independent elements in $\mathscr{C}(G)$ orthogonal to all the rows of mat (L_x).

COROLLARY 4.6. *Suppose the singular set satisfies* $e \in S = S^{-1}$. *Let the coefficient of* e *be* i, *and the other coefficients be one. Then the hermitian element* $\sum_{s_i \in S-\{e\}} s_i$ *contains the element* $-i$ *in its spectrum.*

The following theorem gives examples of groups that contain singular sets S satisfying $e \in S = S^{-1}$. A definition is needed first. Let $n \geqq 2$, and let $F^{(n)}$ be the free group on generators $a_1, a_2, \cdots, a_n$. For any $f \in F^{(n)}$, the *length* of f is:

$$\min \left\{ \sum_{i=1}^{s} | n(i) | : f = a_{i(1)}^{n(1)} a_{i(2)}^{n(2)} \cdots a_{i(s)}^{n(s)} \right\} .$$

THEOREM 4.7. (i) *Let* $F^{(n)}$ *be the free group on generators* $a_1, a_2, \cdots, a_n$; $n \geqq 2$. *Then there is an ordering* $\mathcal{O}$ *of* $F^{(n)}$ *such that the set* $S = \{e = a_0, a_1, a_2, \cdots, a_n, a_1^{-1}, a_2^{-1}, \cdots, a_n^{-1}\}$ *is singular with respect to it.*

(ii) *Let* $G^{(n)}$ *be the free group on generators* $b_1, b_2, \cdots, b_n$; $n \geqq 3$, *each of order two. Then there is an ordering of* $G^{(n)}$ *such that the set* $S = \{e = b_0, b_1, b_2, \cdots, b_n\}$ *is singular with respect to it.*

Proof. (i) The ordering $\mathcal{O}$ is started with $g_i = a_i$, $i = 0, 1, 2, \cdots, n$; and $g_{n+j} = a_j^{-1}$, $j = 1, 2, \cdots, n$. Since the generators are free, each of the sets $Sg_1, Sg_2, \cdots, Sg_{2n}$ contains $2n - 1$ distinct elements of length 2. It is clear that no element of length 2 in Sg_i can equal an element of length 2 in Sg_j, $i \neq j$; and that included in the Sg_i's are all elements of length 2. Now successively adjoin to the set $\{g_0, g_1, \cdots, g_{2n}\}$ the elements of length 2 from $Sg_1, Sg_2, \cdots, Sg_n$ respectively. This gives $g_{2n+1}, g_{2n+2}, \cdots, g_{4n^2}$. Again since the generators are free, each of the sets $Sg_{2n+1}, Sg_{2n+2}, \cdots, Sg_{4n^2}$ contains $2n - 1$ distinct elements of length 3; no element of length 3 in Sg_i can equal an element of length 3 in Sg_j, $i \neq j$; and all the elements of length 3 are included in them. As before successively adjoin the elements of length 3 from $Sg_{2n+1}, Sg_{2n+2}, \cdots, Sg_{4n^2}$. The ordering $\mathcal{O}$, constructed in this manner by then adjoining elements of length 4.5 etc., satisfies the conditions of the theorem. Indeed, we have for any n, $| Sg_n \cap (Sg_0 \cup Sg_1 \cup \cdots \cup Sg_{n-1}) | = 2$, and since $n \geqq 2$, $2 \cdot 2 \leqq 2n + 1 = |S|$.

(ii) The proof is in the same spirit as that in (i). In this case start the ordering with S and successively adjoin the elements from

$$Sg_1 - (Sg_1 \cap S), Sg_2 - (Sg_2 \cap (Sg_1 \cup S)), \cdots, Sg_n - (Sg_n \cap (Sg_{n-1} \cup Sg_{n-2} \cup \cdots \cup S)) , \cdots$$

In this case $| Sg_n \cap (Sg_{n-1} \cup Sg_{n-2} \cup \cdots \cup S) | = 2$, and since $n \geqq 3$, $2 \cdot 2 \leqq n + 1$.

REMARK 4.8. It is not hard to see that for the case of $F^{(n)}$ an element of the form $\alpha e + a_1 + a_2 + \cdots + a_n + a_1^{-1} + a_2^{-1} + \cdots + a_n^{-1}$ where $|\alpha| < 2n - 2$ is singular in $l^1(F^{(n)})$. From this it follows that the hermitian element $x = a_1 + a_2 + \cdots + a_n + a_1^{-1} + a_2^{-1} + \cdots + a_n^{-1}$ contains in its spectrum the closed circle about the origin of radius $2n - 2$.

