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TORSION ENDOMORPHIC IMAGES OF MIXED ABELIAN GROUPS

ELBERT A. WALKER

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In this paper we will answer Fuchs' PROBLEM 32 (a), and the corresponding part of his PROBLEM 33. (See [1], pg. 203.) The statements of these PROBLEMS are the following.

I. "Which are the torsion groups T that are endomorphic images of all groups containing them as maximal torsion subgroups?"

II. "Which are the torsion groups T such that a basic subgroup of T is an endomorphic image of any group G containing T as its maximal torsion subgroup?"

Actually, we will answer question II and the following question which is more general than I.

III. What groups H are endomorphic images of all groups G containing H such that G/H is torsion free?

The solutions will be effected by using some homological results of Harrison [2]. All groups considered here will be Abelian. The definitions and results stated in the remainder of this paragraph are due to Harrison, and may be found in [2]. A reduced group G is *cotorsion* if $\text{Ext}(A, G) = 0$ for all torsion free groups A . If H is a reduced group, then $\text{Ext}(Q/Z, H) = H'$ is cotorsion, where Q and Z denote the additive group of rationals and integers, respectively. Furthermore, H is a subgroup of H' , (that is, there is a natural isomorphism of H into H') and H'/H is divisible torsion free. This implies, of course, that if T is a torsion reduced group, then T is the torsion subgroup of $T' = \text{Ext}(Q/Z, T)$.

Now it is easy to see that *if G is a group such that $\text{Ext}(A, G) = 0$ for all torsion free groups A , then any homomorphic image of G is the direct sum of a cotorsion group and a divisible group.* In fact, let H be a homomorphic image of G . This gives us an exact sequence

$$0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$$

which yields the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(A, K) \rightarrow \text{Hom}(A, G) \rightarrow \text{Hom}(A, H) \rightarrow \\ \text{Ext}(A, K) \rightarrow \text{Ext}(A, G) \rightarrow \text{Ext}(A, H) \rightarrow 0. \end{aligned}$$

If A is any torsion free group, then $\text{Ext}(A, G) = 0$, and so $\text{Ext}(A, H) = 0$. Write $H = D \oplus L$, where D is the divisible part of H . Then L is reduced, and $0 = \text{Ext}(A, D \oplus L) \cong \text{Ext}(A, D) \oplus \text{Ext}(A, L) = \text{Ext}(A, L)$, so that L is cotorsion. Our assertion is proved.

Now we are ready to give the solutions promised earlier. The following theorem settles III.

THEOREM. *The group H is an endomorphic image of every group G containing it such that G/H is torsion free if and only if $H = D \oplus C$, where D is divisible and C is cotorsion. This is equivalent to the assertion that H is a direct summand of every such G .*

Proof. Suppose H is an endomorphic image of every group G containing it such that G/H is torsion free. Let $H = D \oplus C$, where D is divisible and C is reduced. Then C is a subgroup of the cotorsion group $\text{Ext}(Q/Z, C) = C'$ such that C'/C is torsion free, so that H is a subgroup of $D \oplus C' = H'$ such that H'/H is torsion free. Therefore H is an endomorphic image of H' . $\text{Ext}(A, D \oplus C') = 0$ for all torsion free groups A , and as we have just proved, any homomorphic image of $D \oplus C'$ is the direct sum of a cotorsion and a divisible group. It follows that C must be cotorsion.

If $H = D \oplus C$, with D divisible and C cotorsion, then $\text{Ext}(A, H) = 0$ for all torsion free groups A , and hence H is a direct summand of any group G containing it such that G/H is torsion free. If H is a direct summand of any such G , then clearly H is an endomorphic image of any such G . Thus our theorem is proved.

