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WIRTINGER-TYPE INTEGRAL INEQUALITIES

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1. Introduction. The following inequalities (and other similar ones) are known:

(i) if $u'(x) \in L_2$ and $u(0) = 0$, then

$$\int_0^{\pi/2} u^2 dx \leq \int_0^{\pi/2} u'^2 dx \quad [4];$$

(ii) if $u''(x) \in L_2$ and $u(0) = u(\pi) = 0$, then

$$\int_0^{\pi} u^2 dx \leq \int_0^{\pi} u''^2 dx \quad [3];$$

in each case, equality occurs if and only if $u(x) \equiv A \sin x$. P. R. Beesack [1] has generalized these two types of inequalities by considering the underlying differential equations $y'' + py = 0$ and $y^{(2v)} - py = 0$ respectively, together with the equations satisfied by y'/y . In [2], a relation was obtained between the equation $y^{(2n)} - py = 0$ and the inequality

$$(-1)^n \int_a^b p u^2 dx \leq \int_a^b u^{(n)2} dx .$$

In this paper we let Ly be the general self-adjoint linear operator of even order

$$\sum_{i=0}^n (f_i y^{(i)})^{(i)}$$

and extend the methods of [2] to relate the equation

$$(1) \quad Ly = 0$$

and the inequalities

$$(2) \quad 0 \leq \sum_{i=0}^n (-1)^{n+i} \int_a^b f_i u^{(i)2} dx$$

and

$$(3) \quad 0 \geq \int_a^b \frac{1}{f_n} \cdot u^2 dx + (-1)^n \int_a^b \frac{1}{f_0} \cdot u^{(n)2} dx .$$

2. Notation and lemmas. Let $y_i = f_i y^{(i)}$, $v_i = \sum_{k=0}^i y_{n-k}^{(i-k)}$,

$$u_{ij} = v_{n-i} / y^{(j)}, \text{ and } y_{ij} = y^{(i)} / y^{(j)} \quad (i = 0, \dots, n) .$$

Then

$$(4) \quad v_i = v'_{i-1} + y_{n-i} \quad (i = 1, \dots, n).$$

Let $(k_0 \dots k_n)$ be an $(n + 1)$ -tuple consisting of 0's and 1's, such that $\sum_{i=1}^n k_i$ is even. Let

$$(5) \quad c_i = \begin{cases} a, & k_i = 0 \\ b, & k_i = 1 \end{cases}; \quad d_i = \begin{cases} a, & k_{i+1} = 1 \\ b, & k_{i+1} = 0 \end{cases};$$

$$c_i^* = a + b - c_i; \quad p_i = (-1)^{\sum_{j=0}^i k_j}; \quad q_i = (-1)^i p_i; \quad (i = 0, \dots, n).$$

$$d_i^* = a + b - d_i;$$

We now and henceforth assume that (1) has a solution on $[a, b]$ such that

$$(6) \quad p_1 y^{(n-1)}(x) > 0 \quad \text{on } (a, b) \text{ and at } c_1^*;$$

$$p_i y^{(n-i)}(c_i) \geq 0 \quad (i = 2, \dots, n);$$

$$q_i v_i(d_i) \geq 0 \quad (i = 0, \dots, n - 1);$$

and that the $f_i(x) \in L[a, b]$, with $\int_a^b f_0(x) dx \neq 0$, and

$$(7) \quad (-1)^{n+i} f_i(x) \leq 0 \quad \text{on } [a, b] \quad (i = 0, \dots, n - 1);$$

$$f_n(x) \geq 0 \quad \text{on } [a, b].$$

LEMMA 1. *We have*

$$(8) \quad p_i y^{(n-i)}(x) > 0 \quad \text{on } (a, b) \text{ and at } c_i^* \quad (i = 1, \dots, n).$$

Proof. By hypothesis the lemma is true for $i = 1$. Suppose that, for some i such that $1 \leq i \leq n - 1$, the statement holds. Integrating and multiplying by $(-1)^{k_{i+1}}$ we have

$$p_{i+1} y^{(n-i-1)}(x) = p_{i+1} y^{(n-i-1)}(c_{i+1}) + (-1)^{k_{i+1}} \int_{c_{i+1}}^x p_i y^{(n-i)}(t) dt > 0$$

on (a, b) and at c_{i+1}^* . This completes Lemma 1.