REMARK 4.9. For $G^{(2)}$ the theorem is false. One way to see this is to note that $G^{(2)}$ is the semi-direct product of the integers by a group of order two, where the automorphism sends an element to its inverse. Hence by Theorem 3.9, $l^1(G^{(2)})$ is symmetric with respect to the natural involution. Another way of seeing this will be given by Theorem 4.12 (see Remark 4.14).

REMARK 4.10. It is known that the group $F^{(n)}$, $n \geqq 2$ has a complete set of representations by finite groups, and it follows from this that $F^{(n)}$ can be algebraically imbedded in the complete direct sum of these finite groups. By Theorem 3.10 we then have that the natural involution is not hermitian in the group algebra of this complete direct sum. However, we do not know the answer to the involution question for the general case of the restricted direct sum (sequences reducing to the identity from some point on) of finite groups.

REMARK 4.11. Group algebras are A^* algebras in the sense introduced by Rickart [8]. Unfortunately, Hille and Phillips [5: pp 22] have defined an A^* algebra to be a Banach algebra with an hermitian involution. It follows from the above that these two definitions are not the same.

Perhaps the simplest example of an hermitian element with non-real spectrum can be found in the group algebra of the group $G = \{a, b : a^2 = e\}$. The element $x = a + b + b^{-1}$ is hermitian and with respect to an ordering of G constructed in the same fashion as above, the matrix of L_{ie+x} is:

$$\begin{bmatrix}
i & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & i & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & i & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & i & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\
0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & \cdots \\
 & & & & & & \cdots & & & & & &
\end{bmatrix}$$

The element $(i, 1, -1, -i, -i, \cdots) \in \mathscr{C}(G)$ will then be orthogonal to all the rows of this matrix, and hence $-i$ is in the spectrum of x.

Theorem 4.12 is quite special, however it does suffice to show that hermitian elements with finite support in $l^1(G^{(2)})$ have real spectrum.

Let $S = (s_{ij})$; $i, j = 1, 2, \cdots$ be any infinite matrix and define $S^{(n)} = (s'_{ij})$ by the equations $s'_{ij} = s_{ij}$ for $i, j \leqq n$ and $s'_{ij} = 0$ otherwise. $S^{(n)}$ will be called the *principal* $n \times n$ *section of* S. If there is an integer k so that $s_{mn} = 0$ whenever $|m - n| \geqq k$, S will be called a *corridor matrix of width* k.

THEOREM 4.12. *Let* $T = (a_{ij})$ *be an hermitian corridor matrix of width* k, *with* $\sup_{i,j}|a_{ij}| < \infty$. *Then for any real number* ρ, *the operator* $T + \rho iI$ *defined by this matrix maps* l^1 *onto a dense subset of* l^1.

Proof. Note first that since the norm of an operator on l^1 can be computed by taking the sup of the l^1 norms of the rows of its matrix with respect to the usual basis, T is a bounded operator (we will not distinguish between the matrix and the operator it represents) on l^1. Moreover since the matrix of T is hermitian, T can be extended to a bounded operator on l^2. Hence the spectrum of T as an operator on l^2 is real.

Assume there is a sequence $(\varphi_1, \varphi_2, \cdots)$ (not necessarily bounded), that is orthogonal to all the rows of $(T + \rho iI)$. Since $T + \rho iI$ is regular as an operator on l^2, it follows that $\sum_{i=1}^{\infty} |\varphi_i|^2 = \infty$. We are going to show that the sequence $(\varphi_1, \varphi_2, \cdots)$ is in fact unbounded.

Let $l^2_{(n)}$ denote n-dimensional Hilbert space and $\varphi^{(n)} = (\varphi_1, \varphi_2, \cdots, \varphi_n)$. Since T is hermitian, we have for any $y \in l^2_{(n)}$ that $\|(T + \rho iI)_1^{(n)} y\|_2 \geqq \rho \|y\|_2$, where $(T + \rho iI)_1^{(n)}$ denotes the $n \times n$ matrix in the upper left hand portion of the principal $n \times n$ section $(T + \rho iI)^{(n)}$. Let K be any large number, and pick $n_0 > K$ so that $\sum_{i=1}^{n_0} |\varphi_i|^2 > 4K^2M^2k^3\rho^{-2}$ where $M = \sup_{i,j}|t_{ij}|$ and $\operatorname{mat}(T + \rho iI) = (t_{ij})$. Let $(T + \rho iI)_1^{(n_0)} \varphi^{(n_0)} = (a_1, a_2, \cdots, a_{n_0})$, $a_k = \sum_{p=1}^{n_0} a_{kp}\varphi_p$. Since $\varphi = (\varphi_1, \varphi_2, \cdots)$ is orthogonal to all the rows of $T + \rho iI$ we have that $a_1 = a_2 = \cdots = a_{n_0 - k} = 0$. From the l^2 norm inequality above, we have that

$$\|(T + \rho iI)_1^{(n_0)} \varphi^{(n_0)}\|_2^2 \geqq \rho^2 \sum_{i=1}^{n_0} |\varphi_i|^2$$

or

$$\sum_{i=n_0-k+1}^{n_0} |a_i|^2 \geqq \rho^2(4K^2M^2k^3\rho^{-2}) = 4K^2M^2k^3 .$$

