# LEBESGUE DENSITY AS A SET FUNCTION

NATHANIEL F. G. MARTIN

# LEBESGUE DENSITY AS A SET FUNCTION

## N. F. G. Martin

Lebesgue (or metric) density is usually considered as a point function in the sense that a fixed subset of a space $X$ is given and then the value of the density of this set is obtained at various points of the space. Suppose the density is considered in another sense. That is, let a point $x$ of the space be fixed and consider the class $\mathscr{D}(x)$ of all sets whose density exists at this point. Then to each set $E$ in $\mathscr{D}(x)$ we assign the value of its density at $x$, and denote this number by $D_x(E)$. Thus from this point of view the density is a finite set function. It was shown in [2] that if the space $X$ is the real line then the image of $\mathscr{D}(x)$ under $D_x$ is the closed unit interval.

It is evident from the definition of density of sets of real numbers, which we give below, that $D_x$ is a finitely additive, subtractive, monotone, nonnegative set function and the class $\mathscr{D}(x)$ is closed under the formation of complements, proper differences, and disjoint unions. Therefore, if $\mathscr{D}(x)$ were closed under the formation of intersections, $D_x$ would be a finitely additive measure. This however is not the case for if

$$R_n = \left\{ x\colon \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < x < \frac{1}{n}\right\},$$

$$L_n = \left\{ x\colon -\frac{1}{n} < x < -\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)\right\}$$

and

$$L_n^* = \left\{ x\colon -\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < x < -\frac{1}{n+1}\right\},$$

the sets $\bigcup_n (R_n \cup L_n) = E$ and $\bigcup_n (R_n \cup L_n^*) = F$ are members of $D(0)$ but $E \cap F$ is not. In fact $D_0(E) = D_0(F) = \frac{1}{2}$ and the upper density of $E \cap F$ at zero is not less than $\frac{1}{2}$ while the lower density of $E \cap F$ at zero is zero.

In part 1 of this note we prove a theorem which is somewhat of an analogue of the Lebesgue density theorem [3] in the following respect. As noted above $D_x$ is not a finitely additive measure, but we show that the upper density at $x$, $\bar{D}_x$, is a finitely subadditive outer measure defined on the class of all Lebesgue measurable subsets of $X$ and the class of $\bar{D}_x$-measurable sets is the class of all sets whose density exists at $x$ and has the value zero or one. In part 2 a Lebesgue density of a measurable set $E$ on a fixed $F_\sigma$ set of measure zero is defined and a similar result

proven for this function.

1. If $E$ is a measurable subset of the real line $X$ and $I$ is any interval we shall denote the relative Lebesgue measure of $E$ in $I$, $m(E \cap I)/m(I)$, by $\rho(E : I)$.

The upper Lebesgue density of a measurable subset $E$ of $X$ at a point $x \in X$, $\bar{D}_x(E)$, is defined by

$\bar{D}_x(E) = \limsup_{I \to x} \rho(E : I) = \sup \{\limsup_k \rho(E : I_k): I_k \to x\}$ and the lower Lebesgue density of a measurable set $E \subset X$ at a point $x \in X$, $\underline{D}_x(E)$, is defined by

$$\underline{D}_x(E) = \liminf_{I \to x} \rho(E : I) = \inf \{\lim_k \rho(E : I_k): I_k \to x\} \, ,$$

where $I_k \to x$ means the sequence $\{I_k\}$ of intervals converges to $x$ in the sense that $x \in \bar{I}_k$ for all $k$ and $m(I_k) \to 0$ as $k \to \infty$. In the case $\underline{D}_x(E) = \bar{D}_x(E)$ the common value is the Lebesgue density of $E$ at $x$ and will be denoted by $D_x(E)$.

