Pacific Journal of Mathematics

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Vol. 11, No. 2 December 1961

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Introduction. In their paper [3], Hewitt and Zuckerman study the measure algebra $\mathscr{M}(G)$ where G is a topological semigroup of the following type: G is a linearly ordered set topologized with the order topology, is compact in this topology, and multiplication is defined by $xy = \max(x, y)$. In this study, we will suppose that G has the above properties except that compactness will be replaced by local compactness. (See § 8.5 [3]). As the reader will readily observe, we are heavily indebted to Hewitt and Zuckerman for their initial study of these measure algebras. For completeness, we have listed, without proof, a few of their results; they are stated in their paper for compact semigroups but the proofs easily carry over to locally compact semigroups.

In § 2 we study $\hat{G}$ and $\hat{G}_0$. The characterization of the Gel'fand topology on $\hat{G}$ is somewhat simpler than that of Theorem 5.5 [3]. The major result of this study is Theorem 3.4, stating that every closed ideal in $\mathscr{M}(G)$ is the intersection of maximal ideals; i.e., spectral synthesis holds for $\mathscr{M}(G)$. Malliavin [7] has recently shown that spectral synthesis fails for $\mathscr{M}(G)$ when G is a non-compact locally compact commutative group.[1] Theorem 3.4 shows that this result cannot be generalized to locally compact commutative semigroups. In § 4, a generalization of Theorem 6.7 [3] is indicated; see Theorem 4.5. This is used to obtain additional facts about $\mathscr{M}(G)$ (§ 5). In 5.8 we show that our theory is not a special case of the theory of function algebras.

1. Preliminaries.

1.1. We will be concerned with linearly ordered sets; i.e. sets ordered by transitive, irreflexive relations $<$. For elements x and y in such a set X, we define $]x, y[= \{z \in X : x < z < y\}$ and $[x, y] = \{z \in X : x \leqq z \leqq y\}$. The half-open intervals $[x, y[$ and $]x, y]$ are defined analogously. We also define $]-\infty, x[= \{z \in X : z < x\}$ and $]-\infty, x] = \{z \in X : z \leq x\}$ with analogous definitions for $[x, \infty[$, $]x, \infty[$, and $]-\infty, \infty[$. The symbols $-\infty$ and ∞ will never denote actual elements of X. The order topology for X is the topology having the family $\{]-\infty, x[\}_{x \in X} \cup$

Received May 5, 1960. Supported by a National Science Foundation pre-doctoral fellowship. The author is indebted to Professor Edwin Hewitt for his advice and encouragement during the preparation of this paper, which constitutes part of a Ph. D. thesis. Conversations with Dr. Karl R. Stromberg were also helpful. Presented to the American Mathematical Society, January 27, 1960.

[1] Actually Malliavin shows that spectral synthesis fails for $L_1(G)$; the result for $\mathscr{M}(G)$ follows easily from this.

$\{]x, \infty[\}_{x \in X}$ for a sub-base.

For terminology not explained here in measure theory, topology, and harmonic analysis, see [1], [5], and [6], respectively. If A is a subset of B, we will write $A \subseteqq B$; $A \subset B$ will mean that A is a proper subset of B. For sets A and B, we write $A - B = \{x : x \in A, x \notin B\}$ and $A \Delta B = (A - B) \cup (B - A)$. The empty set will be denoted by 0. For any set A, χ_A will denote the characteristic function of A.

1.2. STANDING HYPOTHESES. Let G be a set linearly ordered by the relation $<$. Suppose also that G has the order topology and that under this topology G is locally compact. For $x, y \in G$, we define $xy = \max(x, y)$. With this multiplication G is a locally compact topological semigroup.

1.3. Let $\mathfrak{C}_0(G)$ denote the linear space of all complex-valued continuous functions on G that are arbitrarily small outside of compact sets. For $f \in \mathfrak{C}_0(G)$, let $\|f\| = \max_{x \in G} |f(x)|$. Let $\mathscr{M}(G)$ consist of all countably additive, complex-valued, regular, finite Borel measures on G. Let $\mathfrak{C}_0^*(G)$ be the linear space of all complex-valued bounded linear functionals L on $\mathfrak{C}_0(G)$. For each $L \in \mathfrak{C}_0^*(G)$ there is a unique $\lambda \in \mathscr{M}(G)$ such that

$$L(f) = \int_G f(x) d\lambda(x) \tag{1.3.1}$$

for all $f \in \mathfrak{C}_0(G)$. Also for each $\lambda \in \mathscr{M}(G)$, 1.3.1 defines a member of $\mathfrak{C}_0^*(G)$. Under this correspondence, $\mathscr{M}(G) \cong \mathfrak{C}_0^*(G)$. We will associate L with λ, M with μ, etc.

Let $\lambda \in \mathscr{M}(G)$. Then for Borel sets $E \subseteqq G$, we define

$$|\lambda|(E) = \sup\left\{ \sum_{k=1}^{m} |\lambda(E_k)| : \{E_k\}_{k=1}^{m} \text{ is a Borel partition of } E. \right\} \tag{1.3.2}$$

Then the set-function $|\lambda|$ belongs to $\mathscr{M}(G)$ and

$$\|\lambda\| = |\lambda|(G) = \|L\| \tag{1.3.3}$$

where $L \in \mathfrak{C}_0^*(G)$ is defined by 1.3.1. See [2].

1.4. THEOREM. *Let L and M be in $\mathfrak{C}_0^*(G)$. For all $f \in \mathfrak{C}_0(G)$, let*

$$L*M(f) = \int_G \int_G f(xy) d\lambda(x) d\mu(y) . \tag{1.4.1}$$

Then $L*M \in \mathfrak{C}_0^*(G)$, and

$$\|L*M\| \leqq \|L\| \cdot \|M\| . \tag{1.4.2}$$

1.5. For $\lambda, \mu \in \mathscr{M}(G)$, we define $\lambda*\mu$ to be the unique measure in

$\mathscr{M}(G)$ that corresponds to $L*M \in \mathfrak{C}_0^*(G)$.

1.6. THEOREM. *Under the convolution defined in* 1.5 *and the ordinary linear operations, $\mathscr{M}(G)$ is a commutative Banach algebra.*

We omit the proof; see § 2 [3].

1.7. For $a \in G$, let $\varepsilon_a \in \mathscr{M}(G)$ be defined by

$$\varepsilon_a(E) = \begin{cases} 1 \text{ if } a \in E, \\ 0 \text{ if } a \notin E, \end{cases} \tag{1.7.1}$$

for Borel sets $E \subseteqq G$. For $\lambda \in \mathscr{M}(G)$ and $A \subseteqq G$ a Borel set, $\lambda^A \in \mathscr{M}(G)$ is defined by $\lambda^A(E) = \lambda(A \cap E)$ for all Borel sets $E \subseteqq G$.