Hence some a_{i_0}, $n_0 - k < i_0 \leqq n_0$ is such that $|a_{i_0}|^2 > 4K^2M^2k^2$ or $|a_{i_0}| > 2KMk$. However $a_{i_0} = \sum_{p=1}^{n} t_{i_0 p}\varphi_p$, and since there are at most $2k$ non-zero terms, there is a p_0 with $|t_{i_0 p_0}\varphi_{p_0}| > KM$ and hence $|\varphi_{p_0}| > KM/|t_{i_0 p_0}| \geqq K$. In other words, the sequence $(\varphi_1, \varphi_2, \cdots)$ is unbounded, and it follows that the range of $T + \rho iI$ is dense in l^1.

For $x = \sum_{g \in G} x(g)g \in l^1(G)$, let G_0 denote the subgroup of G generated by $\{g \in G : x(g) \neq 0\}$. Since this set is countable, G_0 is countable. We have:

COROLLARY 4.13. *If $x = x^* \in l^1(G)$, and with respect to the basis in $l^1(G_0)$ defined by some ordering of G_0,* mat (L_x) *is a corridor matrix, then the spectrum of x is real.*

Proof. By Theorem 3.10 it sufficies to look at the spectrum of x as an element of $l^1(G_0)$. Now the theorem above gives that the ranges of L_{ie+x} and L_{-ie+x} are dense in $l^1(G_0)$. But these ranges are also ideals and since they are dense, they must be all of $l_1(G_0)$. This means that there are elements $y_1, y_2 \in l^1(G_0)$ such that $(ie + x)y_1 = e$ and $(-ie + x)y_2 = e$. Applying the involution to the latter equality gives $y_2^*(ie + x) = e$. Hence $ie + x$ has both a right and left inverse, and is hence regular.

REMARK 4.14. Take the ordering $(e, a, b, ab, ba, aba, \cdots)$ in the group $G^{(2)} = \{a, b : a^2 = b^2 = e\}$. Then it is easily seen that mat (L_x) is a corridor matrix whenever $x \in l^1(G^{(2)})$ has finite support. Hence Theorem 4.7 does not hold for $G^{(2)}$.

5. The involution and invariant means.

The main results in this section are Theorem 5.6 and Theorem 5.8. The first theorem gives us some information concerning the involution when the group has an invariant mean, and in the second theorem it is shown that a group containing a singular set cannot have an invariant mean.

A continuous linear functional λ on $\mathscr{C}(G)$ is said to be an *invariant mean*, if it satisfies:

(i) $\lambda(\varphi) \geqq 0, \quad \varphi \geqq 0, \quad \varphi \in \mathscr{C}(G)$;

(ii) $\lambda(\varphi_x) = \lambda(\varphi^x) = \lambda(\varphi)$ where $\varphi_x(y) = \varphi(x^{-1}y)$, and $\varphi^x(y) = \varphi(yx)$;

(iii) $\lambda(I) = 1$ where I is the function identically 1 on G.

Whenever the notation $\lambda(A)$, for A a subset of G, is used, it will mean the number $\lambda(\chi_A)$ where χ_A is the characteristic function of A.

For $\varphi, \psi \in \mathscr{C}(G)$ define a pseudo "inner product" $(\varphi, \psi) = \lambda(\varphi\bar{\psi})$. A few simple properties of this inner product are given in:

LEMMA 5.1. (i) $(\varphi, \psi_1 + \psi_2) = (\varphi, \psi_1) + (\varphi, \psi_2)$;
(ii) $(\varphi, \psi) = \overline{(\psi, \varphi)}$;
(iii) $(\alpha\varphi, \psi) = \alpha(\varphi, \psi)$;
(iv) $(\varphi, \varphi) \geqq 0$;
(v) $|(\varphi, \psi)| \leqq (\varphi, \varphi)^{1/2}(\psi, \psi)^{1/2}$;
where $\varphi, \psi, \psi_1, \psi_2 \in \mathscr{C}(G)$, and α is a complex number.

Proof. (v) will be proved, the other statements following immediately from the definitions.

$$0 \leqq (\varphi - \alpha\psi, \varphi - \alpha\psi) = (\varphi, \varphi) - \alpha(\psi, \varphi) - \bar{\alpha}(\varphi, \psi) + |\alpha|^2(\psi, \psi) .$$

If $(\varphi, \psi) = 0$, (v) is trivial, so assume that $(\varphi, \psi) \neq 0$, and let $\alpha = (\varphi, \varphi)/(\psi, \varphi)$; (v) then follows by direct calculation.