LEMMA 1. *A necessary and sufficient condition that a set $E$ be a member of $\mathscr{D}(x)$ is that*

$$\bar{D}_x(E) + \bar{D}_x(X - E) = 1 \, .$$

*Proof.* The necessity is immediate. To obtain the sufficiency we note that for any interval $I$ containing $x$, $\rho(E : I) + \rho(X - E : I) = 1$ so that $\underline{D}_x(E) + \bar{D}_x(X - E) \geqq 1$. Therefore

$$\bar{D}_x(X - E) \geqq 1 - \underline{D}_x(E) = \bar{D}_x(X - E) + \bar{D}_x(E) - \underline{D}_x(E)$$

and it follows that $\bar{D}_x(E) \leqq \underline{D}_x(E)$.

LEMMA 2. *The set function $\bar{D}_x$ is a finitely subadditive outer measure defined on the class $\mathscr{M}$ of all Lebesgue measurable subsets of the real line.*

*Proof.* It is clear that $\bar{D}_x(\phi) = 0$ and $\bar{D}_x \geqq 0$. Let $E \subset F$ be two sets from $M$. Then since $\rho(E : I) \leqq \rho(F : I)$ for all intervals containing $x$, $\bar{D}_x$ is monotone. Let $E_1, E_2, \cdots, E_n$ be any finite collection of sets from $\mathscr{M}$. Since $\rho(\bigcup_{i=1}^{n} E_i : I) \leqq \sum_{i=1}^{n} \rho(E_i : I)$ for all intervals $I$ containing $x$, we have

$$\bar{D}_x\left(\bigcup_{i=1}^{n} E_i\right) \leqq \sum_{i=1}^{n} \limsup_{I \to x} \rho(E_i : I) = \sum_{i=1}^{n} \bar{D}_x(E_i) \, .$$

Thus $\bar{D}_x$ is a finitely subadditive outer measure.

Let $\mathscr{M}(x)$ denote the class of all sets $E$ such that for every $A \in \mathscr{M}$,

$\bar{D}_x(A) = \bar{D}_x(A \cap E) + \bar{D}_x(A - E)$. Since $\mathscr{M}(x)$ contains $X$ and $\phi \mathscr{M}(x)$ is an algebra (in the sense of Halmos [1]) and the restriction of $\bar{D}_x$ to $\mathscr{M}(x)$ is a finitely additive measure.

LEMMA 3. $\mathscr{M}(x)$ is a subset of $\mathscr{D}(x)$.

Proof. Let $E \in \mathscr{M}(x)$. Since the real line $X$ is a member of $\mathscr{M}$ and $\bar{D}_x(X) = 1$, we have

$$1 = \bar{D}_x(X) = \bar{D}_x(X \cap E) + \bar{D}_x(X - E) = \bar{D}_x(E) + \bar{D}_x(X - E)$$

which by Lemma 1 gives $E \in \mathscr{D}(x)$.

LEMMA 4. If $E \in \mathscr{D}(x)$ and $J$ is any interval with $x$ as one end point then $\bar{D}_x(E \cap J) = D_x(E)$.

Proof. Let $D_x(E) = d$. Since $\bar{D}_x$ is monotone, $d \geq \bar{D}_x(E \cap J)$ and if $\{I_k\}$ is any sequence of intervals converging to $x$, $\limsup_k \rho((E \cap J) : I_k) \leq d$.

Suppose first that $J$ is a bounded interval. If $x$ is the left end point of $J$, denote the right end point by $y$ and let

$$I_n^* = \left\{ z : x \leq z \leq x + \frac{1}{n}(y - x) \right\} ;$$

if $x$ is the right end point of $J$, denote the left end point of $J$ by $y$ and let

$$I_n^* = \left\{ z : x - \frac{1}{n}(x - y) \leq z \leq x \right\} .$$

In either case $I_n^* \to x$ and $\rho(E : I_n^*) = \rho((E \cap J) : I_n^*)$ for all $n$. Therefore, $\lim_n \rho((E \cap J) : I_n^*) = d$ and we have $\bar{D}_x(E \cap J) = D_x(E)$.

