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# PHYSICAL INTERPRETATION AND STRENGTHING OF M. PROTTER'S METHOD FOR VIBRATING NONHOMOGENEOUS MEMBRANES; ITS ANALOGUE FOR SCHRÖDINGER'S EQUATION

JOSEPH HERSCH

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## PHYSICAL INTERPRETATION AND STRENGTHENING OF M. H. PROTTER'S METHOD FOR VIBRATING NONHOMOGENEOUS MEMBRANES; ITS ANALOGUE FOR SCHRÖDINGER'S EQUATION

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The origin of this work lies partly in M. H. Protter's method [7], [8], partly in two papers [3], [5], developing the idea, found in Payne-Weinberger [6], of auxiliary one-dimensional problems; the physical interpretation in § 3 rejoins that of [2] and [4].

1. We consider the first eigenvalue  $\lambda_1$  of a nonhomogeneous membrane with specific mass  $\rho(x, y) \ge 0$  covering a plane domain D and elastically supported (elastic coefficient k(s)) along its boundary  $\Gamma$ :

$$\Delta u + \lambda \rho(x, y)u = 0$$
 in  $D$ ,  $\frac{\partial u}{\partial n} + k(s)u = 0$  along  $\Gamma$ ,

where  $\vec{n}$  is the outward normal.

Every continuous and piecewise smooth function v(x, y) furnishes an upper bound for  $\lambda_1$ : By Rayleigh's principle

$$\lambda_1 = \operatorname{Min}_v rac{D(v) + \oint_F k(s) v^2 ds}{ \iint_D 
ho v^2 dA}$$
 ,

where ds is the length element, dA the element of area, and D(v) the Dirichlet integral  $\iint_{D} \operatorname{grad}^2 v \, dA$ . The Minimum is realized if  $v = u_1(x, y)$  (first eigenfunction, satisfying  $\Delta u_1 + \lambda_1 \rho u_1 = 0$ ).

In the opposite direction, we are here in search of a Maximum principle for  $\lambda_1$ , from which we could calculate lower bounds.

2. Let us consider in D a sufficiently regular vector field  $\vec{p}$  (we shall discuss presently what discontinuities are allowed), satisfying the condition

(1) 
$$ec{p}\cdotec{n}\leq k(s)$$
 along  $\varGamma$ .  
 $\operatorname{grad}^{2}u_{1}+(ec{p}^{2}-\operatorname{div}ec{p})u_{1}^{2}=-\operatorname{div}(u_{1}^{2}ec{p})+\operatorname{grad}^{2}u_{1}+u_{1}^{2}ec{p}^{2}+2u_{1}\operatorname{grad}u_{1}\cdotec{p}$   
 $=-\operatorname{div}(u_{1}^{2}ec{p})+(\operatorname{grad}u_{1}+u_{1}ec{p})^{2}\geq -\operatorname{div}(u_{1}^{2}ec{p})$ .

Let us integrate this inequality:

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$$egin{aligned} &0 \leq D(u_1) + \oint u_1^2 ec p \, \cdot \, ec n \, ds \, + \iint (ec p^2 - \operatorname{div} ec p) u_1^2 dA \ &\leq D(u_1) + \oint k(s) u_1^2 ds \, + \iint (ec p^2 - \operatorname{div} ec p) u_1^2 dA = \iint (\lambda_1 
ho \, + \, ec p^2 - \operatorname{div} ec p) u_1^2 dA \, , \end{aligned}$$

whence the lower bound

(2) 
$$\lambda_1 \ge \inf_B \left( \frac{\operatorname{div} \vec{p} - \vec{p}^2}{\rho} \right).$$

We have equality if (and only if)  $\vec{p} = -\operatorname{grad} u_1/u_1$ , whence the Maximum principle

(3) 
$$\lambda_1 = \operatorname{Max}_{\vec{p}} \cdot \vec{n} \leq k(s) \operatorname{along}_{\Gamma} \operatorname{inf}_{\mathcal{D}} \left( \frac{\operatorname{div} \vec{p} - \vec{p}^2}{\rho} \right).$$

Allowed discontinuities (see also [5]): the same as in Thomson's principle for boundary value problems. — If D is cut into subdomains  $D_1, D_2, \dots, D_n$  by analytic arcs, it is sufficient that the vector field  $\vec{p}$  be continuous and differentiable in each  $D_i$  and that its normal component be continuous across all those analytic arcs; the tangential component need not be continuous. — Other sufficient condition:  $\vec{p} = \{p_1, p_2\}, p_1$  continuous in x and differentiable with respect to  $x, p_2$  continuous in y and differentiable with respect to y.