Let $\mathscr{K} = \{\varphi \in \mathscr{C}(G) : \lambda(|\varphi|) = 0\}$, and $\mathscr{L} = \{\varphi \in \mathscr{C}(G) : \lambda(|\varphi|^2) = 0\}$. We have:

LEMMA 5.2. *$\mathscr{K}$ is equal to $\mathscr{L}$, and is a closed subspace of $\mathscr{C}(G)$.*

Proof. By letting $\psi = I$ and replacing φ by its absolute value in (v) above, we have that, $|(|\varphi|, I)|^2 \leqq (|\varphi|, |\varphi|)(I, I)$ or $(\lambda(|\varphi|)^2 \leqq \lambda(|\varphi|^2)\lambda(I^2) = \lambda(|\varphi|^2)$. Hence $\lambda(|\varphi|^2) = 0$ implies $\lambda(|\varphi|) = 0$, and thus $\mathscr{L} \subset \mathscr{K}$.

Conversely if $\varphi \in \mathscr{K}$, then $|\varphi|^2 \leqq K|\varphi|$ where K is a bound for $|\varphi|$, and it follows that $\mathscr{K} \subset \mathscr{L}$.

Since $|\alpha\varphi + \beta\psi| \leqq |\alpha||\varphi| + |\beta||\psi|$, $\mathscr{K}$ is a subspace of $\mathscr{C}(G)$. Finally for $\varphi_n \in \mathscr{K}$ and $||\varphi_n - \varphi||_\infty \to 0$, it follows from the continuity of λ that $\lambda(|\varphi|) = 0$, and hence $\mathscr{K}$ is closed.

Let $\mathscr{C} - \mathscr{K}$ denote the space of cosets of $\mathscr{C}(G)$ with respect to $\mathscr{K}$. For $\dot{\varphi} \in \mathscr{C} - \mathscr{K}$ let $||\dot{\varphi}||_1 = \lambda(|\varphi|)$: $||\dot{\varphi}||_2 = (\lambda(|\varphi|^2))^{1/2}$, $\varphi \in \dot{\varphi}$; and $(\dot{\varphi}, \dot{\psi}) = \lambda(\varphi\bar{\psi})$, $\varphi \in \dot{\varphi}$, $\psi \in \dot{\psi}$. Then:

LEMMA 5.3. *For $\dot{\varphi}, \dot{\psi} \in \mathscr{C} - \mathscr{K}$.*

(i) *$||\dot{\varphi}||_1$ is well defined and a norm on $\mathscr{C} - \mathscr{K}$;*

(ii) *$(\dot{\varphi}, \dot{\psi})$ is well defined and makes $\mathscr{C} - \mathscr{K}$ into a pre-Hilbert space;*

(iii) $||\dot{\varphi}||_1 \leqq ||\dot{\psi}||_2$.

Proof. (i) Let $\varphi_1, \varphi_2 \in \dot{\varphi}$ so that $\varphi_1 = \varphi_2 + k$ where $k \in \mathscr{K}$. Then $|\varphi_1(g)| = |\varphi_2(g) + k(g)| \leqq |\varphi_2(g)| + |k(g)|$ so that $|\varphi_1| \leqq |\varphi_2| + |k|$. Hence $\lambda(|\varphi_1|) \leqq \lambda(|\varphi_2| + |k|) \leqq \lambda(|\varphi_2|) + \lambda(k|) = \lambda(|\varphi_2|)$. Now by reversing the roles of φ_1 and φ_2, it follows that $||\dot{\varphi}||_1$ is well defined. Also $||\dot{\varphi} + \dot{\psi}||_1 = \lambda(|\varphi + \psi|) \leqq \lambda(|\varphi| + |\psi|) = \lambda(|\varphi|) + \lambda(|\psi|) = ||\dot{\varphi}||_1 + ||\dot{\psi}||_1$, and $||\alpha\dot{\varphi}||_1 = ||(\dot{\alpha\varphi})||_1 = \lambda(|\alpha\varphi|) = |\alpha|\lambda(|\varphi|) = |\alpha|||\dot{\varphi}||$ for $\varphi \in \dot{\varphi}, \psi \in \dot{\psi}$, and α complex. Finally $||\dot{\varphi}||_1 = 0$ implies that $\lambda(|\varphi|) = 0$, and hence that $\dot{\varphi} = 0$. Thus $||\dot{\varphi}||_1$ is a norm on $\mathscr{C} - \mathscr{K}$.