The proofs of the following four lemmas are routine and uninteresting.

1.8. LEMMA. *Let $E \subseteqq G$ be a Borel set and $\lambda \in \mathscr{M}(G)$. Then for any $\varepsilon > 0$, there exist $a, b \in E$ such that*

$$|\lambda|(E \cap \,]-\infty, a[) < \varepsilon \quad \textit{and} \quad |\lambda|(E \cap \,]\, b, \infty[) < \varepsilon . \tag{1.8.1}$$

1.9. LEMMA. *Let X be a linearly ordered set and $U \subseteqq X$ be a finite union of open intervals. Then U is the pairwise disjoint union of open intervals:*

$$U = \bigcup_{i=1}^{m} \,]a_i, b_i[\,,$$

where intervals of the form $[\inf X, b_i[\,, \,]a_i, \sup X]$, *and* $[\inf X, \sup X]$ *are also admissible if* $\inf X$ *or* $\sup X$ *exist. Moreover, $a_i \notin U$ except possibly in the case where $a_i = \inf X$, and $b_i \notin U$ except possibly in the case that $b_i = \sup X$.*

1.10. LEMMA. *Let X be a compact linearly ordered set and $U \subseteqq X$ be an open set. Then U is the pairwise disjoint union of open intervals:*

$$U = \bigcup_{\alpha} \,]a_\alpha, b_\alpha[$$

where intervals of the form $[\inf X, b_\alpha[\,, \,]a_\alpha, \sup X]$, *and* $[\inf X, \sup X]$ *are also admissible. In addition, $a_\alpha \notin U$ except possibly in the case that $a_\alpha = \inf X$, and $b_\alpha \notin U$ except possibly in the case that $b_\alpha = \sup X$.*

1.11. LEMMA. *Let X be a locally compact linearly ordered set. Suppose that $K \subseteqq X$ is compact and that U is an open set such that $K \subseteqq U \subseteqq X$. Then there exist finitely many pairwise disjoint closed compact intervals $\{[a_i, b_i]\}_{i=1}^m$ such that $U \supseteqq \bigcup_{i=1}^m [a_i, b_i] \supseteqq K$. Also there* *exist finitely many pairwise disjoint open intervals* $\{]u_i, v_i[\}_{i=1}^n$ *such that* $U \supseteqq \bigcup_{i=1}^n]u_i, v_i[\supseteqq K$ *and each closed interval* $[u_i, v_i]$ *is compact. Intervals of the form* $[\inf X, v_i[$, $]u_i, \sup X]$, *and* $[\inf X, \sup X]$ *are also admissible whenever* $\inf X$ *or* $\sup X$ *exists.*

2. The spaces $\hat{G}$ and $\hat{G}_0$.

2.1. A Dedekind cut $\{A, B\}$ of G is a pair of subsets of G such that $A \cap B = 0$, $A \cup B = G$, and $x < y$ whenever $x \in A$ and $y \in B$. Let $\hat{G}$ denote the set of all semicharacters of G.

2.2 THEOREM. *Let* $\{A, B\}$ *be a Dedekind cut of* G *such that* $A \neq 0$. *Then the function*

$$\psi_{A,B}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in B, \end{cases} \tag{2.2.1}$$

is a semicharacter of G. *Conversely, every semicharacter on* G *has the form* 2.2.1.

2.3. THEOREM. *Let* $\{A, B\}$ *be a Dedekind cut of* G *such that* $A \neq 0$. *Then the mapping*

$$\pi_A(\lambda) = \lambda(A) = \int_G \psi_{A,B}(x)d\lambda(x) \qquad (\lambda \in \mathscr{M}(G)) \tag{2.3.1}$$

is a homomorphism of $\mathscr{M}(G)$ *onto the complex numbers. Moreover, every homomorphism of* $\mathscr{M}(G)$ *onto the complex numbers has the form* 2.3.1.

Proof. This is essentially proved in Theorems 3.2 and 3.3 [3]; however the proof in [3] that π_A is multiplicative can be simplified. Let $\lambda, \mu \in \mathscr{M}(G)$. According to Theorem 2 [8], $\lambda * \mu(E) = \lambda \times \mu\{(x, y) \in G \times G: xy \in E\}$ for Borel sets $E \subseteqq G$ where $\lambda \times \mu$ is the product measure of λ and μ. Hence if $\{A, B\}$ is a Dedekind cut of G, then

$$\begin{aligned} \pi_A(\lambda * \mu) &= \lambda * \mu(A) = \lambda \times \mu\{(x, y) \in G \times G: \max(x, y) \in A\} \\ &= \lambda \times \mu(A \times A) = \lambda(A)\mu(A) = \pi_A(\lambda)\pi_A(\mu). \end{aligned}$$

2.4. THEOREM. *The Banach algebra* $\mathscr{M}(G)$ *is semisimple.*

Proof. In virtue of 2.3 we need to prove that if $\lambda(A) = 0$ for all Dedekind cuts $\{A, B\}$, then λ is identically zero. Suppose that $\lambda(A) = 0$ for all Dedekind cuts $\{A, B\}$; evidently $\lambda(I) = 0$ for all intervals I. If

λ is not identically zero, then $\lambda(K) \neq 0$ for some compact set $K \subseteqq G$. By regularity there is an open set $U \supseteqq K$ such that $|\lambda|(U - K) < |\lambda(K)|$. For each $x \in K$, let I_x be an open interval such that $x \in I_x \subseteq U$. Let $I_1, \cdots, I_m$ be a finite subset of $\{I_x\}_{x \in K}$ covering K. Let $V = \bigcup_{i=1}^m I_i$; clearly $K \subseteqq V \subseteqq U$. By 1.9, V is the pairwise disjoint union of a finite number of open intervals. Hence $\lambda(V) = 0$. Thus

$$\begin{aligned}|\lambda(V - K)| &= |\lambda(V) - \lambda(K)| \\ &= |\lambda(K)| > |\lambda|(U - K) \geqq |\lambda|(V - K) \geqq |\lambda(V - K)|\end{aligned}$$

which is a contradiction. Hence λ is identically zero.

2.5. Theorem 2.3 identifies completely the homomorphisms of $\mathscr{M}(G)$ onto the complex numbers. Relation 2.3.1 associates each homomorphism π_A of $\mathscr{M}(G)$ with the semicharacter $\psi_{A,B}$. Hence we will usually consider $\hat{G}$ as consisting of the homomorphisms π_A. For $\lambda \in \mathscr{M}(G)$, we define $\hat{\lambda}$ on $\hat{G}$ by

$$\hat{\lambda}(\pi_A) = \pi_A(\lambda) = \lambda(A) \qquad (\pi_A \in \hat{G}); \tag{2.5.1}$$

$\hat{\lambda}$ is the Fourier transform of λ.

