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ON THE ACTION OF A LOCALLY COMPACT GROUP ON E_n

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It is known [2, p. 208] that if a locally compact group acts effectively and differentiably on E_n then it is a Lie group. The object of this note is to show that if the differentiability requirements are replaced by some weaker restrictions, given later on, the theorem is still true. Let G be a locally compact group acting on E_n and let the coordinate functions of the action be given by $f_i(g, x_1, \dots, x_n)$, $1 \le i \le n$. For economy we introduce the following notation

$$Q_{ij}(g,t,x) = \frac{f_i(g,x_1,\cdots,x_j+t,\cdots,x_n) - f_i(g,x_1,\cdots,x_j,\cdots,x_n)}{t}.$$

We denote by $\sigma(Q_{ij}(e,0,x))$ the oscillation of $Q_{ij}(g,t,x)$ at the point (e,0,x).

Before proceeding there is one simple remark to be made on matrices. If $A=(a_{ij})$ is an $n\times n$ matrix such that $|a_{ij}-\delta_{ij}|<(1/n)$ then A is non-singular. If A were singular there would be a vector x such that $\sum_i x_i^2=1$ and Ax=0. From the Schwarz inequality it follows that $x_i^2=(\sum_j (a_{ij}-\delta_{ij})x_j)^2<(1/n)$ and consequently $1=\sum_i x_i^2<1$ which is impossible. If $|a_{ij}-\delta_{ij}|\leq (\alpha/n)$, where $0<\alpha<1$, then the determinant of A is bounded away from zero since the determinant is a continuous function and the set $\{a_{ij}:|a_{ij}-\delta_{ij}|\leq (\alpha/n)\}$ is compact in E_{n2} .

THEOREM 1. If T is a pointwise periodic homeomorphism of E_n then T is periodic.

Proof. [2, p. 224.]

THEOREM 2. If G is a compact, zero dimensional, monothetic group acting effectively on E_n and satisfying

(*)
$$\sigma(Q_{ij}(e, 0, x)) < \frac{\varepsilon}{n}$$
, $0 < \varepsilon < 1$, for each x in E_n ;

then G is a finite cyclic group.

Proof. Since G is monothetic, let a be an element whose powers are dense in G. It is enough to show that there is a power of a which leaves E_n pointwise fixed since the action of G is effective.

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If q is a positive integer we let

$$T_i^q(g, x) = x_i + f_i(g, x) + \cdots + f_i(g^{q-1}, x)$$
.

If $y = (y_i)$ and $x = (x_i)$ let

$$T_{ij}^{q}(g, x, y) = \frac{T_{i}^{q}(g, x_{1}, \dots, x_{j-1}, y_{j}, \dots, y_{n}) - T_{i}^{q}(g, x_{1}, \dots, x_{j}, y_{j+1}, \dots, y_{n})}{y_{j} - x_{j}}$$

for $y_j \neq x_j$ and zero otherwise. If we let y = f(g, x) then we obtain

$$egin{align} f_i(g^q,x) - x_i &= T_i^q(g,y) - T_i^q(g,x) \ &= \sum_{j=1}^n T_{ij}^q(g,x,y) (y_j - x_j) \ &= q \cdot \sum_{i=1}^n rac{1}{q} \ T_{ij}^q(g,x,y) (y_j - x_j) \ . \end{split}$$

Because of the fact that $f_i(e,x)=x_i$ and because of (*) it follows that there is a compact neighborhood U(x) of the identity of G such that if $g, \dots, g^q \in U(x)$ then $|(1/q)T_{ij}^q(g,x,y)-\delta_{i,j}| \leq (\alpha/n), \ 0 < \varepsilon < \alpha < 1$. It follows that if T is the matrix with entries $(1/q)T_{ij}^q(g,x,y)$ then T is non-singular and its determinant is bounded away from zero uniformly in q, so the determinant of the inverse is bounded uniformly in q; thus

$$(f(g, x) - x) = (y - x) = \left(\delta_{ij} \frac{1}{g}\right) \cdot T^{-1} \cdot (f(g^q, x) - x).$$

Since G is monothetic and zero dimensional there is a power of a such that if $g=a^p$ then all the powers of g lie in U(x). Since U(x) is compact it follows that the vectors $f(g^a, x) - x$ are bounded uniformly in g and thus $f(g, x) - x = f(a^p, x) - x = 0$. Hence g is pointwise periodic on g and it follows from Theorem 1 that it is periodic and consequently has a power leaving g pointwise fixed.

From this it follows quickly that if G is a locally compact group acting effectively on E_n and satisfying (*) then it is a Lie group. This follows from the fact that since G is effective it must be finite dimensional [1] and then if G is not a Lie group it must contain a compact, non-finite zero dimensional subgroup H [2, p. 237] which acts effectively. H has small subgroups which act effectively and it follows from Newman's theorem [3, 4] that H cannot have arbitrarily small elements of finite order. Thus H has an element a of infinite order such that the compact subgroup generated by a acts effectively on E_n and satisfies (*) but by Theorem 2 this is impossible.

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