(ii) If $\varphi_1, \varphi_2 \in \dot{\varphi}$ and $\psi_1, \psi_2 \in \dot{\psi}$, then $\varphi_1 = \varphi_2 + k$, $\psi_1 = \psi_2 + l$ where $k, l \in \mathscr{K}$. Then $(\varphi_1, \psi_1) = (\varphi_2 + k, \psi_2 + 1) = (\varphi_2, \psi_2) + (\varphi_2, 1) + (k, \psi_2) + (k, l)$. But $|(\varphi_2, l)|^2 \leqq (\varphi_2, \varphi_2)(l, l) = 0$, $|(k, \psi_2)|^2 \leqq (k, k)(\psi_2, \psi_2) = 0$, and $|(k, l)|^2 \leqq (k, k)(l, l) = 0$, so that $(\dot{\varphi}, \dot{\psi})$ is well defined. If $(\dot{\varphi}, \dot{\varphi}) = 0$, then $\lambda(|\varphi|^2) = 0$ for $\varphi \in \dot{\varphi}$, and by Lemma 5.2, $\varphi \in \mathscr{K}$ or $\dot{\varphi} = 0$. Hence with respect to $(\dot{\varphi}, \dot{\psi})$, $\mathscr{C} - \mathscr{K}$ becomes a pre-Hilbert space.

(iii) $\|\dot\varphi\|_1^2 = (\lambda(|\varphi|))^2 = |(|\varphi|, I)|^2 \leqq (|\varphi|, |\varphi|)(I, I) = \lambda(|\varphi|^2) = \|\dot\varphi\|_2^2$.

$L^1(G, \lambda)$ will denote the completion of $\mathscr{C} - \mathscr{K}$ with respect to $\|\dot\varphi\|_1$; and $L^2(G, \lambda)$ the completion with respect to $\|\dot\varphi\|_2$.

LEMMA 5.4.* *Let $g_1, g_2, \cdots, g_n$ be distinct elements of G. Then there is a subset A of G satisfying $\lambda(A) > 0$, and $Ag_i \cap Ag_j = \phi$, $i \neq j$.*

Proof. Let $\mathscr{A} = \{B \subset G : Bg_i \cap Bg_0 = \phi,\ i = j\}$. $\mathscr{A}$ is then non-empty since $\{e\} \in \mathscr{A}$, and is partially ordered by inclusion. An immediate application of Zorn's lemma gives a maximal element A. Let $C = Ag_1 \cup Ag_2 \cup \cdots \cup Ag_n \cup (\bigcup_{i \neq j} Ag_i g_j^{-1})$. It will be shown that $C = G$. Indeed if $h \in G - C$, let $A' = A \cup \{h\}$. Since A is a maximal element of $\mathscr{A}$, there are indices i_0 and j_0 such that $k \in A'g_{i_0} \cap A'g_{j_0}$ and $k \notin Ag_{i_0} \cap Ag_{j_0}$. Therefore either

(a) $k = hg_{i_0}, k \in Ag_{j_0}$;

(b) $k \in Ag_{i_0},\ k = hg_{j_0}$; or

(c) $k = hg_{i_0},\ k = hg_{j_0}$.

But (c) implies that $g_{i_0} = g_{j_0}$, a contradiction. (a) implies that $k = hg_{i_0}$, $k = ag_{j_0}$ where $a \in A$, and hence $hg_{i_0} = ag_{j_0}$ or $h = ag_{j_0}g_{i_0}^{-1}$ giving $h \in Ag_{j_0}g_{i_0}^{-1}$ which is also a contradiction. The proof that (b) is impossible, is similar to (a). Hence $C = G$, and $\lambda(A) > 0$, since G is then the finite union of sets, each of measure $\lambda(A)$.

Corresponding to an $x \in l^1(G)$, we are now going to define operators on $L^1(G, \lambda)$ and $L^2(G, \lambda)$.

For $\dot\varphi \in \mathscr{C} - \mathscr{K}$ and $g \in G$, $\dot\varphi_g$ will mean the coset in $\mathscr{C} - \mathscr{K}$ containing φ_g. This is well defined since $\varphi, \psi \in \dot\varphi$ imply $\varphi - \psi = k \in \mathscr{K}$. Since $\varphi_g - \psi_g = k_g$ is also in $\mathscr{K}$ it follows that $\dot\varphi_g = \dot\psi_g$. For $x = \sum_{g \in G} x(g)g \in l^1(G)$ define $T_x\dot\varphi = \sum_{g \in G} x(g)\dot\varphi_g$ for $\dot\varphi \in \mathscr{C} - \mathscr{K}$. For $\varphi \in \dot\varphi$ we have,

