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It is known [2, p. 208] that if a locally compact group acts effectively and differentiably on E_n then it is a Lie group. The object of this note is to show that if the differentiability requirements are replaced by some weaker restrictions, given later on, the theorem is still true. Let G be a locally compact group acting on E_n and let the coordinate functions of the action be given by $f_i(g, x_1, \cdots, x_n)$, $1 \leqq i \leqq n$. For economy we introduce the following notation

$$Q_{ij}(g, t, x) = \frac{f_i(g, x_1, \cdots, x_j + t, \cdots, x_n) - f_i(g, x_1, \cdots, x_j, \cdots, x_n)}{t} .$$

We denote by $\sigma(Q_{ij}(e, 0, x))$ the oscillation of $Q_{ij}(g, t, x)$ at the point $(e, 0, x)$.

Before proceeding there is one simple remark to be made on matrices. If $A = (a_{ij})$ is an $n \times n$ matrix such that $|a_{ij} - \delta_{ij}| < (1/n)$ then A is non-singular. If A were singular there would be a vector x such that $\sum_i x_i^2 = 1$ and $Ax = 0$. From the Schwarz inequality it follows that $x_i^2 = (\sum_j (a_{ij} - \delta_{ij})x_j)^2 < (1/n)$ and consequently $1 = \sum x_i^2 < 1$ which is impossible. If $|a_{ij} - \delta_{ij}| \leqq (\alpha/n)$, where $0 < \alpha < 1$, then the determinant of A is bounded away from zero since the determinant is a continuous function and the set $\{a_{ij}: |a_{ij} - \delta_{ij}| \leqq (\alpha/n)\}$ is compact in E_{n^2}.

THEOREM 1. *If T is a pointwise periodic homeomorphism of E_n then T is periodic.*

Proof. [2, p. 224.]

THEOREM 2. *If G is a compact, zero dimensional, monothetic group acting effectively on E_n and satisfying*

$$(*) \qquad \sigma(Q_{ij}(e, 0, x)) < \frac{\varepsilon}{n}, \quad 0 < \varepsilon < 1, \quad \textit{for each } x \textit{ in } E_n;$$

then G is a finite cyclic group.

Proof. Since G is monothetic, let a be an element whose powers are dense in G. It is enough to show that there is a power of a which leaves E_n pointwise fixed since the action of G is effective.

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If q is a positive integer we let

$$T_i^q(g, x) = x_i + f_i(g, x) + \cdots + f_i(g^{q-1}, x) .$$

If $y = (y_i)$ and $x = (x_i)$ let

$$T_{ij}^q(g, x, y) = \frac{T_i^q(g, x_1, \cdots, x_{j-1}, y_j, \cdots, y_n) - T_i^q(g, x_1, \cdots, x_j, y_{j+1}, \cdots, y_n)}{y_j - x_j}$$

for $y_j \neq x_j$ and zero otherwise. If we let $y = f(g, x)$ then we obtain

$$\begin{aligned} f_i(g^q, x) - x_i &= T_i^q(g, y) - T_i^q(g, x) \\ &= \sum_{j=1}^{n} T_{ij}^q(g, x, y)(y_j - x_j) \\ &= q \cdot \sum_{i=1}^{n} \frac{1}{q} T_{ij}^q(g, x, y)(y_j - x_j) . \end{aligned}$$

Because of the fact that $f_i(e, x) = x_i$ and because of (*) it follows that there is a compact neighborhood $U(x)$ of the identity of G such that if $g, \cdots, g^q \in U(x)$ then $|(1/q)T_{ij}^q(g, x, y) - \delta_{ij}| \leqq (\alpha/n)$, $0 < \varepsilon < \alpha < 1$. It follows that if T is the matrix with entries $(1/q)T_{ij}^q(g, x, y)$ then T is non-singular and its determinant is bounded away from zero uniformly in q, so the determinant of the inverse is bounded uniformly in q; thus

$$(f(g, x) - x) = (y - x) = \left(\delta_{ij}\frac{1}{q}\right) \cdot T^{-1} \cdot (f(g^q, x) - x) .$$

Since G is monothetic and zero dimensional there is a power of a such that if $g = a^p$ then all the powers of g lie in $U(x)$. Since $U(x)$ is compact it follows that the vectors $f(g^q, x) - x$ are bounded uniformly in q and thus $f(g, x) - x = f(a^p, x) - x = 0$. Hence a is pointwise periodic on E_n and it follows from Theorem 1 that it is periodic and consequently has a power leaving E_n pointwise fixed.

From this it follows quickly that if G is a locally compact group acting effectively on E_n and satisfying (*) then it is a Lie group. This follows from the fact that since G is effective it must be finite dimensional [1] and then if G is not a Lie group it must contain a compact, non-finite zero dimensional subgroup H [2, p. 237] which acts effectively. H has small subgroups which act effectively and it follows from Newman's theorem [3, 4] that H cannot have arbitrarily small elements of finite order. Thus H has an element a of infinite order such that the compact subgroup generated by a acts effectively on E_n and satisfies (*) but by Theorem 2 this is impossible.

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