Suppose next that $J$ is unbounded. If $x$ is the left end point of $J$ let $I_n^* = \{z : x \leq z \leq z + (1/n)\}$ and if $x$ is the right end point of $J$ let $I_n^* = \{z : x - (1/n) \leq z \leq x\}$. Again we have $I_n^* \to x$ and $\rho(E : I_n^*) = \rho((E \cap J) : I_n^*)$ for all $n$ so that $\bar{D}_x(E \cap J) = D_x(E)$.

LEMMA 5. Let $E \in \mathscr{D}(x)$ and let $J$ be an interval open on the right with right end point at $x$ and $K$ be an interval closed on the left with left end point at $x$. Define the set $A$ by $A = (E \cap K) \cup (J - E)$. Then $\bar{D}_x(A) = \max \{D_x(E), D_x(X - E)\}$.

Proof. Suppose $D_x(X - E) \leq D_x(E) = d$. By Lemma 4, $\bar{D}_x(J - E) = 1 - d \leq d$ and since $\bar{D}_x$ is monotone, $\bar{D}_x(A) \geq \bar{D}_x(E \cap K) = d$.

Let $\varepsilon > 0$ be given. Then there exists a sequence $\{I_k^*\}$ converging to $x$ such that

$$\bar{D}_x(A) < \lim\sup_k \rho(A : I_k^*) + \frac{\varepsilon}{2} \ .$$

For each $k$, let $J_k = I_k^* \cap (J \cup K)$. Since $I_k^* \to x$, $J_k^* \to x$ and $\rho(A : I_k^*) = \rho(A : J_k)$ for all but a finite number of $k$. Therefore

(1) $$\bar{D}_x(A) < \lim\sup_k \rho(A : J_k) + \varepsilon/2 \ .$$

For each interval $J_k$ we have

$$\rho(A : J_k) - d = \rho(K : J_k)[\rho(E : (K \cap J_k)) - d]$$
$$+ \rho(J : J_k)[\rho(X - E : (J \cap J_k)) - d] \ .$$

Since $E \in \mathscr{D}(x)$ and $K \cap J_k \to x$, $\lim_k \rho(E : (K \cap J_k)) = d$. Since $J \cap J_k \to x$, $\lim_k \rho(X - E : (J \cap J_k)) = 1 - d \leq d$. Therefore there exist integers $N_1$ and $N_2$ such that for all $k > N_1$, $\rho(E : (K \cap J_k)) - d < \varepsilon/2$ and for all $k > N_2$, $\rho(X - E : (J \cap J_k)) - d < \varepsilon/2$. Thus for all $k > \max\{N_1, N_2\}$

$$\rho(A : J_k) - d < \frac{\varepsilon}{2}\rho(K : J_k) + \frac{\varepsilon}{2}\rho(J : J_k) = \frac{\varepsilon}{2} \ .$$

Therefore $\lim\sup_k \rho(A : J_k) < d + \varepsilon/2$ and we have by way of equation (1) that $\bar{D}_x(A) < d + \varepsilon$. Since $\varepsilon$ was arbitrary, $\bar{D}_x(A) \leq d$ which completes the proof of the lemma.

THEOREM 1.    *The class $\mathscr{M}(x)$ of $\bar{D}_x$-measurable sets is the class of all sets whose density exists at $x$ and has the value 0 or 1.*

*Proof.* First suppose $E \in \mathscr{M}(x)$ and $D_x(E)=d$. Let $J=\{z: x-1 \leq z < x\}$, $K = \{z: x \leq z \leq x + 1\}$. Define the set $A$ by $A = (E \cap K) \cup (J - E)$. By Lemma 5, $\bar{D}_x(A) = \max\{1 - d, d\}$ and by Lemma 4, $\bar{D}_x(A \cap E) = \bar{D}_x(E \cap K) = d$ and $\bar{D}_x(A - E) = \bar{D}_x(J - E) = 1 - d$. Since $E \in \mathscr{M}(x)$

$$1 = d + 1 - d = \bar{D}_x(A \cap E) + \bar{D}_x(A - E) = \bar{D}_x(A) = \max\{1 - d, d\} \ .$$

Therefore $d = 0$ or 1.