$$\begin{aligned} \| T_x\dot\varphi \|_1 &= \lambda\Big(\Big|\sum_{g \in G} x(g)\varphi_g\Big|\Big) \leqq \lambda\Big(\sum_{g \in G} |x(g)|\,|\varphi_g|\Big) \\ &= \lambda\Big(\sum_{g \in G_1} |x(g)|\,|\varphi_g| + \sum_{g \in G - G_1} |x(g)|\,|\varphi_g|\Big) \end{aligned}$$

where G_1 is some subset of G satisfying $\sum_{g \in G - G_1} |x(g)| < \varepsilon \|\varphi\|_\infty$. Hence

$$\begin{aligned} \| T_x\dot\varphi \|_1 &\leqq \lambda\Big(\sum_{g \in G_1} |x(g)|\,|\varphi_g|\Big) + \lambda\Big(\sum_{g \in G - G_1} |x(g)|\,|\varphi_g|\Big) \\ &\leqq \lambda\Big(\sum_{g \in G_1} |x(g)|\,|\varphi_g|\Big) + \lambda\Big(\|\varphi_g\|_\infty \sum_{g \in G - G_1} |x(g)|\Big) \\ &\leqq \lambda\Big(\sum_{g \in G_1} |x(g)|\,|\varphi_g| + (\varepsilon \|\varphi\|_\infty)\, \|\varphi_g\|_\infty \\ &\leqq \sum_{g \in G_1} |x(g)|\,\lambda(|\varphi_g|) + \varepsilon \leqq \|\dot\varphi_g\|_1\, \|x\| + \varepsilon\,. \end{aligned}$$

* The author is thankful to Professor H. A. Dye who suggested this lemma.

Since ε was arbitrary we have that $\| T_x\dot{\varphi} \|_1 \leqq \| x \| \, \| \dot{\varphi}_g \|_1$.

Now let G_0 be the countable subgroup generated by $\{g : x(g) \neq 0\}$, and let $\{g_1, g_2, \cdots\}$ be an ordering of G_0. Let $x^{(n)} = \sum_{i=1}^{n} x(g_i)g_i$. By Lemma 5.4 there is a subset A of G with $\lambda(A) = d > 0$, and $Ag_i \cap Ag_j = \phi$ for $i \neq j$ and $1 \leqq i, j \leqq n$. Then

$$\| T_x(n)\dot{\chi}_A \|_1 = \lambda\Big(\Big|\sum_{i=1}^{n} x(g_i)(\chi_A)_{g_i}\Big|\Big) = \lambda\Big(\sum_{i=1}^{n} | x(g_i) | \, | (\chi_A)_{g_i} |\Big)$$
$$= \sum_{i=1}^{n} | x(g_i) | \, d = d \sum_{i=1}^{n} | x(g_i) | = \| \dot{\chi}_A \|_1 \, \| x^{(n)} \| \, .$$

Since $\| x^{(n)} - x \| \to 0$ as $n \to \infty$ we have that $\| T_x \|_1 = \| x \|$. Finally since T_x is bounded on $\mathscr{C} - \mathscr{K}$ with respect to $\| \dot{\varphi} \|_1$, it can be extended to the completion $L^1(G, \lambda)$ without increasing its norm. The extension will be denoted by $T_x^{(1)}$.

The operator T_x on $\mathscr{C} - \mathscr{K}$ will now be extended to $L^2(G, \lambda)$. For $\dot{\varphi} \in \mathscr{C} - \mathscr{K}$, and $\varphi \in \dot{\varphi}$ we have

$$\| T_x\dot{\varphi} \|_2^2 = \lambda\Big(\Big| \sum_{g \in G} x(g)\varphi_g \Big|^2\Big) \leqq \lambda\Big(\Big(\sum_{g \in G} | x(g) | \, | \varphi_g |\Big)^2\Big) \, .$$

But $\{| x(g) |^{1/2} : g \in G\} \in l^2(G)$, and so for any $h \in G$ the sequence $\{| x(g) |^{1/2} \, | \varphi_g | \, (h) : g \in G\} \in l^2(G)$. Now

$$\lambda\Big(\Big(\sum_{g \in G} | x(g) |^2 \, | \varphi_g |\Big)^2\Big) = \lambda\Big(\Big(\sum_{g \in G} | x(g) |^{1/2} \, | x(g)^{1/2} \, | \varphi_g |\Big)^2\Big) \, ,$$

and

$$\Big(\sum_{g \in G} | x(g) |^{1/2} \, | x(g) |^{1/2} \, | \varphi_g | \, (h)\Big)^2 \leqq \Big(\sum_{g \in G} | x(g) |\Big)\Big(\sum_{g \in G} | x(g) | \, | \varphi_g |^2(h)\Big) \, ,$$

so that

$$\lambda\Big(\Big(\sum_{g \in G} | x(g) | \, | \varphi_g |\Big)^2\Big) \leqq \| x \| \, \lambda\Big(\sum_{g \in G} | x(g) | \, | \varphi_g |^2\Big)$$