For $\pi_A, \pi_{A'} \in \hat{G}$, we will write $\pi_A < \pi_{A'}$ if and only if $A \subset A'$. Under this ordering, $\hat{G}$ is obviously linearly ordered. Evidently $\hat{G}$ is isomorphic to the maximal ideal space of $\mathscr{M}(G)$. The Gel'fand topology for $\hat{G}$ is the weakest topology for which all the functions $\hat{\lambda}$ are continuous.

Henceforth we will write $\pi_{a]}$ for $\pi_{]-\infty, a]}$ and $\pi_{a[}$ for $\pi_{]-\infty, a[}$ $(a \in G)$.

2.6. DEFINITION. Let $\hat{G}_0 = \hat{G} \cup \{\pi_0\}$ where $\pi_0 < \pi$ for all $\pi \in \hat{G}$.

The symbol π_0 may be taken to correspond to the zero homomorphism of $\mathscr{M}(G)$, the zero semicharacter of G, and the Dedekind cut $\{0, G\}$.

2.7. THEOREM. *The Gel'fand topology on $\hat{G}$ coincides with the order topology.*

Proof. Let $\pi_A \in \hat{G}$ where $A \neq G$, $\lambda \in \mathscr{M}(G)$, and $\varepsilon > 0$. Using 1.8, we can find $b \in A$ and $c \notin A$ such that $|\lambda|(]b, c[) < \varepsilon$. Clearly $\pi_A \in]\pi_{b[}, \pi_{c]}[$. For $\pi_B \in]\pi_{b[}, \pi_{c]}[$, we have

$$\begin{aligned}|\hat{\lambda}(\pi_A) - \hat{\lambda}(\pi_B)| &= |\lambda(A) - \lambda(B)| \\ &= |\lambda(A \mathbin{\Delta} B)| \leqq |\lambda|(A \mathbin{\Delta} B) \leqq |\lambda|(]b, c[) < \varepsilon .\end{aligned}$$

Thus $\hat{\lambda}$ is continuous at $\pi_A \in \hat{G}(A \neq G)$ in the order topology. Similarly $\hat{\lambda}$ is continuous at π_G in the order topology. Hence the Gel'fand topology is weaker than or equivalent to the order topology.

For $b, c \in G$, $b < c$, it is easy to verify that

$$\hat{\varepsilon}_b - \hat{\varepsilon}_c = \chi_{]\pi_{b[},\pi_{c]}[} \quad \text{and} \quad \hat{\varepsilon}_b = \chi_{]\pi_{b[},\pi_G]} .$$

Hence sets of the form

$$(2.7.1) \qquad]\pi_{b[},\pi_{c]}[\quad b < c ,$$

and

$$(2.7.2) \qquad]\pi_{b\,[},\pi_G] ,$$

are open in the Gel'fand topology. All sets of the forms 2.7.1 and 2.7.2 comprise a basis for the order topology. It follows that the order topology on $\hat{G}$ is weaker than or equivalent to the Gel'fand topology on $\hat{G}$.

2.8. THEOREM. *The set $\hat{G}_0$ with the order topology is a totally disconnected compact Hausdorff space. For $\lambda \in \mathscr{M}(G)$, let $\hat{\lambda}$ be defined on $\hat{G}_0$ to agree with $\hat{\lambda}$ on $\hat{G}$ and such that $\hat{\lambda}(\pi_0) = \lambda(0) = 0$. Then $\hat{\lambda}$ is continuous on $\hat{G}_0$.*

Proof. Let $\mathscr{B}$ consist of all subsets of $\hat{G}_0$ of the form:

$$(2.8.1) \qquad]\pi_{a[},\pi_{b]}[\qquad (a < b) ,$$

$$(2.8.2) \qquad [\pi_0, \pi_{b]}[,$$

$$(2.8.3) \qquad]\pi_{a[},\pi_G] .$$

Each set in $\mathscr{B}$ is open and closed and $\mathscr{B}$ is a base for the order topology on $\hat{G}_0$. Hence $\hat{G}_0$ is totally disconnected. The remainder of the proof is omitted.

2.9. DEFINITION. Let I be an interval of $\hat{G}_0$ and let h be a continuous function on $\hat{G}_0$. Then we define:

$$(2.9.1) \quad V(h; I) = \sup\left\{ \sum_{i=1}^{m-1} |h(\pi_{i+1}) - h(\pi_i)| : \pi_1 \leqq \pi_2 \leqq \cdots \leqq \pi_m,\ \pi_i \in I \right\}.$$

In particular, we define $V(h) = V(h\,; \hat{G}_0)$ and say that h has finite variation if $V(h) < \infty$.

2.10. Let h be a continuous function on $\hat{G}_0$ and let $\pi_{A_1} \leqq \pi_{A_2} \leqq \cdots \leqq \pi_{A_k}, \pi_{A_i} \in \hat{G}_0$. Then

$$(2.10.1) \qquad V(h\,;\ [\pi_{A_1}, \pi_{A_k}]) = \sum_{i=2}^{k} V(h\,; [\pi_{A_{i-1}}, \pi_{A_i}]) .$$

Let h be a continuous, real-valued function on $\hat{G}_0$ of finite variation. For $\pi_A \in \hat{G}_0$, let $h_1(\pi_A) = V(h; [\pi_0, \pi_A])$. Let $h_2 = h_1 - h$. Then h_1 and h_2 are continuous, non-decreasing functions on $\hat{G}_0$.

3. The closed ideals of $\mathscr{M}(G)$.

3.1. LEMMA. *Let $\pi_A, \pi_B \in \hat{G}_0$, where $\pi_A \leqq \pi_B$,* and let $\lambda \in \mathscr{M}(G)$.

Then

$$|\lambda|(B-A) = V(\hat{\lambda}\,;\ [\pi_A, \pi_B])\,. \tag{3.1.1}$$

In particular, $\|\lambda\| = |\lambda|(G) = V(\hat{\lambda})$.

Proof. It is easy to show that $V(\hat{\lambda}\,;\ [\pi_A,\pi_B]) \leqq |\lambda|(B-A)$.

