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ON SIMILARITY INVARIANTS OF CERTAIN OPERATORS IN  
 $L_p$

G. K. KALISCH

# ON SIMILARITY INVARIANTS OF CERTAIN OPERATORS IN $L_p$

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The purpose of this paper is to extend the result of Corollary, Theorem 2 of the author's paper on Volterra operators (Annals of Math., 66, 1957, pp. 481-494 quoted as  $A$ ; we shall use the definitions and notations of that paper) to the most general situation applicable: We are dealing with operators  $T_F$  where  $F(x, y) = (y - x)^{m-1} aG(x, y)$  is a function defined on the triangle  $0 \leq x \leq y \leq 1$ , where  $m$  is a positive integer,  $a$  a complex number of absolute value 1,  $G$  is a complex valued function which is continuously differentiable and  $G(x, x)$  is positive real. We recall that if  $f \in L_p [0, 1]$ , then  $(T_F)(f)(x) = \int_x^1 F(x, y)f(y)dy$  is again in  $L_p [0, 1]$ . The only difference from  $A$  is the presence of the constant  $a$  which affects none of results except Theorem 2 and its Corollary. Theorems 1 and 2 of the present paper fill the gap. Theorem 3 shows that differentiability conditions imposed on  $F$  cannot be abandoned entirely—and also that the integral equation (1) of  $A$  cannot be solved unless  $K$  (which corresponds to our  $F$ ) has at least first derivatives near  $y = x$ .

If  $c$  is constant and  $E$  is the function identically equal to 1, we define  $T_E^c$  as  $T_H$  which  $H(x, y) = (y - x)^{c-1}/\Gamma(c)$  (fractional integration of order  $c$ ).

**THEOREM 1.** *Let  $c_1$  and  $c_2$  be complex numbers and let  $r_1$  and  $r_2$  be real numbers such that  $r_i \geq 1$ , then  $c_1 T_E^{r_1}$  is similar to  $c_2 T_E^{r_2}$  if and only if  $c_1 = c_2$  and  $r_1 = r_2$ .*

*Proof.* The first part of the Proof of Theorem 2 of  $A$  applies and implies that  $r_1 = r_2$  ( $= r$ ) and  $|c_1| = |c_2|$ . Thus suppose that  $c_1 T_E^r$  is similar to  $c_2 T_E^r$  or that  $c T_E^r$  is similar to

$$(1) \quad T_E^r = P c T_E^r P^{-1} \text{ for } |c| = 1$$

where  $P$  is a bounded linear transformation of  $L_p [0, 1]$  onto itself with the bounded linear inverse  $P^{-1}$ . If  $T$  is similar to  $S = P T P^{-1}$ , then  $f(T)$  is similar to

$$(2) \quad f(S) = P f(T) P^{-1}$$

for polynomials and even analytic functions  $f$ . Let

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$$f(z) = \sum_{i=0}^{\infty} a_i z^{i+1}$$

Then

$$f(cT_B^r) = \sum_{i=0}^{\infty} a_i c^{i+1} T_B^{r(i+1)} = T_{g_1(y-x)}$$

where  $g_1(t) = ct^{r-1}g(ct^r)$  where we have written  $t$  for  $y - x$  and where

$$g(z) = \sum_{i=0}^{\infty} b_i z^i$$

with  $b_i = a_i/\Gamma(r(i + 1))$ . Equations (1) and (2) imply that  $\|f(T_B^r)\| \leq \|P\| \|P^{-1}\| \|f(cT_B^r)\|$ . The definition of the norm of a linear transformation in a Banach space implies the following inequality:

$$\|f(T_B^r)\| = \|T_{ct^{r-1}g(ct^r)}\| \geq \left\| \int_x^1 (y - x)^{r-1} g((y - x)^r) k(y) dy \right\|_p$$

for all  $k \in L_p [0, 1]$  such that  $\|k\|_p = 1$ . On the other hand, Lemma 2 of *A* implies that

$$\|T_{ct^{r-1}g(ct^r)}\| \leq \|ct^{r-1}g(ct^r)\|_1 = \|t^{r-1}g(ct^r)\|_1.$$

Thus if  $k(y) = 1$ , we obtain

$$\begin{aligned} L &= \left\| \int_x^1 (y - x)^{r-1} g((y - x)^r) dy \right\|_p \leq \|f(T_B^r)\| \\ (3) \quad &\leq \|P\| \|P^{-1}\| \|f(cT_B^r)\| \\ &\leq \|P\| \|P^{-1}\| \|t^{r-1}g(ct^r)\|_1 = R. \end{aligned}$$