and hence

$$\lambda\Big(\Big(\sum_{g \in G} | x(g) | \, | \varphi_g |^2\Big) \leqq \| x \| \, \lambda\Big(\sum_{g \in G} | x(g) | \, | \varphi_g |^2\Big) = \| x \| \sum_{g \in G} | x(g) | \, \lambda(| \varphi_g |^2)$$
$$= \| x \| \, \| x \| \, \| \dot{\varphi}_g \|_2^2 \, ,$$

so

$$\| T_x\dot{\varphi} \|_2^2 \leqq \| x \|^2 \, \| \dot{\varphi}_g \|_2^2 \quad \text{or} \quad \| T_x \|_2 \leqq \| x \| \, .$$

We can therefore extend T_x to a bounded operator on $L^2(G, \lambda)$ without increasing its norm. These results are summarized in the following theorem.

THEOREM 5.5. *The operator* T_x *on* $\mathscr{C} - \mathscr{K}$ *defined by* $T_x\dot{\varphi} = \sum_{g \in G} x(g)\dot{\varphi}_g$, *where* $x = \sum_{g \in G} x(g)g \in l^1(G)$, *can be uniquely extended to a bounded operator* $T_x^{(1)}(T_x^{(2)})$ *on* $L^1(G, \lambda)(L^2(G, \lambda))$, *and* $\| T_x^{(2)} \|_2 \leqq \| T_x^{(1)} \|_1 = \| x \|$.

THEOREM 5.6. *Let G have an invariant mean λ, and let $x = x^* \in l^1(G)$. If there is a $\varphi \in \mathscr{C}(G)$ whose nullspace contains the range of $iI + L_x$, then $\lambda(| \varphi |) = 0$.*

Proof. Since the nullspace of φ contains the range of $iI + L_x$, we have in particular that $\varphi((iI + L_x)(h)) = 0$ for all $h \in G$. Let $\varphi = \{\varphi(g)\}$ and $x = \sum_{g \in G} x(g)g$. Then

$$L_x h = \sum_{g \in G} x(g)gh = \sum_{g \in G} x(gh^{-1})g ,$$

and

$$(iI + L_x)(h) = ih + \sum_{g \in G} x(gh^{-1})g = (i + x(e))(h) + \sum_{g \neq h} x(gh^{-1})g .$$

Hence

$$0 = (i + x(e))\varphi(h) + \sum_{g \neq h} x(gh^{-1})\varphi(g) = (i + x(e))\varphi(h) + \sum_{g \neq e} x(g)\varphi(gh) .$$

Taking complex conjugates and letting $\psi(g) = \overline{\varphi(g)}$ we have,

$$0 = (-i + \overline{x(e)})\overline{\varphi(h)} + \sum_{g \neq e} \overline{x(g)}\,\overline{\varphi(gh)} = (-i + \overline{x(e)})\psi(h) + \sum_{g \neq e} \overline{x(g^{-1})}\psi(g^{-1}h)$$
$$= (-i + x(e))\psi(h) + \sum_{g \neq e} x(g)\psi_g(h)$$

for all $h \in G$, since $x = x^*$.

On the other hand,

$$\begin{aligned}((-iI + T_x^{(2)})\psi)(h) &= -i\psi(h) + \sum_{g \in G} x(g)\psi_g(h) \\ &= -i\psi(h) + x(e)\psi(h) + \sum_{g \neq e} x(g)\psi_g(h) \\ &= (-i + x(e))\psi(h) + \sum_{g \neq e} x(g)\psi_g(h) = 0\end{aligned}$$

for all $h \in G$. This means that $\overline{(-iI + T_x^{(2)})(\psi)} \in \mathscr{K}$, and since $T_x^{(2)}$ is an hermitian operator on $L^2(G, \lambda)$, we must have $\dot{\psi} = 0$. Therefore $\lambda(| \varphi |) = \lambda(| \psi |) = 0$.

We now show that the existence of a singular set (not necessarily inverse closed) in a countable group, implies that the group does not have an invariant mean. For this purpose we make essential use of a theorem due to Følner [2].