Let $\varepsilon > 0$. Let $E_1, \cdots, E_m$ be pairwise disjoint non-void Borel sets whose union is $B-A$. For $i = 1, \cdots, m$, let $K_i \subseteqq E_i$ be a compact set for which $|\lambda|(E_i - K_i) < \varepsilon/m$. By induction (and using the second part of 1.11) we obtain pairwise disjoint open sets $U_1, \cdots, U_m$ such that

(i) $K_i \subseteqq U_i \subseteq \bar{U}_i \subseteqq G - (\bigcup_{j=i+1}^m K_j \cup \bigcup_{j=1}^{i-1} \bar{U}_j)$,

(ii) $|\lambda|(U_i - K_i) < \varepsilon/m$,

(iii) U_i is a finite union of pairwise disjoint open intervals ;

$i = 1, \cdots, m$. Now $\bigcup_{i=1}^m U_i$ is the finite union of pairwise disjoint open intervals, say $\{I'_j\}_{j=1}^r$, such that each I'_j is a subset of some U_i. For $j = 1, \cdots, r$, let $I_j = I'_j \cap (B-A)$. Evidently $\bigcup_{j=1}^r I_j = \bigcup_{i=1}^m (U_i \cap (B - A))$; we may suppose that each I_j is non-void. Let $A_{2j} = \{x \in G : x \leqq y$ for some $y \in I_j\}$ $(j = 1, \cdots, r)$. Relabelling if necessary, we may suppose that $A_2 \subset A_4 \subset \cdots \subset A_{2r}$. Let $A_{2j-1} = \{x \in G : x < y$ for all $y \in I_j\}$. Then $\pi_A \leqq \pi_{A_1} < \pi_{A_2} \leqq \pi_{A_3} < \pi_{A_4} \leqq \cdots < \pi_{A_{2r}} \leqq \pi_B$ and $I_j = A_{2j} - A_{2j-1}$ for $j = 1, \cdots, r$. Now

$$\begin{aligned} V(\hat{\lambda}\,;\ [\pi_A, \pi_B]) &\geqq \sum_{i=1}^{2r-1} |\hat{\lambda}(\pi_{A_{i+1}}) - \hat{\lambda}(\pi_{A_i})| = \sum_{i=1}^{2r-1} |\lambda(A_{i+1} - A_i)| \\ &\geqq \sum_{j=1}^{r} |\lambda(I_j)| \geqq \sum_{i=1}^{m} |\lambda(U_i \cap (B-A))| \end{aligned}$$

whereas

$$\begin{aligned} \sum_{i=1}^m |\lambda(E_i)| &= \sum_{i=1}^m |\lambda(E_i - K_i) + \lambda(U_i \cap (B-A)) \\ &- \lambda((U_i \cap (B-A)) - K_i)| \leqq 2\varepsilon + \sum_{i=1}^m |\lambda(U_i \cap (B-A))| \end{aligned}$$

so that

$$\sum_{i=1}^m |\lambda(E_i)| \leqq 2\varepsilon + V(\hat{\lambda}\,;\ [\pi_A, \pi_B]).$$

It follows that $|\lambda|(B-A) \leqq V(\hat{\lambda}\,;\ [\pi_A, \pi_B])$ since $\{E_i\}_{i=1}^m$ and ε are arbitrary.

3.2. LEMMA. *Let R be an interval of $\hat{G}_0$ of the form* 2.8.1 *or* 2.8.3. *Suppose that $\lambda \in \mathscr{M}(G)$ and that $\hat{\lambda}(\pi) \neq 0$ for all $\pi \in R$. Then there exists a $\nu \in \mathscr{M}(G)$ such that*

$$\hat{\nu}(\pi) = \begin{cases} \dfrac{1}{\hat{\lambda}(\pi)} & \text{for } \pi \in R\,, \\ 0 & \text{for } \pi \notin R\,. \end{cases} \tag{3.2.1}$$

Proof. Suppose that $R =]\pi_{x[}\,, \pi_{y]}[$ and let $X = [x, y[$. Evidently X is a locally compact subsemigroup of G. Throughout this proof, elements of $\hat{X}$ will be denoted by $\tilde{\pi}$; whenever the symbol $\tilde{\pi}_A$ occurs, it is tacitly assumed that $A \subseteqq X$ and that $\{A, X - A\}$ is a Dedekind cut of X. The functions $\hat{\lambda}$ will be considered defined on $\hat{G}$ or $\hat{X}$ rather than $\hat{G}_0$ or $\hat{X}_0$. For Borel sets $E \subseteqq X$, let $\tilde{\lambda}(E) = \lambda(E \cap X) + \lambda(] - \infty, x[)\,\varepsilon_x(E)$. We have $\tilde{\lambda} \in \mathscr{M}(X)$. We now show that

$$\hat{\tilde{\lambda}}(\tilde{\pi}_A) = \hat{\lambda}(\pi_{A \cup]-\infty, x[}) \text{ for } \tilde{\pi}_A \in \hat{X}\,. \tag{3.2.2}$$

Indeed $\hat{\tilde{\lambda}}(\tilde{\pi}_A) = \tilde{\lambda}(A) = \lambda(A \cap X) + \lambda(] - \infty, x[)\,\varepsilon_x(A) = \lambda(A) + \lambda(]-\infty, x[) = \lambda(A \cup] - \infty, x[) = \hat{\lambda}(\pi_{A\cup]-\infty, x[})$. Since $\pi_{A\cup]-\infty, x[} \in R$ whenever $\tilde{\pi}_A \in \hat{X}$, it follows from 3.2.2. that

$$\hat{\tilde{\lambda}}(\tilde{\pi}_A) \neq 0 \quad \text{for} \quad \tilde{\pi}_A \in \hat{X}\,. \tag{3.2.3}$$

By Theorem 4.15.1 (9) [4], $\tilde{\lambda} \in \mathscr{M}(X)$ has an inverse $\tilde{\nu} \in \mathscr{M}(X)$. For Borel sets $E \subseteqq G$, let

$$\nu(E) = \tilde{\nu}(E \cap X) - \tilde{\nu}(X)\varepsilon_y(E)\,.$$

Evidently $\nu \in \mathscr{M}(G)$. It is now routine to verify 3.2.1.

If $R =]\pi_{x[}\,, \pi_G]$, we let $X = [x, \infty[$ and repeat the preceding proof with the appropriate modifications.

3.3. NOTATION. For subsets A and B of G (or $\hat{G}_0$), we write $A < B$ if $x \in A$ and $y \in B$ imply $x < y$ and $A \leqq B$ if $x \in A$ and $y \in B$ imply $x \leqq y$. Note, in particular, that $0 < A$ and $A < 0$ for any set A. Let $P = \{\pi_1, \cdots, \pi_m\}$ be a finite subset of $\hat{G}_0$ where $\pi_1 < \pi_2 < \cdots < \pi_m$. We will sometimes write $\sum(\hat{\lambda}\,; P)$ for $\sum_{i=1}^{m-1} |\hat{\lambda}(\pi_{i+1}) - \hat{\lambda}(\pi_i)|$, $\lambda \in \mathscr{M}(G)$,.