Two properties of a "good" concurrent vector field: One should try to construct  $\vec{p}$  such that  $\vec{p} \cdot \vec{n} = k(s)$  along  $\Gamma$  and  $(\operatorname{div} \vec{p} - \vec{p}^2)/\rho =$ const in D (such is the case for the extremal field  $\operatorname{-grad} u_1/u_1$ ); the examples calculated in [5] show that such a "good" field may be easy to construct.

REMARK. For a fixed boundary  $(u = 0 \text{ along } \Gamma)$ ,  $k \equiv \infty$  and condition (1) falls off. — A "good" field will then be singular along  $\Gamma$ .

#### 3. A physical interpretation.

3.1. One verifies immediately that the nonhomogeneous membrane upon D, with specific mass  $= \lambda_1 \rho(x, y)$  and elastic coefficient k(s), vibrates with ground eigenfrequency 1:  $\Delta u_1 + 1 \cdot (\lambda_1 \rho) u_1 = 0$ .

We shall presently establish the following theorem: Given an admissible vector field  $\vec{p}$  in *D*, the nonhomogeneous membrane with specific mass  $\tilde{\rho}(x, y) = \operatorname{div} \vec{p} - \vec{p}^2$  in *D* and elastic coefficient  $\tilde{k}(s) = \vec{p} \cdot \vec{n}$  along  $\Gamma$ , vibrates with ground frequency  $\geq 1$ .

The inequality (2) follows as a corollary: according to two general principles regarding vibrating systems (cf. [1], pp. 354 and 357), a homo-

geneous membrane with specific mass  $\leq \tilde{\rho}$  and elastic coefficient  $k(s) \geq \tilde{k}(s)$  vibrates a fortiori with ground frequency  $\geq 1$ ; whence (2).

3.2. The above theorem will be established by proving the following statement to be true: If we *cut* the membrane (specific mass  $\tilde{\rho}(x, y) = \operatorname{div} \vec{p} - \vec{p}^2$ , elastic coefficient  $\tilde{k}(s) = \vec{p} \cdot \vec{n}$ ) into slices  $D_j$  of infinitesimal breadth along all trajectories of  $\vec{p}$ , it then vibrates with ground frequency 1.

Indeed: Each slice  $D_j$  has the first eigenfrequency 1: Call s the arc length along the trajectory (measured from an arbitrary origin on  $D_j$ ); we define in  $D_j$  a function  $f(x, y) = f(s) = c_j \exp\left\{-\int_{s=0}^s \vec{p} \cdot \vec{ds}\right\}$ ,  $c_j > 0$  arbitrary. Then grad  $f = -f\vec{p}$ ;

$$arDelta f = -f \operatorname{div} ec p - ec p \cdot \operatorname{grad} \, f = (ec p^2 - \operatorname{div} ec p) f = - \widetilde{
ho} f$$
 ,

 $\frac{\partial f}{\partial n} = -(\vec{p} \cdot \vec{n})f = \begin{cases} -\tilde{k}f \text{ on } \Gamma_j \text{ (infinitesimal part of } \Gamma \text{ bounding } D_j); \\ 0 \text{ along the cuts;} \end{cases}$ 

f > 0 in D. Thus, our function f is the first eigenfunction of the vibrating slice  $D_j$  with specific mass  $\tilde{\rho}$ , free along the cuts and with elastic coefficient  $\tilde{k}$  on  $\Gamma_j$ ; its first eigenfrequency is 1, because  $\Delta f + 1 \cdot \tilde{\rho}f = 0$ : this proves the theorem and justifies our physical interpretation of (2).—The light in which the Maximum principle is viewed here, is in agreement with [2] and [4].

