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MULTIPLICATION OPERATORS

R. CREIGHTON BUCK

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1. Introduction. The prototype for partially ordered linear spaces is $C[X]$, the space of all real valued continuous functions on a topological space X , with the natural ordering defined by: $f \geq 0$ if and only if $f(x) \geq 0$ for all $x \in X$. If V is a real linear space with a partial order defined by a suitable positive cone P , then V has a canonical embedding in a function space $C[X]$.

The containing space $C[X]$ has a more elaborate structure than did the original space V ; in particular, $C[X]$ is an algebra. If we take any aspect of $C[X]$, we may ask how it appears when transferred back to V . This paper deals with one aspect of this.

Among the linear operators on $C[X]$, an interesting class that arises in many contexts is the class of multiplication operators. These are defined by:

$$T(f) = g \quad \text{where} \quad g(x) = \phi(x)f(x) \quad x \in X,$$

and where ϕ is a specific member of $C[X]$.

The central result in this paper is a simple characterization, in terms of order, of the linear operators on V which become multiplication operators when V is represented in a function space $C[X]$. This in turn yields a new and more transparent proof of the Stone-Krein theorem on ordered algebras.

2. A simpler case. Let V be a real linear space. We assume that there is a convex cone P with vertex at 0 which defines an order relation \leq in V by $x \leq y$ if and only if $y - x \in P$. On P , we impose three conditions:

- (1) $P \cap -P = \{0\}$
- (2) P is generating
- (3) P is linearly closed in V .

The second condition implies that every element $x \in V$ is the difference of positive elements; the third condition requires that every line meet P in a (possibly unbounded) closed interval. Note that we do not impose any further lattice properties on V , nor do we assume that there is an order unit. If V' denotes the dual space of V , consisting of all linear functionals on V , then V' has a natural partial ordering derived from that of V . A functional L is said to be positive if $L(x) \geq 0$ for

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all $x \geq 0$; the positive cone in V' is P' . The space V' will not in general obey all the properties (1), (2), (3).

Let $\mathcal{L}(V)$ denote the algebra of all linear transformations on V . We single out a subclass $\mathfrak{A} \subset \mathcal{L}(V)$ consisting of the order-bounded transformations:

DEFINITION 1. An operator $T \in \mathcal{L}(V)$ is order bounded if there is a constant r such that

$$(4) \quad -rx \leq Tx \leq rx \quad \text{for all } x \geq 0 \text{ in } V.$$

We observe that \mathfrak{A} is a subalgebra of $\mathcal{L}(V)$ containing the identity operator I ; for, if T_1 and T_2 are in \mathfrak{A} , with associated constants r_1 and r_2 , then it follows readily from (4) that $T_1 T_2$ obeys (4) with $r = 3r_1 r_2$. We wish to show that V has function space representations in which the algebra \mathfrak{A} becomes multiplication operators. We will prove this first under the strong restriction that V has an "order unit", and then remove this restriction.

Let us suppose that there is an element $e \in V$ such that $e \geq 0$ and

$$(5) \quad \text{for every } x \geq 0, \text{ there is } \lambda > 0 \text{ such that } x \leq \lambda e.$$

This restriction can be described geometrically: the point e is a radially interior point of P , so that every line thru e meets P in a line segment containing e as interior point.

THEOREM 1. *Let V be a partially ordered linear space obeying (1), (2), (3) and (5). Let \mathfrak{A} be the order bounded operators on V . Then there is a compact set Γ and an order preserving representation $\theta: x \rightarrow \hat{x}$ of V onto a subspace of $C[\Gamma]$, and an isomorphism $\bar{\theta}: T \rightarrow \hat{T}$ of \mathfrak{A} into the multiplication operators on $C[\Gamma]$ such that*

$$\theta(Tx) = \hat{T}\hat{x}$$

for all $x \in V, T \in \mathfrak{A}$.