For $\pi_A \in \hat{G}_0$, let $I_A = \{\lambda \in \mathscr{M}(G)\colon \lambda(A) = 0\}$. Note that $I_0 = \mathscr{M}(G)$. Since each $I_A(\pi_A \in \hat{G})$ is the kernel of the homomorphism π_A, the set $\{I_A\}_{\pi_A \in \hat{G}}$ is precisely the set of all regular maximal closed ideals in $\mathscr{M}(G)$.

The following theorem characterizes the closed ideals in $\mathscr{M}(G)$.

3.4. THEOREM. *Let $I \subseteqq \mathscr{M}(G)$ be a closed ideal. Let $H = \{\pi \in \hat{G}_0\colon \hat{\lambda}(\pi) = 0$ for all $\lambda \in I\}$. Then H is closed in $\hat{G}_0$ and*

$$I = \bigcap_{\pi_A \in H} I_A\,. \tag{3.4.1}$$

Proof. Obviously $H = \bigcap_{\lambda \in I} (\hat{\lambda})^{-1}(0)$ is closed and $I \subseteqq \bigcap_{\pi_A \in H} I_A$.

Let λ be a fixed element of $\bigcap_{\pi_A \in H} I_A$. Let $Z = \{\pi \in \hat{G}_0 : \hat{\lambda}(\pi) = 0\}$. Clearly Z is closed in $\hat{G}_0$, $H \subseteqq Z$, and $\pi_0 \in Z$. By Lemma 1.10, the complement Z' of Z in $\hat{G}_0$ is a pairwise disjoint union of open intervals:

$$Z' = \bigcup_\alpha]\pi_{A_\alpha}, \pi_{B_\alpha}[$$

where one of these intervals may be of the form $]\pi_{A_\alpha}, \pi_G]$. Moreover, $\pi_{A_\alpha} \in Z$ for all α and $\pi_{B_\alpha} \in Z$ for all α except possibly when $\pi_{B_\alpha} = \pi_G$. We assume in the following that $\pi_G \notin Z'$; elementary modifications are necessary when $\pi_G \in Z'$.

We first prove

$$V(\hat{\lambda}) = \sum_\alpha V(\hat{\lambda}\,; [\pi_{A_\alpha}, \pi_{B_\alpha}]) . \tag{3.4.2}$$

Using 3.1, we have $\sum_\alpha V(\hat{\lambda}\,; [\pi_{A_\alpha}, \pi_{B_\alpha}]) = \sum_\alpha |\lambda|(B_\alpha - A_\alpha) \leqq |\lambda|(G) = V(\hat{\lambda})$. Let $\pi_1 < \pi_2 < \cdots < \pi_m$, $\pi_i \in \hat{G}_0$, and call this partition P'. Let $P = P' \cup \{\pi_G\}$. Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be precisely those α such that $]\pi_{A_{\alpha_i}}, \pi_{B_{\alpha_i}}[\cap P \neq 0$. For this paragraph we write A_i for A_{α_i} and B_i for B_{α_i}. We may suppose that $]\pi_{A_i}, \pi_{B_i}[<]\pi_{A_{i+1}}, \pi_{B_{i+1}}[$ $(i = 1, \cdots, k-1)$. For $i = 1, \cdots, k$, let $P_i =]\pi_{A_i}, \pi_{B_i}[\cap P$. Let $Z_0 = [\pi_0, \pi_{A_1}] \cap P$. For $i = 1, \cdots, k-1$, let $Z_i = [\pi_{B_i}, \pi_{A_{i+1}}] \cap P$. Let $Z_k = [\pi_{B_k}, \pi_G] \cap P$. Clearly some or all of the Z_i may be void. Evidently we have:

(i) $P = Z_0 \cup P_1 \cup Z_1 \cup P_2 \cup \cdots \cup P_{k-1} \cup Z_{k-1} \cup P_k \cup Z_k$;
(ii) $Z_0 < P_1 < Z_1 < P_2 < \cdots < P_{k-1} < Z_{k-1} < P_k < Z_k$;
(iii) $Z \cap P = \bigcup_{i=0}^k Z_i$;
(iv) $P_i \subseteqq]\pi_{A_i}, \pi_{B_i}[$ $(i = 1, \cdots, k)$;
(v) the intervals given in (iv) are pairwise disjoint.

Now let $P^* = P \cup \{\pi_{A_1}, \pi_{B_1}, \pi_{A_2}, \pi_{B_2}, \cdots, \pi_{A_k}, \pi_{B_k}\}$. Clearly $Z_0 \leqq \{\pi_{A_1}\} < P_1 < \{\pi_{B_1}\} \leqq Z_1 \leqq \{\pi_{A_2}\} < P_2 < \cdots \leqq Z_{k-1} \leqq \{\pi_{A_k}\} < P_k < \{\pi_{B_k}\} \leqq Z_k$. Using the notation established in 3.3, we now get

$$\sum_{i=1}^{m-1} |\hat{\lambda}(\pi_{i+1}) - \hat{\lambda}(\pi_i)| = \sum(\hat{\lambda}\,; P') \leqq \sum(\hat{\lambda}\,; P^*)$$
$$= \sum_{i=1}^{k} \sum(\hat{\lambda}\,; \{\pi_{A_i}\} \cup P_i \cup \{\pi_{B_i}\}) .$$

By 2.9, we have $\sum(\hat{\lambda}\,; \{\pi_{A_i}\} \cup P_i \cup \{\pi_{B_i}\}) \leqq V(\hat{\lambda}\,; [\pi_{A_i}, \pi_{B_i}])$ for $i = 1, \cdots, k$. Combining these inequalities, we obtain

$$\sum_{i=1}^{m-1} |\hat{\lambda}(\pi_{i+1}) - \hat{\lambda}(\pi_i)| \leqq \sum_{i=1}^{k} V(\hat{\lambda}; [\pi_{A_i}, \pi_{B_i}]) \leqq \sum_\alpha V(\hat{\lambda}\,; [\pi_{A_\alpha}, \pi_{B_\alpha}]) .$$

Since the partition P' was arbitrary, we have $V(\hat{\lambda}) \leqq \sum_\alpha V(\hat{\lambda}\,; [\pi_{A_\alpha}, \pi_{B_\alpha}])$ and hence 3.4.2 is proved.