## 4. An inequality of M. H. Protter.

Let  $\vec{p} = \frac{\vec{t}}{a} - \frac{\operatorname{grad} a}{2a}$ , where  $\vec{t}(x, y)$  is a vector field and a(x, y) > 0 a scalar field. Then

$$\operatorname{div} ec{p} - ec{p}^2 = rac{\operatorname{div} ec{t}}{a} - rac{ec{t}^2}{a^2} - rac{\varDelta a}{2a} + rac{\operatorname{grad}^2 a}{4a^2} \geq rac{\operatorname{div} ec{t}}{a} - rac{ec{t}^2}{a^2} - rac{\varDelta a}{2a} \; .$$

For a membrane with fixed boundary, Condition (1) falls off, so we have by (2)

(4) 
$$\lambda_1 \ge \inf_{D} \left[ \frac{\operatorname{div} \vec{t} - \frac{\vec{t}^2}{a} - \frac{4a}{2}}{a\rho} \right].$$

This is M. H. Protter's inequality [7], [8] (if we write  $\vec{t} = \{P, Q\}$ ) —although he requires P(x, y) and Q(x, y) to be  $C^1$  in D, which is unnecessarily restrictive (cf. also [5] and [3]): P might be discontinuous in y and Q in x. M. H. Protter indicates in [8] very interesting applications of (4) to comparison theorems between ground eigenfrequencies of two non-homogeneous membranes spanning the same domain D.

Critical remark.—In the proof of (4) we neglected the positive term  $\operatorname{grad}^2 a/4a^2$ : equality is impossible in (4) unless  $a(x, y) = \operatorname{const}$ , in which case (4) reduces back to (2) with  $\vec{p} = \vec{t}/a$ .

5. Strengthening of Protter's inequality. Let first (a little more generally)  $\vec{p} = \frac{\vec{t}}{a} + \vec{v}$  with  $\vec{t}(x, y)$ ,  $\vec{v}(x, y)$ , a(x, y) > 0; div  $\vec{p} - \vec{p}^2 = \frac{\operatorname{div} \vec{t}}{a}$  $-\frac{\vec{t}^2}{a^2} + \operatorname{div} \vec{v} - \vec{v}^2 - \frac{\operatorname{grad} a}{a^2} \cdot \vec{t} - 2\frac{\vec{v}}{a} \cdot \vec{t}$ ; in order that the two last terms may cancel everywhere, let (with Protter)  $\vec{v} = -\frac{\operatorname{grad} a}{2a} = -\frac{\operatorname{grad} \sqrt{a}}{\sqrt{a}}$ ; then div  $\vec{v} - \vec{v}^2 = -\frac{4\sqrt{a}}{\sqrt{a}}$ ; let  $\sqrt{a(x, y)} = b(x, y) > 0$  in D, i.e.  $\vec{p} = \frac{\vec{t}}{b^2} - \frac{\operatorname{grad} b}{b}$ ; div  $\vec{p} - \vec{p}^2 = \frac{\operatorname{div} \vec{t}}{b^2} - \frac{\vec{t}^2}{b^4} - \frac{4b}{b}$ . Under the condition (5)  $\frac{\vec{t} \cdot \vec{n}}{b^2} - \frac{1}{b} \frac{\partial b}{\partial n} \leq k(s)$  (identically satisfied if  $k \equiv \infty$ ),

we have the lower bound

(6) 
$$\lambda_1 \ge \inf_{D} \left[ \frac{1}{\rho} \left( \frac{\operatorname{div} \vec{t}}{b^2} - \frac{\vec{t}^2}{b^4} - \frac{\mathcal{A}b}{b} \right) \right] \quad ,$$

with equality whenever  $\frac{\vec{t}}{b^2} - \frac{\operatorname{grad} b}{b} = -\frac{\operatorname{grad} u}{u}$ , as no term has been neglected.—If, for example, we take  $\vec{t} \equiv 0$ , we get an inequality of Barta-Pólya  $\lambda_1 \geq \inf_D \left(-\frac{\Delta b}{\rho b}\right)$ .—[In fact, if  $\frac{\partial b}{\partial n} + k(s)b = 0$  on  $\Gamma, \lambda_1$  is comprised between the two Barta-Pólya bounds

$$\inf_{D}\left(-\frac{\varDelta b}{\rho b}\right) \leq \lambda_{1} \leq \sup_{D}\left(-\frac{\varDelta b}{\rho b}\right).$$

The expression in square brackets in (6) is larger than that in (4), because

$$-\frac{\varDelta a}{2a} = -\frac{\varDelta (b^2)}{2b^2} = -\frac{\operatorname{div} (b \operatorname{grad} b)}{b^2} = -\frac{\varDelta b}{b} - \frac{\operatorname{grad}^2 b}{b^2};$$

this does not diminish M. H. Protter's merit, as his inequality (4)

contains (2) as a special case, whence (6) follows.

