Pacific Journal of Mathematics

AN INEQUALITY FOR LOGARITHMIC CAPACITIES

HEINZ RENGGLI

Vol. 11, No. 1 November 1961

AN INEQUALITY FOR LOGARITHMIC CAPACITIES

HEINZ RENGGLI

1. Introduction. In his work on capacities, G. Choquet proved that for many capacities the inequality of strong subadditivity holds [1]. It is the purpose of this note to show that a similar inequality holds for logarithmic capacities. More precisely we shall prove the

THEOREM. *Let A and B be compact sets in the complex z-plane E. By $C(S)$ we denote the logarithmic capacity* [2] *of a given compact set S, $S \subset E$, where we agree to put $C(S) = 0$ whenever $S = \phi$. Then*

$$C(A \cup B) \cdot C(A \cap B) \leqq C(A) \cdot C(B) .$$

2. Proof of the theorem. Let S, $S \subset E$, be a compact set whose boundary consists of a finite number of analytic arcs. By S^* we denote that component of $E - S$ which is unbounded. Then Green's function of S^* is defined by the properties: it is harmonic in S^*, vanishes at the finite boundary points of S^* and has a logarithmic singularity at infinity. We will denote this function by $g_S(z, \infty)$.

First we shall deal with the case when the respective boundaries of A, B and $A \cap B$ consist of a finite number of non-degenerate analytic arcs. We remark that the difference $g_{A\cap B}(z, \infty) - g_A(z, \infty)$ is harmonic in A^*, $A^* \subset (A \cap B)^*$, and at infinity. It is furthermore non-negative on the boundary of A^* and hence non-negative in A^* by the maximum principle. Similarly $g_{A\cup B}(z, \infty) \geqq g_B(z, \infty)$ holds in B^*, $B^* \subset (A \cap B)^*$.

The function

$$h(z) = g_{A\cup B}(z, \infty) + g_{A\cap B}(z, \infty) - g_A(z, \infty) - g_B(z, \infty)$$

is harmonic in $(A \cup B)^*$ and at infinity. From $(A \cup B)^* = A^* \cap B^*$ it follows that the boundary points of $(A \cup B)^*$ belong either to the boundary of A^* or to the boundary of B^*. Therefore $g_{A\cup B}(z, \infty)$ and either $g_A(z, \infty)$ or $g_B(z, \infty)$ vanish at these boundary points. With the aid of the remark made above we get the result that $h(z)$ is non-negative in $(A \cup B)^*$.

Therefore

$$g_A(z, \infty) + g_B(z, \infty) \leqq g_{A\cup B}(z, \infty) + g_{A\cap B}(z, \infty)$$

holds in $(A \cup B)^*$. From this general inequality and using the fact that

Received December 16, 1959. Presented to the Amer. Math. Soc. in Jan. 1958, where a less elementary proof based on an inequality for extremal lengths was given.

$$\lim_{z\to\infty} \{g_S(z, \infty) - \log |z|\}$$

is the constant $\gamma(S)$ of Robin [2] we deduce

$$\gamma(A) + \gamma(B) \leqq \gamma(A \cup B) + \gamma(A \cap B) .$$

But

$$C(S) = \exp \{-\gamma(S)\}$$

by definition. Hence our theorem is proven for the special case.

The general case follows by the usual approximation techniques [2].

REFERENCES

1. G. Choquet, *Theory of capacities*, Ann. Inst. Fourier, Grenoble **5** (1953–54), 131–295.
2. R. Nevanlinna, *Eindeutige analytische Funktionen*, Grundlehren 46, 2. Aufl., Springer, 1953.

RUTGERS UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Reprinted 1966 in the United States of America

Pacific Journal of Mathematics
Vol. 11, No. 1 November, 1961