LEMMA 2. *We have*

$$(9) \quad q_i v_i(x) \geq 0 \quad \text{on } [a, b], > 0 \text{ at } d_i^* \quad (i = 0, \dots, n - 1).$$

Proof. We proceed by induction on i ($i = n - 1, \dots, 1, 0$). Now $v'_{n-1}(x) = v_n(x) - y_0 = -y_0$, so

$$q_{n-1} v_{n-1}(x) = q_{n-1} v_{n-1}(d_{n-1}) - (-1)^{1+k_n} \int_{d_{n-1}}^x (-1)^n f_0 p_n y dt \geq 0;$$

since $|y| > 0$ and $\int_a^b f_0(x)dx \neq 0$, the inequality is strict at d_{n-1}^* .

Now suppose that, for some i ($n - 1 \geq i \geq 1$), the statement holds. Then, integrating (4) and multiplying by q_{i-1} ,

$$q_{i-1}v_{i-1}(x) = q_{i-1}v_{i-1}(d_{i-1}) + (-1)^{1+k_i} \int_{d_{i-1}}^x q_i v_i dt - (-1)^{1+k_i} \int_{d_{i-1}}^x (-1)^i f_{n-i} p_i y^{(n-i)} dt ,$$

so $q_{i-1}v_{i-1}(x) \geq 0$ on (a, b) and >0 at d_{i-1}^* . This completes Lemma 2.

3. The formal identity. Since (at least formally)

$$u_{ii} = v'_{n-i-1}/y^{(i)} + f_i ,$$

we have

$$(10) \quad u_{ii} = u'_{i+1,i} + u_{i+1,i+1}y_{i+1,i}^2 + f_i .$$

Now we use (10) and induction to derive the formal identity

$$(11) \quad 0 = \sum_{i=0}^{n-1} (-1)^{n+i} \left\{ u_{i+1,i} u^{(i)2} \Big|_a^b + \int_a^b u_{i+1,i+1} (u^{(i+1)} - y_{i+1,i} u^{(i)})^2 dx \right\} + \sum_{i=0}^n (-1)^{n+i} \int_a^b f_i u^{(i)2} dx ;$$

then we will justify the formal steps.

First,

$$\int_a^b u'_{i+1,i} u^{(i)2} dx = u_{i+1,i} u^{(i)2} \Big|_a^b - \int_a^b 2u_{i+1,i} u^{(i)} u^{(i+1)} dx = u_{i+1,i} u^{(i)2} \Big|_a^b - \int_a^b 2u_{i+1,i+1} y_{i+1,i} u^{(i)} u^{(i+1)} dx ,$$

so

$$(12) \quad \int_a^b (u'_{i+1,i} + u_{i+1,i+1}y_{i+1,i}^2) u^{(i)2} dx = u_{i+1,i} u^{(i)2} \Big|_a^b + \int_a^b u_{i+1,i+1} (u^{(i+1)} - y_{i+1,i} u^{(i)})^2 dx - \int_a^b u_{i+1,i+1} u^{(i+1)2} dx .$$

Since $v_n(x) \equiv Ly \equiv 0$, $u_{00}(x) \equiv 0$; using (10) and (12) with $i = 0$,

$$0 = u_{10} u^2 \Big|_a^b + \int_a^b u_{11} (u' - y_{10} u)^2 dx + \int_a^b f_0 u^2 dx - \int_a^b u_{11} u^2 dx .$$

Suppose that, for some k such that $1 \leq k \leq n - 1$,

$$(13) \quad 0 = \sum_{i=0}^{k-1} (-1)^i \left\{ u_{i+1,i} u^{(i)2} \Big|_a^b + \int_a^b u_{i+1,i+1} (u^{(i+1)} - y_{i+1,i} u^{(i)2}) dx \right\} + \sum_{i=0}^{k-1} (-1)^i \int_a^b f_i u^{(i)2} dx + (-1)^k \int_a^b u_{\kappa k} u^{(k)2} dx .$$

Using (10) and (12) with $i = k$, and substituting for the last term in (13), we obtain (13) with k replaced by $k + 1$. Hence (13) holds for $k = 1, \dots, n$; with $k = n$, using the fact that $u_{nn} \equiv f_n$, and multiplying by $(-1)^n$, we have (11).