#### 6. Applications.

6.1. The inequalities obtained by M. H. Protter in [8] may be sharpened by using (6) instead of (4).

6.2. Small variation of the elastic coefficient along the boundary. First case:  $\rho(x, y)$ , k(s);  $\lambda_1$ ,  $u_1(x, y)$ . Second case:  $\tilde{\rho}(x, y) = \rho(x, y)$ ,  $\tilde{k}(s) = k(s) + \varepsilon K(s)$ ;  $\tilde{\lambda}_1$ ,  $\tilde{u}_1(x, y)$ . By Rayleigh's principle,

(7) 
$$\tilde{\lambda}_1 \leq \frac{D(u_1) + \oint \tilde{k} u_1^2 ds}{\iint \rho u_1^2 dA} = \lambda_1 + \varepsilon Q , \text{ where } Q = \frac{\oint K u_1^2 ds}{\iint \rho u_1^2 dA}$$

We now introduce  $b = u_1(x, y)$  into (6):

 $\widetilde{\lambda}_1 \geq \lambda_1 + \inf_D \left\{ \frac{1}{\rho} \left( \frac{\operatorname{div} \vec{t}}{u_1^2} - \frac{\vec{t}^2}{u_1^4} \right) \right\} \text{ under the condition } \frac{\vec{t} \cdot \vec{n}}{u_1^2} \leq \varepsilon K(s),$ whence  $\iint \operatorname{div} \vec{t} dA = \oint \vec{t} \cdot \vec{n} ds \leq \varepsilon \oint K u_1^2 ds = \varepsilon Q \iint \rho u_1^2 dA.$  There exists a vector field  $\vec{t}$  such that

div  $\vec{t} = \varepsilon Q \rho(x, y) u_1^2$  in D and  $\vec{t} \cdot \vec{n} = \varepsilon K(s) u_1^2$  along  $\Gamma$ : indeed, we can even impose the supplementary condition rot  $\vec{t} = 0$ ,  $\vec{t} = \text{grad } v$ ; v (determined up to an additive constant) is the solution of the Poisson-Neumann problem

$$\varDelta v = arepsilon Q 
ho(x, y) u_1^2 ext{ in } D ext{ and } rac{\partial v}{\partial n} = arepsilon K(s) u_1^2 ext{ along } \Gamma ext{ .}$$

Clearly, v and  $\vec{t}$  are proportional to  $\varepsilon$ . Thus,

(7') 
$$\widetilde{\lambda}_1 \geq \lambda_1 + \varepsilon Q - \sup_D \left( \frac{\vec{t}^2}{\rho u_1^4} \right) = \lambda_1 + \varepsilon Q - O(\varepsilon^2) .$$

(7) and (7') give

(7") 
$$\widetilde{\lambda}_1 = \lambda_1 + \varepsilon Q - O(\varepsilon^2)$$

The first perturbation calculus gives  $\tilde{\lambda}_1 = \lambda_1 + \epsilon Q$ ; we thus verify that this is the tangent to the exact curve  $\tilde{\lambda}_1 = \tilde{\lambda}_1(\epsilon)$ .

6.3. Small variation of the specific mass  $\rho(x, y)$ . First case:  $\rho(x, y)$ , k(s);  $\lambda_1$ ,  $u_1(x, y)$ . Second case:  $\tilde{\rho}(x, y) = \rho(x, y) + \varepsilon \sigma(x, y)$ ,  $\tilde{k}(s) = k(s)$ ;  $\tilde{\lambda}_1$ ,  $\tilde{u}_1(x, y)$ . By Rayleigh's principle,