THEOREM (Følner). *A group G has an invariant mean if and only* *if for any finite set F and $\varepsilon > 0$, there exists a finite set A of G such that $|A \cap xA|/|A| > 1 - \varepsilon$ for all $x \in F$.*

LEMMA 5.7. *Let F be a finite subset of a group G such that $[F]$ is infinite. Then if there is a finite set A with $|A \cap xA|/|A| > 1 - \varepsilon$ for all $x \in F$, then $|A| \geqq 1/\varepsilon$.*

Proof. Let $F = \{f_1, f_2, \cdots, f_s\}$, $|A| = r$, and assume that $r < 1/\varepsilon$. Then $\varepsilon < 1/r$, and for any $f_i \in F$, $|A \cap f_iA| > (1 - \varepsilon)r > (1 - 1/r)r = r - 1$, and hence $A \cap f_iA = A$ or $f_iA = A$ $(i = 1, 2, \cdots, s)$. It follows that $gA = A$ for any $g \in [F]$. Now since A is finite and $[F]$ infinite, there must exist elements $a_0 \in A$, $g_1, g_2 \in F$, $g_1 \neq g_2$ such that $g_1a_0 = g_2a_0$. But this gives $g_1 = g_2$, and we have contradicted the assumption that $|A| < 1/\varepsilon$.

THEOREM 5.8. *If G contains a singular set F with respect to the ordering $\{g_1, g_2, \cdots\}$, then G does not possess an invariant mean.*

Proof. Since F is singular there exists an integer t_0 such that $2|Fg_t \cap (Fg_1 \cup Fg_2 \cup \cdots \cup Fg_{t-1})| \leqq |F| = s$ for $t > t_0$. Assume G does have an invariant mean, so that Fϕlner's condition is satisfied. Let $\varepsilon = 1/72t_0(s - 1)$. Then there exists a finite set A with $|A| = r$, and $|A \cap f_iA| > (1 - \varepsilon)|A|$ for any $f_i \in F$. From Lemma 5.7 $|A| \geqq 1/\varepsilon = 72t_0(s - 1) > 6t_0$. Let $A = \{g_{n_1}, g_{n_2}, \cdots, g_{n_r} : n_1 < n_2 < \cdots < n_r\}$. Then $2|Fg_{n_t} \cap (Fg_{n_1} \cup Fg_{n_2} \cup \cdots \cup Fg_{n_{t-1}})| \leqq s$ for $t > t_0$.
Consider the matrix

$$\begin{bmatrix} f_1g_{n_1} & f_1g_{n_2} & \cdots & f_1g_{n_r} \\ f_2g_{n_1} & f_2g_{n_2} & \cdots & f_2g_{n_r} \\ \cdots & \cdots & & \cdots \\ f_sg_{n_1} & f_sg_{n_2} & \cdots & f_sg_{n_r} \end{bmatrix}$$

and let B denote the set of distinct elements of this matrix. We are first going to get an upper bound for $|B|$ by counting the elements of the matrix row by row, and then a lower bound for $|B|$ by counting them column by column. It will turn out that these bounds are incompatible and the proof completed.

The kth row of the matrix is simply f_kA, and $|f_jA - (f_iA \cap f_jA)| < 3r\varepsilon$. Indeed $|A \cap f_iA)| > (1 - \varepsilon)r$ implies $|A - (A \cap f_iA)| < r\varepsilon$ so that

$$A = ((A - (A \cap f_iA)) \cap (A - (A \cap f_jA))) \cup ((A - (A \cap f_iA)) \cap (A \cap f_jA)) \\ \cup ((A \cap f_iA) \cap (A - (A \cap f_jA))) \cup ((A \cap f_iA) \cap (A \cap f_jA)) = A_1 \cup A_2 \cup A_3 \cup A_4 ,$$

where the A_i's are disjoint. Therefore $r = |A| = |A_1| + |A_2| + |A_3| + |A_4| < r\varepsilon + r\varepsilon + r\varepsilon + |A_4|$, and $|(f_iA \cap f_jA)| \geqq |A \cap f_iA \cap f_jA| = |A_4| > r - 3r\varepsilon$ or $|f_jA - (f_iA \cap f_jA)|3r\varepsilon$. Now the first row of the matrix has

r elements and, as has just been shown, each additional row adds less than $3r\varepsilon$ additional distinct elements. Adding, we have $|B| < r + (s-1)3r\varepsilon$.

The first t_0 columns obviously contain at least s distinct elements, and from the singularity condition it follows that each additional column from $t_0 + 1$ through r adds at least $s/2$ distinct elements to B. Hence $|B| \geqq s + (r - t_0)s/2$.

Therefore $s + (r - t_0)s/2 < r + (s-1)3r\varepsilon = r + (s-1)3r/72t_0(s-1) = r + r/24t_0$. Since $r = |A| \geqq 1/\varepsilon > 6t_0$ we have $r - t_0 > 5r/6$. Hence $s + (5r/6)(s/2) < s + (r - t_0)s/2 < r + r/24t_0$. Since $s \geqq 3$, $s(1 + 5r/12) \geqq 3 + 5r/4$ so that $3 + 5r/4 < r + r/24t_0$ or $12 + 5r < 4r + r/6t_0 = r(4 + 1/6t_0) < 5r$, and we have obtained the desired contradiction.

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