LEMMA 3. Let $u(x)$ be a function such that

$$(14) \quad u^{(n)} \in L_2[a, b]; u^{(i)}(c_{n-i}) = 0 \quad (i = 0, \dots, n - 1) .$$

(Note that (14) implies that the zero of $u^{(i)}$ at c_{n-i} is of order ≥ 1 ($i = 0, \dots, n - 2$) and $> \frac{1}{2}$ ($i = n - 1$)). Then (11) is valid.

Proof. Our concern is with possible zeros of $y^{(i)}$ ($i = 0, \dots, n - 1$) on $[a, b]$; by Lemma 1, the only possible zero of $y^{(i)}$ is at c_{n-i} . Let i be such that $0 \leq i \leq n - 1$, and suppose that $y^{(i)}$ has a zero of order r at c_{n-i} . Then $r \leq n - i$. For if $r > n - i$ then $y^{(i+k)}(c_{n-i}) = 0$ ($k = 1, \dots, n - i$), and so $c_{n-i} = c_{n-i-1} = \dots = c_1$; thus $y^{(n)}(c_1) = 0$. But, by Lemma 2, $v_0(c_1) \neq 0$ (since $c_1 = d_0^*$), and $v_0(x) = f_n(x) y^{(n)}(x)$. Thus $r \leq n - i$. Now, since $c_{n-i} = \dots = c_1$, $u^{(i)}$ has a zero of order $\geq r$ at c_{n-i} ($i = 0, \dots, n - 2$), and of order $> \frac{1}{2}$ ($i = n - 1$). The lemma now follows, as does the fact (to be used in the proof of Lemma 5) that $u_{i+1,i}(c_{n-i}) u^{(i)2}(c_{n-i}) = 0$ ($i = 0, \dots, n - 1$).

LEMMA 4. On $[a, b]$, $(-1)^{n+i-1} u_{ii}(x) \leq 0$ ($i = 1, \dots, n$).

Proof. By Lemmas 1 and 2,

$$\begin{aligned} (-1)^{n+i-1} u_{ii} &= (-1)^{n+i-1} \cdot (-1)^{n-i} \cdot q_{n-i} v_{n-i} / p_{n-i} y^{(i)} \\ &= -q_{n-i} v_{n-i} / p_{n-i} y^{(i)} \leq 0 . \end{aligned}$$

LEMMA 5. $(-1)^{n+i} u_{i+1,i} u^{(i)2} \Big|_a^b \leq 0$ ($i = 0, \dots, n - 1$).

Proof. Since $c_j = d_{j-1}^*$,

$$(-1)^{n+i} u_{i+1,i} u^{(i)2} \Big|_a^b = (-1)^{n+i+1+k_{n-i}} u_{i+1,i} u^{(i)2} \Big|_{d_{n-i-1}^*} .$$

Evaluation at c_{n-i} gives zero, and

$$(-1)^{n+\ell+k}n^{-i}u_{i+1,i} = -q_{n-i-1}v_{n-i-1}/p_{n-i}y^{(\ell)} \leq 0$$

on $[a, b]$ and so at d_{n-i-1} .

4. The inequality. We now state

THEOREM 1. Let $f_i(x) \in L[a, b]$ ($i = 0, \dots, n$), with $\int_a^b f_0(x)dx \neq 0$. Let $f_i(x)$ ($i = 0, \dots, n$) satisfy (7), and let $y(x)$ be a solution of (1) which satisfies (6). Let $u(x)$ satisfy (14). Then

$$(2) \quad 0 \leq \sum_{i=0}^n (-1)^{n+i} \int_a^b f_i(x)u^{(i)2}(x)dx .$$

Further, equality obtains if and only if $u(x) \equiv cy(x)$ and (6) is modified to make $q_i v_i(d_i) = 0$ ($i = 0, \dots, n - 1$).