Let $\varepsilon > 0$. We shall ultimately show that there is a $\mu \in I$ such that $\|\lambda - \mu\| \leqq 3\varepsilon$. Since ε is arbitrary and I is closed, this will prove that

$\lambda \in I$. It will then follow that $\bigcap_{\pi_A \in H} I_A \subseteq I$, completing the proof. By 3.4.2, there exist $\alpha_1, \cdots, \alpha_m$ such that $\sum_{i=1}^m V(\hat{\lambda}; [\pi_{A_{\alpha_i}}, \pi_{B_{\alpha_i}}]) + \varepsilon \geqq V(\hat{\lambda})$. We shall henceforth write A_i for A_{α_i} and B_i for B_{α_i}. Then

$$V(\hat{\lambda}) - \sum_{i=1}^m V(\hat{\lambda}; [\pi_{A_i}, \pi_{B_i}]) \leqq \varepsilon . \tag{3.4.3}$$

We may suppose that $A_1 \subset B_1 \subseteqq A_2 \subset B_2 \subseteqq \cdots \subseteqq A_m \subset B_m$. By 1.8, there exist $x_i, y_i \in B_i - A_i$ such that

$$|\lambda|((B_i - A_i) - [x_i, y_i]) \leqq \frac{\varepsilon}{m} \qquad (i = 1, \cdots, m) . \tag{3.4.4}$$

Let $U_i =]\pi_{x_i[}, \pi_{y_i]}[$; obviously U_i is open and closed. Note also that $U_i \subseteqq]\pi_{A_i}, \pi_{B_i}[\subseteqq Z'$. Let $U = \bigcup_{i=1}^m U_i$; U is open and closed (and hence compact). Also $U \subseteqq Z' \subseteqq H'$ where H' denotes the complement of H in $\hat{G}_0$. Thus for each $\pi_A \in U$, there is a $\lambda_A \in I$ such that $\lambda_A(A) = \hat{\lambda}_A(\pi_A) \neq 0$. Note that $\pi_0 \notin U$ since $\pi_0 \in H$ and $\pi_G \notin U$ since $\pi_G \notin Z'$. By the continuity of $\hat{\lambda}_A$ on $\hat{G}_0$ and Theorem 2.8, there exists an open and closed set V_A such that

(a) $\pi_A \in V_A$;

(b) $\pi \in V_A$ implies $\hat{\lambda}_A(\pi) \neq 0$;

(c) $V_A \subseteqq U$;

(d) V_A has the form 2.8.1.[2]

Since U is compact and $\bigcup_{\pi_A \in U} V_A = U$, there is a finite set $\{V_{A_i}\}_{i=1}^p$ such that $\bigcup_{i=1}^p V_{A_i} = U$.

For $V_{A_i} =]\pi_{a_i[}, \pi_{b_i]}[$, let $V_{A_i}^- = [\pi_0, \pi_{a_i]}[$ and $V_{A_i}^+ =]\pi_{b_i[}, \pi_G]$. Let $\mathscr{V}$ be the family of sets consisting of all V_{A_i}, $V_{A_i}^-$, and $V_{A_i}^+$. For $\pi \in U$, let $R_\pi = \bigcap \{V \in \mathscr{V} : \pi \in V\}$. Clearly there exist only finite many distinct R_π — say $\{R_i\}_{i=1}^k$.

The following assertions are easily shown:

(a′) $\bigcup_{i=1}^k R_i = U$;

(b′) each R_i has the form 2.8.1[3];

(c′) the family $\{R_i\}_{i=1}^k$ is pairwise disjoint;

(d′) for each i, there exists a $\lambda_i \in I$ such that $\pi \in R_i$ implies $\hat{\lambda}_i(\pi) \neq 0$.

By Lemma 3.2[3], there are $\nu_i \in \mathscr{M}(G)$ such that

$$\hat{\nu}_i(\pi) = \begin{cases} \dfrac{1}{\hat{\lambda}_i(\pi)} & \text{if } \pi \in R_i , \\ 0 & \text{if } \pi \notin R_i ; \end{cases}$$

$i = 1, \cdots, k$. Let $\mu = \sum_{i=1}^k \lambda_i * \nu_i * \lambda$; clearly $\mu \in I$. Evidently

[2] If $\pi_G \in Z'$, then V_A can be of the form 2.8.3.

[3] If $\pi_G \in Z'$, then R_i can be of the form 2.8.3.

$$\hat{\mu}(\pi) = \begin{cases} \hat{\lambda}(\pi) & \text{if } \pi \in U\,, \\ 0 & \text{if } \pi \notin U\,. \end{cases}$$

We observe that

$$(\hat{\lambda} - \hat{\mu})(\pi) = \begin{cases} 0 & \text{if } \pi \in U_i =]\pi_{x_i[}, \pi_{y_i]}[\,, \\ \hat{\lambda}(\pi) & \text{if } \pi = \pi_{x_i[} \text{ or } \pi = \pi_{y_i]}\,. \end{cases}$$

Using this, Lemma 3.1, and relation 3.4.4, we have

$$\begin{aligned} (3.4.5) \qquad V(\hat{\lambda} - \hat{\mu}\,;\, [\pi_{x_i[}, \pi_{y_i]}]) &= |\hat{\lambda}(\pi_{x_i[})| + |\hat{\lambda}(\pi_{y_i]})| \\ &= |\hat{\lambda}(\pi_{x_i[}) - \hat{\lambda}(\pi_{A_i})| + |\hat{\lambda}(\pi_{B_i}) - \hat{\lambda}(\pi_{y_i]})| \\ &\leqq V(\hat{\lambda}\,; [\pi_{A_i}, \pi_{x_i[}]) + V(\hat{\lambda}\,; [\pi_{y_i]}, \pi_{B_i}]) \\ &\leqq |\lambda|(]-\infty, x_i[- A_i) + |\lambda|(B_i -]-\infty, y_i]) \\ &= |\lambda|((B_i - A_i) - [x_i, y_i]) \leqq \frac{\varepsilon}{m}\,. \end{aligned}$$

We also have from 3.1 that

$$\begin{aligned} (3.4.6) \qquad & V(\hat{\lambda}\,; [\pi_{y_i]}, \pi_{B_i}]) + V(\hat{\lambda}\,; [\pi_{A_i}, \pi_{x_i[}]) \\ &= |\lambda|((B_i - A_i) - [x_i, y_i]) \leqq \frac{\varepsilon}{m}\,. \end{aligned}$$

Using 2.10, 3.4.5, and 3.4.6, we obtain

$$\begin{aligned} (3.4.7) \quad V(\hat{\lambda} - \hat{\mu}; [\pi_{A_i}, \pi_{B_i}]) &= V(\hat{\lambda} - \hat{\mu}; [\pi_{A_i}, \pi_{x_i[}]) + V(\hat{\lambda} - \hat{\mu}; [\pi_{x_i[}, \pi_{y_i]}]) \\ &+ V(\hat{\lambda} - \hat{\mu}; [\pi_{y_i]}, \pi_{B_i}]) = V(\hat{\lambda}\,; [\pi_{A_i}, \pi_{x_i[}]) \\ &+ V(\hat{\lambda} - \hat{\mu}; [\pi_{x_i[}, \pi_{y_i]}]) + V(\hat{\lambda}\,; [\pi_{y_i]}, \pi_{B_i}]) \leqq \frac{2\varepsilon}{m}\,. \end{aligned}$$