Next let $E$ be a set whose density at $x$ is zero or one. Let $A$ be any Lebesgue measurable set and suppose $D_x(E) = 0$. Since $\bar{D}_x$ is monotone, $\bar{D}_x(A \cap E) \leq D_x(E) = 0$ and hence $\bar{D}_x(A \cap E) = 0$. Since $\bar{D}_x$ is an outer measure

$$\bar{D}_x(A - E) \geq \bar{D}_x(A) - \bar{D}_x(E) = \bar{D}_x(A) \ ,$$

and since $\bar{D}_x$ is monotone $\bar{D}_x(A) \geq \bar{D}_x(A - E)$. Therefore $\bar{D}_x(A) = \bar{D}_x(A \cap E) + \bar{D}_x(A - E)$ and $E$ is in $\mathscr{M}(x)$. In case $D_x(E) = 1$ the above argument with $E$ replaced by $X - E$ gives the desired result.

2.    Suppose that $Z$ represents an $F_\sigma$ set of measure zero. Define

the upper Lebesgue density of a measurable set $E$ or $Z$ by

$$\bar{D}_z(E) = \sup \{\bar{D}_x(E) : x \in Z\}$$

and the lower Lebesgue density of $E$ or $Z$ by

$$\underline{D}_z(E) = \inf \{\underline{D}(E) : x \in Z\}.$$

If $\bar{D}_z(E) = \underline{D}_z(E)$ we will say that the Lebesgue density of $E$ on $Z$, denoted by $D_z(E)$, exists and has the common value of $\bar{D}_z(E)$ and $\underline{D}_z(E)$. It is clear that if the density of $E$ exists on $Z$ then the density exists at every point of $Z$ and has the same value at each point. In [2] it was shown that for any number $d$ such that $0 < d < 1$, there exists a set $E$ such that $D_z(E) = d$. Thus if $\mathscr{D}(Z)$ denotes the class of all sets whose density on $Z$ exists, $D_z$ is a set function which maps $\mathscr{D}(Z)$ onto the closed unit interval. It is clear that $D_z$ will have the same properties as $D_x$ where $x$ is any point in $Z$.

LEMMA 7. $\bar{D}_z$ is a finitely subadditive outer measure defined on the class $\mathscr{M}$.

Proof. The lemma follows immediately from the monotoniety and subadditivity of $\bar{D}_x$ and the definition of $\bar{D}_z$.

Let $\mathscr{M}(Z)$ denote the class of all sets $E$ such that $E \in \mathscr{M}$ and for every $A \in \mathscr{M}$, $\bar{D}_z(A) = \bar{D}_z(A \cap E) + \bar{D}_z(A - E)$. Then $\mathscr{M}(Z)$ is an algebra and the restriction of $\bar{D}_z$ to $\mathscr{M}(Z)$ is a finitely additive measure.

LEMMA 8. $\mathscr{M}(Z)$ is a subset of $\mathscr{D}(Z)$.

Proof. Let $E \in \mathscr{M}(Z)$. The real line $X$ is in $\mathscr{M}$ so we have

$$1 = \bar{D}_z(X) = \bar{D}_z(E) + \bar{D}_z(X - E) \geq \sup \{\bar{D}_x(E) + \bar{D}_x(X - E) : x \in Z\}$$

and

$$\bar{D}_x(E) + \bar{D}_x(X - E) \leq 1$$

for all $x \in Z$. But for any $x \in Z$, $\bar{D}_x$ is subadditive so that $\bar{D}_x((E) + \bar{D}_x(X - E) \geq 1$. Therefore $\bar{D}_x(E) + \bar{D}_x(X - E) = 1$ for all $x \in Z$ and by Lemma 1, the density of $E$ exists at every point of $Z$. Hence $\underline{D}_x(E) + \bar{D}_x(X - E) = 1$ for all $x$ in $Z$ and

$$\underline{D}_z(E) + \bar{D}_z(X - E) \geq \inf \{\underline{D}(E) + \bar{D}_x(E) : x \in Z\}$$
$$= 1 = \bar{D}_z(E) + \bar{D}_z(X - E).$$

Since $\bar{D}_z$ if finite, $\underline{D}_z(E) \geq \bar{D}_z(E)$ and it follows that $E \in \mathscr{D}(Z)$.