We shall find a family of functions  $g_v$  (and correspondingly  $f_v$ ) depending on a positive parameter  $v$  such that if we use the notations  $L_v$  and  $R_v$  for the corresponding left and right hand sides of (3),  $L_v \rightarrow \infty$  and  $R_v \rightarrow 0$  as  $v \rightarrow \infty$  contradicting the inequality (3): this contradiction then proves our theorem.

Let us first consider the case where the real part of  $c$ ,  $Re(c)$ , is less than 0. Let  $g_v(t) = \exp(vt)$ . Since  $T_B^r$  is generalized nilpotent for  $r \geq 1$ , the corresponding function  $f_v(T_B^r)$  exists and (1) indeed implies (2) for  $S = T_B^r$  and  $T = cT_B^r$ . Then

$$R_v = \|t^{r-1}g_v(ct^r)\|_1 = \int_0^1 |t^{r-1} \exp(vct^r)| dt$$

and  $R_v \rightarrow 0$  as  $v \rightarrow \infty$ . On the other hand

$$L_v = (1/r^p) \int_0^1 (\exp(v(1-x)) - 1/v)^p dx \rightarrow \infty$$

as  $v \rightarrow \infty$ . If finally  $Re(c) \geq 0$  and  $c \neq 1$ , then there exist a positive

integer  $n$  such that  $Re(c^n) < 0$ . But then (1) implies that  $c^n T_E^{nr}$  is similar to  $T_E^{nr} = P c^n T_E^{nr} P^{-1}$  which contradicts the preceding result and the proof of the theorem is complete.

**THEOREM 2.** *Let  $F(x, y) = (y - x)^{m-1} a G(x, y)$  satisfy, in addition to the general hypotheses stated above, one of the following:*

- (1)  $G$  is analytic in a suitable region and  $m$  is arbitrary;
- (2)  $G(x, y) = G(y - x)$ ,  $G(0) \neq 0$ ,  $G \in C^2$  and  $m$  is arbitrary;
- (3)  $G \in C^2$  and  $m = 1$ . Let  $A$  be a complex number. Then  $AI + T_F$  and  $AI + T_F^*$  are similar to the unique operator  $AI + caT_E^m$  and  $AI + c\bar{a}T_E^m$  respectively where  $c = \left(\int_0^1 (G(u, u)^{1/m} du)\right)^m$ .

Here  $I$  is the identity operator and  $T_K^*$ , the adjoint of  $T_K$ , is defined by

$$(T_K^*)(f)(x) = \int_0^x \overline{K(y, x)} f(y) dy .$$

*Proof.* Note first that  $A$  implies that  $AI + T_F$  is similar to  $AI + caT_E^m$  and that  $AI + T_F^*$  is similar to  $AI + c\bar{a}T_E^{*m}$  (see Cor. Theorem 2 of A). Observe next that  $T_E^* f(x) = \int_0^x f(y) dy$  and

$$T_E^{*m} f(x) = (1/\Gamma(m)) \int_0^x (x - y)^{m-1} f(y) dy$$

and that if  $(S_{1-x} f)(x) = f(1 - x)$  then  $S_{1-x}$  is an isometry of  $L_p[0, 1]$  onto itself and  $S_{1-x} T_E^m S_{1-x}^{-1} = T_E^{*m}$ . It remains to show uniqueness. Suppose that  $A_1 I + c_1 a_1 T_E^{m_1}$  is similar to  $A_2 I + c_2 a_2 T_E^{m_2}$ . Then  $A_1 = A_2$  (because of the complete continuity of  $T_E$ ) and  $c_1 a_1 T_E^{m_1}$  is similar to  $c_2 a_2 T_E^{m_2}$  which by Theorem 1 implies that  $c_1 = c_2$ ,  $a_1 = a_2$ ,  $m_1 = m_2$ .