We used the fact that $\hat{\mu}$ is zero on $[\pi_{A_i}, \pi_{x_i[}]$ and $[\pi_{y_i]}, \pi_{B_i}]$ since these sets are disjoint from U. Finally, using 2.10, 3.1, and 3.4.7, we get

$$\begin{aligned} ||\lambda - \mu|| &= V(\hat{\lambda} - \hat{\mu}) = V(\hat{\lambda} - \hat{\mu}; [\pi_{B_m}, \pi_G]) + V(\hat{\lambda} - \hat{\mu}; [\pi_0, \pi_{A_1}]) \\ &+ \sum_{i=2}^{m} V(\hat{\lambda} - \hat{\mu}; [\pi_{B_{i-1}}, \pi_{A_i}]) + \sum_{i=1}^{m} V(\hat{\lambda} - \hat{\mu}; [\pi_{A_i}, \pi_{B_i}]) \\ &= V(\hat{\lambda}\,; [\pi_{B_m}, \pi_G]) + V(\hat{\lambda}\,; [\pi_0, \pi_{A_1}]) + \sum_{i=2}^{m} V(\hat{\lambda}\,; [\pi_{B_{i-1}}, \pi_{A_i}]) \\ &+ \sum_{i=1}^{m} V(\hat{\lambda} - \hat{\mu}; [\pi_{A_i}, \pi_{B_i}]) \leqq |\lambda|(G - B_m) + |\lambda|(A_1) + \sum_{i=2}^{m} |\lambda|(A_i - B_{i-1}) \\ &+ 2\varepsilon = |\lambda|(G) - \sum_{i=1}^{m} |\lambda|(B_i - A_i) + 2\varepsilon = V(\hat{\lambda}) \\ &- \sum_{i=1}^{m} V(\hat{\lambda}\,; [\pi_{A_i}, \pi_{B_i}]) + 2\varepsilon\,. \end{aligned}$$

Now applying 3.4.3, we obtain $||\lambda - \mu|| \leq 3\varepsilon$. This completes the proof.

3.5. EXAMPLES. Let $G =]0, 1[$ and $\lambda \in \mathscr{M}(G)$ be ordinary Lebesgue measure. Then the ideal $I = \{\lambda*\mu + \alpha\lambda: \mu \in \mathscr{M}(G)$ and α is a complex number$\}$ is dense in $\mathscr{M}(G)$ since $\hat{\lambda}$ vanishes only at π_0; I is the ideal generated by λ. If $G = [0, 1]$ and λ is Lebesgue measure, then $I = \{\lambda*\mu: \mu \in \mathscr{M}(G)\}$ is the ideal generated by λ and I is dense in $\{\lambda \in \mathscr{M}(G): \lambda(\{0\}) = 0\}$.

4. The Herglotz-Bochner theorem for $\mathscr{M}(G)$.

This section generalizes § 6 [3].

4.1. DEFINITION. Let h be any bounded, real-valued, nondecreasing function on $\hat{G}_0$. Let Δ denote a partition $\{t_k\}_{k=0}^m$ of G where $t_0 < t_1 < \cdots < t_m$. For an arbitrary complex-valued function f on G, let

$$S(f, \Delta) = f(t_0)\,[h(\pi_{t_0]}) - h(\pi_{t_0[})] + \sum_{k=1}^{m} f(t_k)\,[h(\pi_{t_k]}) - h(\pi_{t_{k-1}]})] \,.$$

4.2. THEOREM. *Let $f \in \mathfrak{C}_0(G)$ and h be as in 4.1. Then there exists a unique number $L(f)$ such that for every $\varepsilon > 0$ there exists a Δ_0 as in 4.1 with the property that $|L(f) - S(f, \Delta)| \leqq \varepsilon$ for all $\Delta \supseteqq \Delta_0$. We write this relation as $L(f) = \lim_\Delta S(f, \Delta)$.*

4.3. THEOREM. *The function L defined in 4.2 for all $f \in \mathfrak{C}_0(G)$ is a bounded nonnegative linear functional on $\mathfrak{C}_0(G)$.*

4.4. DEFINITION. Let h be a continuous function on $\hat{G}_0$ and let $\pi_A, \pi_B \in \hat{G}_0$, $\pi_A < \pi_B$. Then we define

$$V_c(h; [\pi_A, \pi_B]) = \sup\Big\{\sum_{i=1}^{m} V(h; [\pi_{x_i[}, \pi_{y_i]}]): \quad x_1 \leqq y_1 < x_2 \leqq y_2 < \cdots < x_m \leqq y_m, \quad \pi_A \leqq \pi_{x_1[}, \pi_{y_m]} \leqq \pi_B,\ [x_i, y_i] \ \text{compact}\Big\} \,. \tag{4.4.1}$$

In particular, we define $V_c(h) = V_c(h; [\pi_0, \pi_G])$. We also define

$$V_c(h; [\pi_A, \pi_A]) = 0 \tag{4.4.2}$$

for $\pi_A \in \hat{G}_0$.

4.5. Let h be a real-valued continuous function on $\hat{G}_0$ having finite variation and let $\pi_{A_1} \leqq \pi_{A_2} \leqq \cdots \leqq \pi_{A_k}$. Then

$$V_c(h; [\pi_{A_1}, \pi_{A_k}]) = \sum_{i=2}^{k} V_c(h; [\pi_{A_{i-1}}, \pi_{A_i}]) \,. \tag{4.5.1}$$

4.6. THEOREM. *Let h be a continuous function on $\hat{G}_0$ having finite* *variation and such that* $h(\pi_0) = 0$. *Then there exists a* $\lambda \in \mathscr{M}(G)$ *such that* $\hat{\lambda} = h$ *if and only if*

$$V(h) = V_c(h) \tag{4.6.1}$$

The proof is a tedious lengthy extension of the proof of Theorem 6.7 [3] and uses 4.2, 4.3, 3.1, 4.5, and 1.11 in the case that h is non-decreasing. The general case is proved by applying 2.10.

4.7. EXAMPLES. Let G be the real line under the usual ordering. Then a function h on $\hat{G}_0$ is the Fourier transform of some measure $\lambda \in \mathscr{M}(G)$ if and only if h is continuous, has finite variation, and $h(\pi_0) = 0$.