Proof. The Theorem follows immediately from the lemmas, except for the last statement, which follows from the fact that equality obtains if and only if $u^{(i+1)}(x) \equiv y_{i+1,i}(x)u^{(i)}(x)$ ($i = 0, \dots, n - 1$) and $v_i(d_i) = 0$ ($i = 1, \dots, n$).

5. The reciprocal inequality. We now derive a set of inequalities which includes (3); we prove

THEOREM 2. Let the $f_i(x)$ ($i = 0, \dots, n$) and $y(x)$ satisfy the hypothesis of Theorem 1; in addition, let $f_i(x) \equiv 0$ or $f_i(x) \neq 0$ on $[a, b]$ ($i = 0, \dots, n$). Let $u(x)$ satisfy

$$(15) \quad u^{(n)} \in L_2[a, b]; \quad u^{(i)}(d_i) = 0 \quad (i = 0, \dots, n - 1) .$$

Then, for each k ($1 \leq k \leq n$) such that $f_{n-k}(x) \neq 0$,

$$(16) \quad 0 \geq \int_a^b \frac{1}{f_n(x)} u^2(x)dx + (-1)^k \int_a^b \frac{1}{f_{n-k}(x)} u^{(k)2}dx .$$

Proof. The proof is similar to that of Theorem 1, so we present it here in less detail. Let $r_{ij} = y^{(n-i)}/v_j$; then, formally,

$$(17) \quad r_{ii} = r'_{i+1,i} + r_{i+1,i}v_{i+1}/v_i - r_{i+1,i}^2 f_{n-i-1} .$$

Thus

$$(18) \quad \int_a^b r_{ii}u^{(i)2}dx = r_{i+1,i}u^{(i)2}\Big|_a^b + \int_a^b r_{i+1,i+1}\left(u^{(i+1)} - \frac{v_{i+1}u^{(i)}}{v_i}\right)^2dx \\ - \int_a^b f_{n-i-1}r_{i+1,i}^2u^{(i)2}dx - \int_a^b r_{i+1,i+1}u^{(i+1)2}dx \quad (i=0, \dots, n-2) ,$$

and

$$\begin{aligned}
 (19) \quad \int_a^b r_{ii} u^{(i)^2} dx &= r_{i+1,i} u^{(i)^2} \Big|_a^b - \int_a^b \frac{1}{f_{n-i-1}} (u^{(i+1)} - r_{i+1,i} f_{n-i-1} u^{(i)})^2 dx \\
 &+ \int_a^b r_{i+1,i} \frac{v_{i+1}}{v_i} u^{(i)^2} dx + \int_a^b \frac{1}{f_{n-i-1}} u^{(i+1)^2} dx \\
 &\qquad\qquad\qquad (i = 0, \dots, n - 1).
 \end{aligned}$$

Repeated application of (18) to $\int_a^b r_{00} u^2 dx$ gives

$$\begin{aligned}
 \int_a^b \frac{1}{f_n} u^2 dx &= \sum_{i=0}^{k-2} (-1)^i \left\{ r_{i+1,i} u^{(i)^2} \Big|_a^b + \int_a^b r_{i+1,i+1} \left(u^{(i+1)} - \frac{v_{i+1}}{v_i} u^{(i)} \right)^2 dx \right. \\
 &\quad \left. - \int_a^b f_{n-i-1} r_{i+1,i}^2 u^{(i)^2} dx \right\} + (-1)^{k-1} \int_a^b r_{k-1,k-1} u^{(k-1)^2} dx ;
 \end{aligned}$$