Otherwise described, the diagram

$$\begin{array}{ccc} V & \xrightarrow{\theta} & C[\Gamma] \\ \downarrow T & & \downarrow \hat{T} \\ V & \xrightarrow{\theta} & C[\Gamma] \end{array}$$

commutes. Corresponding to T , there is a function $\phi \in C[\Gamma]$ such that if $Tx = y$, then $\hat{y}(p) = \phi(p)\hat{x}(p)$, for all $p \in \Gamma$.

COROLLARY 1. \mathfrak{A} is a commutative subalgebra of $\mathcal{L}(V)$.

The method we use will be to construct certain appropriate real homomorphisms of \mathfrak{A} . Recall first the important notion of a minimal positive element (See Brelot [3] for background.)

DEFINITION 2. An element $u \geq 0$ in V is said to be minimal if $0 \leq x \leq u$ implies that $x = \lambda u$ for some real λ .

This can be described geometrically: u is minimal if the ray ρ generated by u is extremal in P , and this is so if u cannot be expressed as the midpoint of two points in P that are not on ρ . In contrast with the situation for finite dimensional spaces, a cone P in a general linear space will usually have no extremal rays (or minimal elements). This is the case for $C[X]$ when X is the line, but is not the case if X is discrete. The dual cone P' of positive linear functionals on V can be better behaved; however, if V is the space $L[0, 1]$, neither P nor P' have extremal rays.

LEMMA 1. *If P is the positive cone in a space V and P contains a radially interior point, then P' has a separating family of extremal rays.*

This is more or less familiar. (See Bonsall [2], Kadison [8], Kelly [9].) One defines a norm in V by

$$\|x\| = \inf \{ \text{all } r \text{ with } -re \leq x \leq re \} .$$

Let D be the functionals L on V such that $\|L\| \leq 1$ and $L(e) = 1$. This is then a w^* compact convex set in the dual space of $\langle V, \|\cdot\| \rangle$. Invoking the Krein-Milman theorem, D has extreme points L_0 whose convex hull is dense in D . These are in fact minimal positive elements in V' , generating extremal rays in P' . Moreover, if $L_0(x) = 0$ for all L_0 , then $x = 0$.

The key to the proof of Theorem 1 is the observation that minimal elements of P will yield homomorphism of \mathfrak{A} onto the reals. If $T \in \mathfrak{A}$, then by (4) there is a number r such that

$$(6) \quad 0 \leq rx + Tx \leq 2rx \quad \text{all } x \geq 0 .$$

Let $x = u$, a minimal element of P . Then, we see at once that u is an eigenvector for T . Denoting the corresponding eigenvalue by $\lambda(T)$, we have $Tu = \lambda(T)u$, holding for all $T \in \mathfrak{A}$. But, it then follows that $T \rightarrow \lambda(T)$ is a homomorphism of \mathfrak{A} onto the real field k ; for, given T_1 and T_2 , we have

$$\begin{aligned} \lambda(T_1 T_2)u &= T_1 T_2(u) \\ &= T_1(\lambda(T_2)u) \\ &= \lambda(T_1)\lambda(T_2)u . \end{aligned}$$

Unfortunately, except in unusual cases, P will not have any minimal elements. Let us go over to the adjoint algebra $\mathfrak{A}^* \subset \mathcal{L}(V')$ consisting of all operators T^* for $T \in \mathfrak{A}$. T^* is defined on V' , the dual space of V , by:

$$(7) \quad T^*(L)(x) = L(Tx) \quad \begin{array}{l} \text{all } L \in V' \\ \text{all } x \in V, \end{array}$$

and the mapping $T \rightarrow T^*$ is an anti-isomorphism of \mathfrak{A} onto \mathfrak{A}^* . From (7) and (5), we see that if T obeys (4), then

$$(8) \quad -rL \leq T^*(L) \leq rL \quad \text{all } L \geq 0.$$

Thus, \mathfrak{A}^* is an algebra of order-bounded operators on the partially ordered space V' . By Lemma 1, since P was assumed to have an order unit e , there are many minimal elements L_0 in P' .