Condition 4.6.1 is not always satisfied by continuous functions h on $\hat{G}_0$ having finite variation and satisfying $h(\pi_0) = 0$. Let $G = [0, 1] \times]0, 1[$ where $(a, b) < (c, d)$ if $a < c$ or if $a = c$ and $b < d$. Let h on $\hat{G}_0$ be defined by

$$h(\pi_A) = \sup \{a \in [0, 1]\colon (a, x) \in A \text{ for some } x \in \;]0, 1[\} .$$

The function h is continuous, $V(h) = 1$, and $V_c(h) = 0$. The linear functional L obtained from h in 4.3 turns out to be the zero functional.

5. Some consequences of the Herglotz-Bochner theorem.

Theorems 5.1 and 5.2 are routine applications of 4.6.

5.1. THEOREM. *Let* ϕ *be a continuous function from a subset* $H \supseteqq \{0\}$ *of the complex plane to the complex plane such that* $\phi(0) = 0$ *and*

(5.1.1) *for every* $M > 0$, *there exists a* $K_M > 0$ *such that*
$$|\phi(z) - \phi(w)| \leqq K_M\,|z - w| \text{ for } z, w \in H,\ |z| \leqq M,\ |w| \leqq M .$$

(I.e., ϕ *satisfies a Lipschitz condition for arbitrarily large disks.) Then for every* $\lambda \in \mathscr{M}(G)$ *for which (range* $\hat{\lambda}) \subseteqq H$, *there exists a* $\nu \in \mathscr{M}(G)$ *such that* $\hat{\nu} = \phi\circ\hat{\lambda}$.

5.2. THEOREM. *Let* ϕ *be a continuous function from* $[0, \infty[$ to $[0, \infty[$ *that is non-decreasing, absolutely continuous on all intervals* $[0, M]$, *and such that* $\phi(0) = 0$. *Then for every nonnegative measure* $\lambda \in \mathscr{M}(G)$ *there exists a nonnegative* $\nu \in \mathscr{M}(G)$ *such that* $\hat{\nu} = \phi\circ\hat{\lambda}$.

5.3. COROLLARY. *Let* $\lambda \in \mathscr{M}(G)$. *Then there exists a* $\nu \in \mathscr{M}(G)$ *such that* $\hat{\nu}(\pi) = |\hat{\lambda}(\pi)|$ *for all* $\pi \in \hat{G}_0$.

5.4. COROLLARY. *Let* $\lambda \in \mathscr{M}(G)$. *Then there exists a* $\nu \in \mathscr{M}(G)$ *such that* $\hat{\nu}(\pi) = \overline{\hat{\lambda}(\pi)}$ *for all* $\pi \in \hat{G}_0$; *here* $\bar{z}$ *denotes the complex conju-*

gate of z. In other words, $\mathscr{M}(G)$ is self-adjoint (see page 88 [6]).

5.5. COROLLARY. *Let $\lambda \in \mathscr{M}(G)$ be a nonnegative measure. Then there exists a nonnegative $\nu \in \mathscr{M}(G)$ such that $\nu * \nu = \lambda$.*

5.6. It is natural to ask whether Theorem 5.2 is valid for more general measures λ; one might hope that the result would be valid at least for $\lambda \in \mathscr{M}(G)$ for which $\hat{\lambda}$ is nonnegative. If this were the case, 5.5 would also generalize. However, we will see in 5.7 that this is not the case whenever G is infinite. Theorem 5.7 also shows that the Lipschitz condition assumed for ϕ in 5.1 cannot be replaced by absolute continuity. (The function $\phi(x) = \sqrt{x}$ is absolutely continuous on all intervals $[0, M]$ but does not satisfy 5.1.1.)

5.7. THEOREM. *Suppose that G is infinite. Then there exists a $\lambda \in \mathscr{M}(G)$ such that $\hat{\lambda}$ is nonnegative on $\hat{G}_0$ and such that $\lambda \neq \nu * \nu$ for all $\nu \in \mathscr{M}(G)$.*

Proof. Suppose G has an infinite subset $\{x_i\}_{i=1}^{\infty}$ such that $x_i < x_{i+1}$ for all i. Let λ be the discrete measure defined by

$$\lambda(\{x_n\}) = \begin{cases} \dfrac{1}{n^2} & \text{if } n \text{ odd}\,, \\ -\dfrac{1}{(n-1)^2} & \text{if } n \text{ even}\,. \end{cases}$$

It can be shown that λ satisfies the conclusions of the theorem. If G does not have an infinite subset as above, then G has an infinite subset $\{x_i\}_{i=1}^{\infty}$ such that $x_i > x_{i+1}$ for all i. This case is treated in a similar manner.

5.8. It is evident from 5.7 that $\mathscr{M}(G)$ (G infinite) is not isomorphic as an algebra to the algebra $\mathfrak{C}_0(X)$ for any locally compact space X. In the contrary case, $\mathscr{M}(G)$ would be isomorphic to $\mathfrak{C}_0(\hat{G})$ and the isomorphism would be $\lambda \to \hat{\lambda}$. However, if $h \in \mathfrak{C}_0(\hat{G})$ is nonnegative, then for some $h_0 \in \mathfrak{C}_0(\hat{G})$, we have $h_0^2 = h$.

Finally, the result of 8.3 [3] holds for locally compact G. That is,

5.9. THEOREM. *A measure $\lambda \in \mathscr{M}(G)$ is idempotent if and only if λ is of the form:*

$$\lambda = \varepsilon_{c_0} - \varepsilon_{c_1} + \cdots + (-1)^k \varepsilon_{c_k} \tag{5.9.1}$$

where $c_0 < c_1 < \cdots < c_k$.

REFERENCES

1. P. R. Halmos, *Measure theory*, D. van Nostrand Co., New York, 1950.
2. E. Hewitt, *Remarks on the inversion of Fourier-Stieltjes transforms*, Ann. of Math., (2) **57** (1953), 458-474.
3. E. Hewitt and H. S. Zuckerman, *Structure theory for a class of convolution algebras*, Pacific J. Math., **7** (1957), 913-941.
4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, (revised edition) A. M. S. Colloquium Publ. XXXI, New York, 1957.
5. J. L. Kelley, *General topology*, D. van Nostrand Co., New York, 1955.
6. L. H. Loomis, *An introduction to abstract harmonic analysis*, D. van Nostrand Co., New York, 1953.
7. P. Malliavin, *Impossibitié de la synthèse spectrale sur les groupes abéliens non compacts*, Publications Mathématiques, Institut des hautes études scientifiques, no. 2, 1959.
8. K. Stromberg, *A note on the convolution of regular measures*, Math. Scand. **7** (1959), 347-352.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 11, No. 2 December, 1961