application of (19) to the last term gives

$$\begin{aligned}
 (20) \quad \int_a^b \frac{1}{f_n} u^2 dx &= \sum_{i=0}^{k-1} (-1)^i r_{i+1,i} u^{(i)^2} \Big|_a^b \\
 &+ \sum_{i=0}^{k-2} (-1)^i \left\{ \int_a^b r_{i+1,i+1} \left(u^{(i+1)} - \frac{v_{i+1}}{v_i} u^{(i)} \right)^2 dx \right. \\
 &\quad \left. - \int_a^b f_{n-i-1} r_{i+1,i}^2 u^{(i)^2} dx \right\} \\
 &+ (-1)^{k-1} \left\{ \int_a^b r_{k,k-1} \frac{v_k}{v_{k-1}} u^{(k-1)^2} dx \right. \\
 &\quad \left. - \int_a^b \frac{1}{f_{n-k}} (u^{(k)} - r_{k,k-1} f_{n-k} u^{(k-1)})^2 dx \right. \\
 &\quad \left. + \int_a^b \frac{1}{f_{n-k}} u^{(k)^2} dx \right\} \qquad\qquad\qquad (k = 1, \dots, n).
 \end{aligned}$$

We now show that, if $f_{n-k}(x) \neq 0$, (20) is valid. Let a v_i have a zero of order r ; such a zero must be at d_i . Now, $r \leq n - i$. For we have

$$v'_j = q_{j+1} (q_{j+1} v_{j+1} + (-1)^j f_{n-j-1} p_{j+1} y^{(n-j-1)}) ;$$

since $y^{(n-j-1)}(d_j) \neq 0$, if $v'_j(d_j) = 0$ then $f_{n-j-1} \equiv 0$, and $v'_j \equiv v_{j+1}$. Thus, if $r > n - i$, $v_i^{(n-i-1)} = v_{n-1}$ and also $v_i^{(n-i)} = v_n \equiv 0$. The first of these implies that $v_i^{(n-i)} = v'_{n-1} = v_n - y_0 = -y_0 \neq 0$, a contradiction. Further, we have $d_i = \dots = d_{i+r-1}$, so $u^{(i)}$ has a zero of order greater than $r - \frac{1}{2}$ at d_i . This suffices to justify (20). We note in addition that $r_{i+1,i}(d_i) u^{(i)^2}(d_i) = 0$ ($i = 0, \dots, n - 1$).

Now by hypothesis $(-1)^{i+1} f_{n-i-1} \leq 0$ ($i = 0, \dots, n - 1$). Lemma 4 implies that $(-1)^i r_{i+1,i+1} \leq 0$ ($i = 0, \dots, n - 2$). Finally,

$$(-1)^i r_{i+1,i} u^{(i)2} \Big|_a^b = - \frac{p_{i+1} y^{(n-i-1)} u^{(i)2} \Big|_{a_i}^{a_i^*}}{q_i v_i} ;$$

evaluation at d_i^* gives a non-positive quantity; evaluation at d_i gives zero. Hence the inequality (16) follows from (20).

6. Concluding remarks. If we want (16) for only one particular value of k ($k < n$), we need correspondingly less hypotheses on $y(x)$ and its derivatives, $u(x)$ and its derivatives, and $f_i(x)$ ($i = 0, \dots, n$), since only $k + 1$ of the functions in each of these sets are actually involved in any of the proofs.

Since $(-1)^{n-i} f_i(x) \leq 0$, from (2) we may delete any combination of terms excluding the last, and to the right-hand side of (16) we may add any terms of the form

$$(-1)^j \int_a^b \frac{1}{f_{n-j}} u^{(j)2} dx \quad (i \leq j \neq k) .$$

Thus, e.g., (2) implies

$$0 \leq (-1)^k \int_a^b f_{n-k} u^{(k)2} dx + \int_a^b f_n u^{(n)2} dx ,$$

which perhaps corresponds more obviously to (16) than does (2).

Finally, the set of allowed values of $(k_0 \dots k_n)$ can be split into halves such that one half, together with the inequality $Ly \geq 0$, and also the other half, together with $Ly \leq 0$, will produce the inequalities.

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