Let D be the convex cross-section of P' consisting of all $L \geq 0$ with $L(e) = 1$. Each extreme point of D is a minimal positive element in P' and generates an extremal ray; let Γ be the closure of the set of extreme points in D , in the w^* topology arising from the natural norm topology on V . By the simple argument given above, each $L_0 \in \Gamma$ yields a real homomorphism λ_{L_0} of \mathfrak{A}^* , defined by the equation

$$T^*(L_0) = \lambda_{L_0}(T^*)L_0.$$

Since \mathfrak{A}^* is (anti) isomorphic to \mathfrak{A} , λ_{L_0} in turn defines a real homomorphism h_{L_0} of \mathfrak{A} ; using (7), this takes the explicit form:

$$(10) \quad L_0(Tx) = h_{L_0}(T)L_0(x) \quad \begin{array}{l} \text{all } x \in V \\ \text{all } T \in \mathfrak{A} \\ \text{all } L_0 \in \Gamma \end{array}$$

By Lemma 1, the functionals L_0 separate V so that the collection of homomorphisms h_{L_0} separate \mathfrak{A} . We may conclude that \mathfrak{A} is isomorphic to a product of fields k , and is therefore commutative; this proves the corollary.

To complete the proof of Theorem 1, we examine (10). We first represent V in $C[\Gamma]$, mapping x onto $\theta(x) = \hat{x}$ where $\hat{x}(L_0) = L_0(x)$ for all $L_0 \in \Gamma$. Since $L_0(e) = 1$ for all L_0 , \hat{e} is the constant function 1; in fact, the mapping θ is one-to-one and order preserving. For fixed $T \in \mathfrak{A}$, define a function ϕ on Γ by

$$(11) \quad \phi(L_0) = h_{L_0}(T).$$

Let $Tx = y$; then, (10) can be rewritten as:

$$(12) \quad \hat{y}(L_0) = \phi(L_0)\hat{x}(L_0).$$

The representation θ is such that every order-bounded operator T is carried into a multiplication operator on $C[\Gamma]$, and the correspondence is an isomorphism of \mathfrak{A} with a subalgebra of $\mathcal{L}(C[\Gamma])$, and in fact, with a subalgebra of $C[\Gamma]$ itself.

3. The Krein-Stone theorem. Before removing the assumption that V possesses an order unit e , we insert an immediate application

of our results. (See Stone [14], Krein [10], Kadison [8]).

THEOREM 2. *Let A be a real algebra with unit e and having a partial order such that if $x \geq 0, y \geq 0$, then $x + y \geq 0$ and $xy \geq 0$. Assume further that, as a linear space, A obeys restrictions (1), (2), (3) and (5). Then, A is commutative and can be represented as a subalgebra of a function algebra $C[X]$.*

Proof. Consider the left regular representation of A . This sends $a \in A$ into the operator $U_a \in \mathcal{L}(A)$ where $U_a(x) = ax$ for all $x \in A$. Since A has a unit, this is an isomorphism of A onto a subalgebra $\bar{A} \subset \mathcal{L}(A)$. By virtue of (5), we can choose r depending upon a so that $-re \leq a \leq re$. If $x \geq 0$, then $-rx \leq ax \leq rx$ so that U_a is an order bounded operator on the linear space $\langle A, + \rangle$. Hence, $\bar{A} \subset \mathfrak{A}$, and since this is a commutative algebra, so is A .

As a matter of fact, it is not necessary in this proof to assume that A is even associative, since this too can be deduced from the representation. Since $U_a U_b = U_b U_a$, it follows that $a(bx) = b(ax)$ for all $x \in A$; with $x = e$, we find that A is commutative. Then, $a(bc) = a(cb)$ while $b(ac) = (ac)b$ and A is associative.

Conversely, we note that Corollary 1 follows from Theorem 2, since \mathfrak{A} itself is an ordered algebra, with I as unit.