THEOREM 2. The class of all $\bar{D}_z$-measurable sets is the class of

*all sets from $\mathscr{D}(Z)$ which are mapped onto 0 or 1 by $D_z$.*

**Proof.** Let $\mathscr{K} = \{E : E \in D(Z) \text{ and } D_z(E) = 0 \text{ or } 1\}$. If $E \in \mathscr{K}$ we may show that $E \in \mathscr{M}(Z)$ exactly as was done in Theorem 1.

Suppose $E \in \mathscr{M}(Z)$. By Lemma 8, $E \in \mathscr{D}(Z)$ and hence $D_z(E) = D_x(E) = d$ for all $x \in Z$. Let $x_1$ be any point in $Z$ and let $J = \{z : z < x_1\}$, $K = \{z : z \geq x_1\}$. Define the set $A$ by $A = (J - E) \cup (E \cap K)$. Then by Lemmas 4 and 5, $\bar{D}_{x_1}(A) = \max \{d, 1 - d\}$, $\bar{D}_{x_1}(A \cap E) = d$, and $\bar{D}_{x_1}(A - E) = 1 - d$. Since $A \in \mathscr{M}$ and $E \in \mathscr{M}(Z)$,

$$\sup \{\bar{D}_x(A) : x \in Z\} = \sup \{\bar{D}_x(A \cap E) + \bar{D}_x(A - E) : x \in Z\} .$$

Let $\varepsilon > 0$ be given. Then there exists an $x_2 \in Z$ such that

$$\bar{D}_{x_2}(A) + \varepsilon > \sup \{\bar{D}_x(A \cap E) + \bar{D}_x(A - E) : x \in Z\}$$
$$\geq \bar{D}_{x_1}(A \cap E) + \bar{D}_{x_1}(A - E) = 1 .$$

Suppose $x_2 < x_1$. Then $\bar{D}_{x_2}(A) = D_{x_2}(X - E)$ and $1 - d + \varepsilon > 1$. Since $\varepsilon$ was arbitrary and $1 - d \leq 1$ we have $1 - d = 1$ and $d = 0$.

Suppose $x_2 > x_1$. Then $\bar{D}_{x_2}(A) = D_{x_2}(E)$ and $d + \varepsilon > 1$. Since $\varepsilon$ was arbitrary and $d \leq 1$ we have $d = 1$.

Suppose $x_2 = x_1$. Then $\bar{D}_{x_2}(A) = \max \{d, 1 - d\}$, and $\max \{d, 1 - d\} + \varepsilon > 1$. Since $\varepsilon$ was arbitrary $\max \{d, 1 - d\} \geq 1$. But both $d$ and $1 - d$ do not exceed 1 so that $d = 0$ or 1.

Therefore $E$ is in $\mathscr{K}$ and we have $\mathscr{M}(Z) = \mathscr{K}$.

## BIBLIOGRAPHY

1. P. R. Halmos, *Measure Theory*, Princeton, New Jersey, (1950).
2. N. F. G. Martin, *A note on metric density of sets of real numbers*, to be published in Proc. Amer. Math. Soc., 11 (1960), 344–347.
3. S. Saks, *Theory of the integral*, Trans. by L. C. Young, Warsaw, (1937).

IOWA STATE UNIVERSITY AND
UNIVERSITY OF VIRGINIA

# PACIFIC JOURNAL OF MATHEMATICS

# Pacific Journal of Mathematics

## Vol. 11, No. 2  December, 1961