**THEOREM 3.** *The linear transformation  $T_E + T_E^{1+a}$  where  $0 < a < 1$  of  $L_p[0, 1]$  into itself is not similar to any linear transformation  $cT_E^r$  for complex  $c$  and real  $r \geq 1$ .*

*Proof.* Preliminaries. 1. If two linear transformations  $S$  and  $T$  are similar, i.e., if there exists  $P$  such that  $S = PTP^{-1}$ , then there exists a constant  $K$  such that

$$(4) \quad 1/K \leq \| T^n \| \| S^n \| \leq K ,$$

for all positive integers  $n$ . It suffices to take  $K = \| P \| \| P^{-1} \|$ .

2. The following inequality is a consequence of the fact that if  $0 \leq F_1(x, y) \leq F_2(x, y)$  then  $\| T_{F_1} \| \leq \| T_{F_2} \|$ :

$$(5) \quad \|(T_E + T_E^{1+a})^n\| \geq n \|T_E^{n+a}\|$$

for all positive integers  $n$ .

3. Our next task is to find estimates for  $\|T_E^n\|$ . An estimate from above is the following:

$$(6) \quad \|T_E^n\| \leq 1/(n\Gamma(n)p^{1/p})$$

for all positive integers  $n$ . An estimate from below is furnished by the following Proposition:

Given the real positive number  $e$  there exists a positive number  $K = K(e)$  and a positive integer  $N = N(e)$  such that for all integers  $n \geq N$ ,

$$(7) \quad \|T_E^n\| \geq K/(n^{1+e}\Gamma(n)).$$

*Proof of (6).* If  $f \in L_p[0, 1]$ ,

$$T_E^n f(x) = \int_x^1 [(y - x)^{n-1}/\Gamma(n)]f(y)dy .$$

If  $(1/p) + (1/q) = 1$ , Hölder's inequality yields

$$\begin{aligned} \int_x^1 (y - x)^{n-1}f(y)dy &\leq \left(\int_x^1 (y - x)^{(n-1)q}dy\right)^{1/q} \|f\|_p \\ &= (1 - x)^{((n-1)q+1)/q} \|f\|_p / ((n - 1)q + 1)^{1/q} \end{aligned}$$

so that

$$\begin{aligned} \|T_E^n f\|_p^p &= \int_0^1 |(T_E^n f)(x)|^p dx \\ &= (1/\Gamma(n))^p \int_0^1 \left| \int_x^1 (y - x)^{n-1}f(y)dy \right|^p dx \\ &\leq (1/\Gamma(n))^p (1/((n - 1)q + 1)^{p/q}) \int_0^1 (1 - x)^{((n-1)p+(p/q))} dx \|f\|_p^p \\ &= (1/\Gamma(n))^p (1/((n - 1)q + 1)^{p/q}) (1/((n - 1)p + (p/q) + 1)) \|f\|_p^p \end{aligned}$$

which implies that

$$\|T_E^n\| \leq (1/\Gamma(n))(1/((n - 1)q + 1)^{1/q})(1/((n - 1)p + (p/q) + 1)^{1/p})$$

which in turn implies (6).

*Proof of (7).* We first observe that elementary considerations concerning the gamma function imply that given  $c$  such that  $0 < c < 1$  and given a positive real number  $d$  there exists an integer  $N$  depending on  $c$  and  $d$  such that for all integers  $n \geq N$

$$(8) \quad \Gamma(n + c) < (n + c)^{e+a}\Gamma(n) .$$

Consider next the function  $f(x) = r(1 - x)^{-s} \in L_p [0, 1]$  such that  $\|f\|_p = 1$ , i.e.,  $r^p = 1 - sp$  and  $0 < s < 1/p$ . Then

$$T_E^n f(x) = r\Gamma(1 - s)(1 - x)^{n-s}/\Gamma(n + 1 - s)$$

and

$$\|T_E^n\| \geq r\Gamma(1 - s)/\Gamma(n + 1 - s)(p(n - s) + 1)^{1/p} .$$

We now choose  $s$  (and hence  $r$ ) such that for the positive real number  $e$  of (7),  $0 < (1/p) - s < e$  and then we choose  $d$  such that  $0 < d < e + s - (1/p)$  and finally by virtue of (8) we obtain  $N$  as a function of  $e$  such that for all integers  $n \geq N$ ,  $\Gamma(n + 1 - s) < (n + 1 - s)^{1-s+a}\Gamma(n)$  whence

$$\|T_E^n\| \geq r\Gamma(1 - s)/(n + 1 - s)^{1-s+a}\Gamma(n)(p(n - s) + 1)^{1/p}$$

which upon choosing  $K = K(e)$  properly implies (7).