The torsion group T is a direct summand of every group containing it as its maximal torsion subgroup if and only if $T = D \oplus B$, with D divisible and B of bounded order. (See [1], pg. 187.) Thus, by our theorem, we see that *the torsion group T is an endomorphic image of every group containing it as its maximal torsion subgroup if and only if $T = D \oplus B$, with D divisible and B of bounded order.*

The solution of II goes as follows. Suppose a basic subgroup of T is an endomorphic image of every group G in which T is the maximal torsion subgroup. Let $T = D \oplus B$, with D divisible and B reduced. Then a basic subgroup of T must be an endomorphic image of $D \oplus B' = D \oplus \text{Ext}(Q/Z, B)$. Therefore a basic subgroup of T must be cotorsion, since it is reduced, and since it is torsion, it is of bounded order. (See [1], pg. 187. The remark by Harrison in [2], pg. 371 is incorrectly worded.) Writing T as $D \oplus B$, we see that a basic subgroup of B is a basic subgroup of T . But any two basic subgroups of T are isomorphic, and if B has a basic subgroup of bounded order, then B must be of bounded order. In fact, the only basic subgroup of B is B itself. Thus $T = D \oplus B$, with D divisible and B of bounded order. If $T = D \oplus B$, with D divisible and B of bounded order, then B is a basic subgroup of T . Now $D \oplus B$, and hence B , is a direct summand of any G in which T is the maximal torsion subgroup. Therefore B is an endomorphic image of any such G , and hence any basic subgroup of T

is such an endomorphic image. Thus we see that *the answers to questions I and II are the same.*

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1. L. Fuchs, *Abelian Groups*, Budapest, 1958.
2. D. K. Harrison, *Infinite Abelian groups and homological methods*, *Annals of Math.*, **69** (1959), 366-391.

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A. A. Albert, <i>Generalized twisted fields</i>	1
Richard Arens, <i>Operational calculus of linear relations</i>	9
John Herbert Barrett, <i>Disconjugacy of a self-adjoint differential equation of the fourth order</i>	25
Paul Richard Beesack, <i>Hardy's inequality and its extensions</i>	39
Julius Rubin Blum and David Lee Hanson, <i>On invariant probability measures. II</i>	63
Robert Allen Bonic, <i>Symmetry in group algebras of discrete groups</i>	73
R. Creighton Buck, <i>Multiplication operators</i>	95
Jack Gary Ceder, <i>Some generalizations of metric spaces</i>	105
Meyer Dwass, <i>Random crossings of cumulative distribution functions</i>	127
Albert Edrei, Wolfgang H. J. Fuchs and Simon Hellerstein, <i>Radial distribution and deficiencies of the values of a meromorphic function</i>	135
William Cassidy Fox, <i>Harmonic functions with arbitrary local singularities</i>	153
Theodore Thomas Frankel, <i>Manifolds with positive curvature</i>	165
Avner Friedman, <i>A strong maximum principle for weakly subparabolic functions</i>	175
Watson Bryan Fulks and J. O. Sather, <i>Asymptotics. II. Laplace's method for multiple integrals</i>	185
Adriano Mario Garsia and Eugene Richard Rodemich, <i>An embedding of Riemann surfaces of genus one</i>	193
Irving Leonard Glicksberg, <i>Weak compactness and separate continuity</i>	205
Branko Grünbaum, <i>On a conjecture of H. Hadwiger</i>	215
Frank J. Hahn, <i>On the action of a locally compact group on E_n</i>	221
Magnus R. Hestenes, <i>Relative hermitian matrices</i>	225
G. K. Kalisch, <i>On similarity invariants of certain operators in L_p</i>	247
Yitzhak Katznelson and Walter Rudin, <i>The Stone-Weierstrass property in Banach algebras</i>	253
Samir A. Khabbaz, <i>The subgroups of a divisible group G which can be represented as intersections of divisible subgroups of G</i>	267
Marvin Isadore Knopp, <i>Construction of a class of modular functions and forms</i>	275
Charles Alan McCarthy, <i>Commuting Boolean algebras of projections</i>	295
T. M. MacRobert, <i>Transformations of series of E-functions</i>	309
Heinz Renggli, <i>An inequality for logarithmic capacities</i>	313
M. S. Robertson, <i>Applications of the subordination principle to univalent functions</i>	315
David Sachs, <i>Partition and modulated lattices</i>	325
Frank S. Scalora, <i>Abstract martingale convergence theorems</i>	347
Elbert A. Walker, <i>Torsion endomorphic images of mixed Abelian groups</i>	375
Morgan Ward, <i>The prime divisors of Fibonacci numbers</i>	379
Charles R. B. Wright, <i>On the nilpotency class of a group of exponent four</i>	387