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In this paper we will answer Fuchs' PROBLEM 32 (a), and the corresponding part of his PROBLEM 33. (See [1], pg. 203.) The statements of these PROBLEMS are the following.

I. "Which are the torsion groups T that are endomorphic images of all groups containing them as maximal torsion subgroups?"

II. "Which are the torsion groups T such that a basic subgroup of T is an endomorphic image of any group G containing T as its maximal torsion subgroup?"

Actually, we will answer question II and the following question which is more general than I.

III. What groups H are endomorphic images of all groups G containing H such that G/H is torsion free?

The solutions will be effected by using some homological results of Harrison [2]. All groups considered here will be Abelian. The definitions and results stated in the remainder of this paragraph are due to Harrison, and may be found in [2]. A reduced group G is *cotorsion* if $\text{Ext}(A, G) = 0$ for all torsion free groups A. If H is a reduced group, then $\text{Ext}(Q/Z, H) = H'$ is cotorsion, where Q and Z denote the additive group of rationals and integers, respectively. Furthermore, H is a subgroup of H', (that is, there is a natural isomorphism of H into H') and H'/H is divisible torsion free. This implies, of course, that if T is a torsion reduced group, then T is the torsion subgroup of $T' = \text{Ext}(Q/Z, T)$.

Now it is easy to see that *if G is a group such that* $\text{Ext}(A, G) = 0$ *for all torsion free groups A, then any homomorphic image of G is the direct sum of a cotorsion group and a divisible group.* In fact, let H be a homomorphic image of G. This gives us an exact sequence

$$0 \to K \to G \to H \to 0$$

which yields the exact sequence

$$\begin{aligned} 0 \to \text{Hom}(A, K) \to \text{Hom}(A, G) \to \text{Hom}(A, H) \to \\ \text{Ext}(A, K) \to \text{Ext}(A, G) \to \text{Ext}(A, H) \to 0 \ . \end{aligned}$$

If A is any torsion free group, then $\text{Ext}(A, G) = 0$, and so $\text{Ext}(A, H) = 0$. Write $H = D \oplus L$, where D is the divisible part of H. Then L is reduced, and $0 = \text{Ext}(A, D \oplus L) \cong \text{Ext}(A, D) \oplus \text{Ext}(A, L) = \text{Ext}(A, L)$, so that L is cotorsion. Our assertion is proved.

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Now we are ready to give the solutions promised earlier. The following theorem settles III.

THEOREM. *The group H is an endomorphic image of every group G containing it such that G/H is torsion free if and only if $H = D \oplus C$, where D is divisible and C is cotorsion. This is equivalent to the assertion that H is a direct summand of every such G.*

Proof. Suppose H is an endomorphic image of every group G containing it such that G/H is torsion free. Let $H = D \oplus C$, where D is divisible and C is reduced. Then C is a subgroup of the cotorsion group $\operatorname{Ext}(Q/Z, C) = C'$ such that C'/C is torsion free, so that H is a subgroup of $D \oplus C' = H'$ such that H'/H is torsion free. Therefore H is an endomorphic image of H'. $\operatorname{Ext}(A, D \oplus C') = 0$ for all torsion free groups A, and as we have just proved, any homomorphic image of $D \oplus C'$ is the direct sum of a cotorsion and a divisible group. It follows that C must be cotorsion.

If $H = D \oplus C$, with D divisible and C cotorsion, then $\operatorname{Ext}(A, H) = 0$ for all torsion free groups A, and hence H is a direct summand of any group G containing it such that G/H is torsion free. If H is a direct summand of any such G, then clearly H is an endomorphic image of any such G. Thus our theorem is proved.

The torsion group T is a direct summand of every group containing it as its maximal torsion subgroup if and only if $T = D \oplus B$, with D divisible and B of bounded order. (See [1], pg. 187.) Thus, by our theorem, we see that *the torsion group T is an endomorphic image of every group containing it as its maximal torsion subgroup if and only if $T = D \oplus B$, with D divisible and B of bounded order.*

The solution of II goes as follows. Suppose a basic subgroup of T is an endomorphic image of every group G in which T is the maximal torsion subgroup. Let $T = D \oplus B$, with D divisible and B reduced. Then a basic subgroup of T must be an endomorphic image of $D \oplus B' = D \oplus \operatorname{Ext}(Q/Z, B)$. Therefore a basic subgroup of T must be cotorsion, since it is reduced, and since it is torsion, it is of bounded order. (See [1], pg. 187. The remark by Harrison in [2], pg. 371 is incorrectly worded.) Writing T as $D \oplus B$, we see that a basic subgroup of B is a basic subgroup of T. But any two basic subgroups of T are isomorphic, and if B has a basic subgroup of bounded order, then B must be of bounded order. In fact, the only basic subgroup of B is B itself. Thus $T = D \oplus B$, with D divisible and B of bounded order. If $T = D \oplus B$, with D divisible and B of bounded order, then B is a basic subgroup of T. Now $D \oplus B$, and hence B, is a direct summand of any G in which T is the maximal torsion subgroup. Therefore B is an endomorphic image of any such G, and hence any basic subgroup of T

is such an endomorphic image. Thus we see that *the answers to questions* I *and* II *are the same.*

References

1. L. Fuchs, *Abelian Groups*, Budapest, 1958.
2. D. K. Harrison, *Infinite Abelian groups and homological methods*, Annals of Math., **69** (1959), 366-391.

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