After these preliminaries, we turn to the proof of the theorem. We distinguish several cases. Let  $T = T_E + T_E^{1+a}$ .

*Case 1.*  $|c| \leq 1$ . Consider

$$h_n = \|(cT_E^n)^n\|/\|T^n\| \leq \|T_E^n\|/(n\|T_E^{n+a}\|)$$

where we have used (5) and the fact that  $r \geq 1$ . Take now positive real numbers  $e$  and  $d$  such that  $a + e + d < 1$ . Then there exists by (7) a positive constant  $K$  and an integer  $N$  such that for all integers  $n \geq N$

$$(9) \quad \begin{aligned} h_n &\leq (n + a)^{1+e}\Gamma(n + a)/(n^2\Gamma(n)p^{1/p}K) \\ &\leq (n + a)^{1+e+a+d}\Gamma(n)/(n^2\Gamma(n)p^{1/p}K) \end{aligned}$$

where we have made use of (8) and (6). The last inequality implies that  $h_n \rightarrow 0$  which in conjunction with (4) implies the truth of our theorem in the case under consideration.

*Case 2.*  $r < 1$ . Using the notations and making similar choices as under Case 1, (9) becomes

$$h_n \leq |c|^n(n + a)^{1+e+a+d}\Gamma(n)/(n^2r\Gamma(rn)p^{1/p}K)$$

which, since  $|c|^n\Gamma(n)/\Gamma(rn)$  is bounded (in fact converges to 0) for  $r > 1$  as  $n \rightarrow \infty$ , again proves the truth of the theorem in the present case.

*Case 3.*  $r = 1$ ,  $|c| > 1$ . This time we consider the quotient

$$\begin{aligned}
 k_n &= \| T^n \| \| (c T_E)^n \| \\
 (10) \quad &\leq \sum_{i=0}^n \binom{n}{i} \| T_E^{n+a(n-i)} \| \| (c |c|^n \| T_E^n \|) \\
 &\leq ((n^{1+e} \Gamma(n) / (|c|^n K p^{1/p})) \sum_{i=0}^n \binom{n}{i} / (\Gamma(n + a(n - i) + 1)) ,
 \end{aligned}$$

which is valid for sufficiently large  $n$ ; again we used (6) and (7).

In order to complete the proof of our theorem, we need the following fact:

Given any positive real number  $e$  and given the positive real number  $a < 1$ , there exists an integer  $N = N(e; a)$  such that for all integers  $i$  and  $n$  such that  $0 \leq i \leq n \leq N$

$$(11) \quad \Gamma(n) / \Gamma(n + a(n - i) + 1) \leq 2e^{n-i} .$$

*Proof.* The case  $i = 0$  results from elementary considerations about the gamma function. If  $i = 1$ , we find  $N_1$  so that (11) is valid for  $i = 0$  and  $n \geq N_1$ . We then find  $N_2$  so that (8) is true for some arbitrary but fixed  $d$ , for  $c = a$  and for  $n \geq N_2$ . Then  $\Gamma(n) / \Gamma(n + (n - 1)a + 1) \leq (\Gamma(n) / \Gamma(n + na + 1)) / (n + na + 1)^{a+a}$  which for  $n \geq \max(N_1, N_2, e^{-1/a}) = N_3$  implies (11) for  $i = 2$  and  $n \geq N_3$ . The remaining cases are settled by induction (except  $i = n$  which is obvious); note that we never have to go above  $N_3$  at any point. This completes the proof of (11).

The proof is now completed by substituting (11) into (10):

$$k_n \leq 2n^{1+e} (1 + e_1)^n / |c|^n K p^{1/p}$$

where  $e_1$  is the constant  $e$  of (11). Thus  $k_n \rightarrow 0$  upon proper choice of  $e_1$  and our theorem is again true in view of (4). This completes the proof of Theorem 3.

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