Other proofs which have been given for this result rely upon the construction of appropriate real homomorphisms h of A . These are linear functionals on $\langle A, + \rangle$ which are multiplicative and obey $h(e) = 1$. It is natural to look for these among the extreme points of an appropriate convex set D in the dual space of $\langle A, + \rangle$. Since any finite set of distinct real homomorphisms of A are linearly independent, the collection of h are precisely the extreme points of the convex set D_0 which they generate. Unfortunately, we cannot obtain D_0 directly. Instead, one selects a $D \supset D_0$, easily described, and then proves $D = D_0$. For example, the method adopted in Tate [15], Kadison [8] and Kelley [9] is to select D as all functionals L on $\langle A, + \rangle$ such that $L(e) = 1$ and $L(x^2) \geq 0$ for all $x \in A$. We note that the proof of $D = D_0$ depends strongly upon the hypotheses on A ; one can construct a finite dimensional algebra B for which D is a closed disc, having a circle for its extreme points, but such that B has no proper real homomorphisms.

4. Reduction of the general case. Suppose now that V is not assumed to satisfy (5). This is true for example, of the space $C_0[R]$ of functions with compact support, continuous on the real line R . We reduce this case to the previous one. Let e be an element in P and form

$$(13) \quad V(e) = \{\text{all } x \in V \text{ such that for some } \lambda, -\lambda e \leq x \leq \lambda e\}.$$

This is a linear subspace of V ; it inherits a partial order from V , and in its positive cone $P \cap V(e)$, the element e is an order unit. Suppose that $T \in \mathfrak{A}$. Then, from (4), if $x \in V(e)$, then for the appropriate λ , we have

$$-3\lambda re \leq Tx \leq 3\lambda re .$$

Thus, $V(e)$ is left invariant under all operators $T \in \mathfrak{A}$. Accordingly, if we restrict \mathfrak{A} to $V(e)$, we obtain a representation of \mathfrak{A} in $\mathcal{L}(V(e))$. Applying Theorem 1 to the resulting algebra, we find that \mathfrak{A} is commutative in its action on $V(e)$, and also obtain a representation (homomorphic) of \mathfrak{A} as multiplication operators on an appropriate function space $C[\Gamma_e]$. Finally, as e ranges over P , the subspaces $V(e)$ cover V , and we have proved the following result:

THEOREM 3. *Let V be a partially ordered linear space obeying (1), (2) and (3), but not necessarily (5). Let \mathfrak{A} be its algebra of order bounded operators. Then, \mathfrak{A} is commutative, and corresponding to any positive element e in V , there is a compact set Γ_e , an order preserving linear representation θ of $V(e)$ into $C[\Gamma_e]$ and a homomorphism $\bar{\theta}$ of \mathfrak{A} into the multiplication operators on $C[\Gamma_e]$ such that $\theta(Tx) = \bar{\theta}(T)\theta(x)$ for all $x \in V(e)$ and $T \in \mathfrak{A}$.*

A footnote to this is in order. Although we have shown that the algebra \mathfrak{A} is commutative, we have not shown that it need contain more than the multiples of the identity operator I . This can in fact, happen, although it does not in most of the interesting cases discussed in the next section. A glance at the finite dimensional case will be helpful. Let P be a polyhedral cone in n -space, and let u_1, u_2, \dots, u_N generate its extremal rays. Each u_j is an eigenvector for all the order bounded operators $T \in \mathfrak{A}$, and in turn generates real homomorphisms h_j of \mathfrak{A} , with

$$T(u_j) = h_j(T)u_j .$$

Suppose that the $\{u_j\}_1^N$ are such that $N > n$ and every set of n is independent. Then, it follows that all the h_j coincide on \mathfrak{A} . Since together they define a faithful representation of \mathfrak{A} , we conclude that \mathfrak{A} consists exactly of the scalar multiples of I . In contrast, if $N = n$, and the u_j form a basis, then \mathfrak{A} becomes the algebra of diagonal matrices; these, of course, are the multiplication operators in this representation.

5. Examples. In this section, we give a number of interesting illustrations of Theorem 3, together with a counterexample to show the necessity of the assumption that P is a linearly closed cone.

First, choose V as the space $C_0[X]$ of all real valued continuous functions on the locally compact space X which vanish at infinity. With

the usual ordering ($f \geq 0$ means $f(p) \geq 0$ for all $p \in X$) this is a partially ordered linear space satisfying the hypotheses of Theorem 3. Note in particular that $C_0[X]$ does not have an order unit. What are the order bounded operators on $C_0[X]$? Applying Theorem 3, we choose any $e \geq 0$ in $C_0[X]$ and form the subspace $V(e)$. By (13), $f \in V(e)$ if and only if f/e is a bounded function on X . Thus, $V(e)$ is isomorphic to the space of bounded continuous functions on the open support O_e of e . The set Γ_e is the Čech compactification of O_e , which contains O_e densely. Any point $p \in O_e$ defines a minimal functional L_p on $V(e)$ so that by (10) and (12),

$$(14) \quad L_p(Tf) = (Tf)(p) = \phi(p)f(p)$$

for all $p \in O_e$ and any $T \in \mathfrak{A}$. If X is σ -compact, we can take e so that $O_e = X$, and we find that the only order bounded transformations on $C_0[X]$ are those defined as point-wise multiplication by *bounded* continuous functions ϕ on X . If X is not σ -compact, we arrive at the same conclusion by varying e .

We note that if V is $C[X]$ itself, a simple and direct characterization of the order bounded operators is available. Using the fact that if $f(p_0) = 0$, then we may write $f = f_1 - f_2$ where $f_i \geq 0$ and $f_i(p_0) = 0$, it readily follows from the characteristic property of T that $(Tf)(p_0) = 0$. Applying this to $f = g - g(p_0)$, we have $Tg = \phi g$ where $\phi = T(1)$.

Another interesting special case is obtained by taking V as the space H of all bounded harmonic functions on an open domain Ω . The constant function is an order unit for H so that we do not need the full machinery of Theorem 3. The extremal rays in P are generated by the R. S. Martin minimal functions (see Brelot [3]) and H is represented as a subspace of the space of continuous functions on the ideal boundary Γ of Ω . The order bounded transformations are represented in turn as $C[\Gamma]$ itself; for any $T \in \mathfrak{A}$, Tf is the harmonic function $g \in H$ which is described by the (abstract) Dirichlet problem $g|_{\Gamma} = \phi f|_{\Gamma}$ where ϕ is the function in $C[\Gamma]$ corresponding to T . Note that T is not a multiplication on Ω itself. With Ω chosen as the unit disc and $\phi(x, y) = x$, we have $T(1) = x$, $T(y) = xy$, but $T(x) = (1/2)\{x^2 - y^2 + 1\}$, and $T(xy) = (1/4)\{3x^2y - y^3 + y\}$.

A somewhat more complicated illustration is provided by the space $C[X: E]$ of all bounded functions f on a locally compact space X with values in a fixed partially ordered linear space E . We order this by saying $f \geq g$ when $f(p) \geq g(p)$ for all $p \in X$. We shall also assume that E has an order unit e and require that each f be continuous when E is given the norm topology associated with e . If $v \in E$, denote by \bar{v} the constant function on X with value v . Note that \bar{v} is then an order unit for $C[X: E]$. To apply Theorem 3, we must determine minimal functionals in the dual space of V . We can find one associated with each point

$p_0 \in X$ and any minimal functional θ on E ; define L_0 on $C[X: E]$ by $L_0(f) = \theta(f(p_0))$. The following argument proves that L_0 is indeed minimal. Suppose $0 \leq L \leq L_0$. Then, for any $v \geq 0$ in E , $0 \leq L(\bar{v}) = \theta(v)$. Thus, $v \rightarrow L(\bar{v})$ is a positive linear functional on E which is dominated by θ . Since θ is minimal on E , there is a constant ρ such that $L(\bar{v}) = \rho\theta(\bar{v}) = \rho L_0(\bar{v})$ for all $v \geq 0$ in E (and thus for all $v \in E$). Suppose now that $f \in C[X: E]$ with $f(p) \leq f(p_0)$ for all $p \in X$; we shall say that such a function f takes a maximum value at p_0 and that $f \in \mathcal{F}_{p_0}$. Setting $v = f(p_0)$, we have $\bar{v} - f \geq 0$ so that $0 \leq L(\bar{v} - f) \leq L_0(\bar{v} - f)$. But, $L_0(\bar{v} - f) = \theta(v - f(p_0)) = 0$ so that $L(f) = L(\bar{v}) = \rho L_0(\bar{v}) = \rho L_0(f)$. Thus, $L = \rho L_0$ on the linear span of the special class \mathcal{F}_{p_0} . Consider now a general function $F \in C[X: E]$; since F is bounded, $\|F(p)\| \leq M$ for all $p \in X$. Define g, g_1 , and g_2 on X by:

$$\begin{aligned} g(p) &= F(p) - F(p_0) \\ g_1(p) &= \frac{1}{2}\{2\|g(p)\|e + g(p)\} & p \in X. \\ g_2(p) &= \frac{1}{2}\{2\|g(p)\|e - g(p)\} \end{aligned}$$

One sees that $g_i \geq 0$ and $g_i(p_0) = 0$, with $\|g_i(p)\| \leq 3M$ for all $p \in X$. Moreover,

$$g(p) = \{4M - g_2(p)\} - \{4M - g_1(p)\}$$

for all $p \in X$, so that $g \in \mathcal{F}_{p_0} - \mathcal{F}_{p_0}$. We conclude that $L(F) = \rho L_0(F)$, so that L_0 is indeed a minimal positive functional on $C[X: E]$.

Let Γ be the set of extreme points in the set D of functionals α on E with $\alpha \geq 0$ and $\alpha(e) = 1$. Applying Theorem 3, we find that any order bounded operator T has the property that

$$(15) \quad \alpha(T(f)(p_0)) = \alpha(T(\bar{v})(p_0))\alpha(f(p_0))$$

for all $f \in C[X: E]$, $p_0 \in X$ and $\alpha \in \Gamma$. If we represent the functions f in $C[X: E]$ as functions f on $X \times \Gamma$, then

$$\bar{\theta}(Tf)(p, \alpha) = \phi(p, \alpha)f(p, \alpha)$$

for all (p, α) .

The original space $C[X: E]$ is not an algebra, but is a module over the algebra $C[X]$. Formula (9) shows immediately that any order bounded transformation on $C[X: E]$ is in fact *algebraic*. If $\psi \in C[X]$ and $f \in C[X: E]$, then $T(\psi f) = \psi T(f)$. For,

$$\begin{aligned} \alpha(T(\psi f)(p)) &= \phi(p, \alpha)\alpha(\psi(p)f(p)) \\ &= \psi(p)\phi(p, \alpha)\alpha(f(p)) \\ &= \psi(p)\alpha(T(f)(p)) \\ &= \alpha(\psi(p)T(f)(p)) \end{aligned}$$

for each $p \in X$ and $\alpha \in \Gamma$.

Finally, we use a familiar example to show that the most crucial hypothesis on the partially ordered linear space V in Theorem 1 and 3 is that P be *linearly closed*. Take for V the space of all polynomials, with the ordering: $a_0 + a_1x + \cdots + a_mx^m > 0$ if $a_m > 0$. P satisfies the first and second requirements, but is not linearly closed; in fact

$$\lambda(x^2) + (1 - \lambda)(-x) \in P \quad \text{only if } \lambda > 0.$$

There is no order unit. We can still introduce the algebra \mathfrak{A} of order bounded transformations on V . It is easy to see, however, that \mathfrak{A} is *not* commutative. Let T be defined on V by $T(x^n) = q_n$ where q_n is a polynomial of degree less than n . Then, $I \pm T \geq 0$ so that $T \in \mathfrak{A}$. In particular, $T_1 = x(d^2/dx^2)$ and $T_2 = d/dx$ are in \mathfrak{A} ; however, $T_1T_2 \neq T_2T_1$. In this example, the reason for this can be traced to the fact that P is so large that there are too many positive linear operators on V , (and no non-degenerate positive linear functionals).

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