(8) 
$$\widetilde{\lambda}_1 \leq \frac{D(u_1) + \oint k(s)u_1^2 ds}{\iint \widetilde{\rho} u_1^2 dA} = \frac{\lambda_1}{1 + \varepsilon R}$$
, where  $R = \frac{\iint \sigma u_1^2 dA}{\iint \rho u_1^2 dA}$ 

We now introduce again  $b = u_1(x, y)$  into (6):

 $\tilde{\lambda}_1 \ge \inf_D \left\{ \frac{1}{\tilde{
ho}} \left( \frac{\operatorname{div} \vec{t}}{u_1^2} - \frac{\vec{t}^2}{u_1^4} + \lambda_1 \rho \right) \right\}$  under the condition  $\vec{t} \cdot \vec{n} \le 0$  along  $\Gamma$ ; we want to use a vector field  $\vec{t}$  such that  $\vec{t} \cdot \vec{n} = 0$  along  $\Gamma$  and  $\frac{1}{\tilde{
ho}} \left( \frac{\operatorname{div} \vec{t}}{u_1^2} + \lambda_1 \rho \right) = c = \operatorname{const} \operatorname{in} D$ , so  $\operatorname{div} \vec{t} = u_1^2 (c \tilde{\rho} - \lambda_1 \rho)$ ; the constant cis determined by the condition

whence

$$c=rac{\lambda_1}{1+arepsilon R} ext{ ; } ext{ div } ec{t}=\lambda_1 u_1^2 \Bigl(rac{
ho+arepsilon\sigma}{1+arepsilon R}-
ho \Bigr)=arepsilon\lambda_1 u_1^2 rac{\sigma-
ho R}{1+arepsilon R} ext{ ; }$$

such a vector field  $\vec{t}$  exists: we can even request that it be a gradient field;  $\vec{t} = O(\varepsilon)$ .

(8') 
$$\widetilde{\lambda}_1 \geq \frac{\lambda_1}{1+\varepsilon R} - \sup_D \left(\frac{\overline{t}^2}{\widetilde{\rho}u_1^4}\right) = \frac{\lambda_1}{1+\varepsilon R} - O(\varepsilon^2) .$$

(8) and (8') give

(8") 
$$\widetilde{\lambda}_1 = \frac{\lambda_1}{1 + \varepsilon R} - O(\varepsilon^2) .$$

## 7. Schrödinger's equation.

7.1. We consider an equation of Schrödinger's type in 3-space:

with some boundary conditions not specified here, but which must permit partial integrations analogous to those of § 2;  $W = \frac{2m}{\bar{h}^2} V(x, y, z)$ ,  $\lambda_1 = \frac{2m}{\bar{h}^2} E_1$ , where V is the potential, and  $E_1$  the lowest energy level. Rayleigh's principle:

(10) 
$$\lambda_1 = \operatorname{Min}_v \frac{D(v) + \iiint W(x, y, z) v^2 d\tau}{\iiint v^2 d\tau} ,$$

with, possibly, a supplementary term at the numerator, owing to the boundary conditions;  $d\tau$  is the volume element.—The Minimum is realized for the first eigenfunction  $u_1(x, y, z)$  of the differential equation.

7.2. An argument almost identical to that of §2 (cf. also [5]) gives the Maximum principle:

(11) 
$$\lambda_1 = \operatorname{Max}_{\vec{p}} \inf_{\mathcal{D}} \{ W(x, y, z) + \operatorname{div} \vec{p} - \vec{p}^2 \}$$

where the concurrent vector fields  $\vec{p}$  must satisfy corresponding boundary conditions.—The Maximum is realized for  $\vec{p} = -\text{grad } u_1/u_1$ .—Allowed discontinuities: cf. § 2 (continuity of the normal derivative, etc.).—To get a good lower bound, one should try to construct a vector field  $\vec{p}$  such that  $W(x, y, z) + \text{div } \vec{p} - \vec{p}^2 = \text{const.}$ 

7.3. A physical interpretation.—For expository purposes, we shall consider here equation (9) for 2 dimensions only.—This is exactly the equation of a vibrating homogeneous membrane covering a plane domain D, on which each area element dxdy (at the point (x, y)) is pulled towards its equilibrium position u = 0 by a weak spring of infinitesimal elastic coefficient W(x, y)dxdy.—We suppose that the membrane's boundary  $\Gamma$  is also elastically supported with elastic coefficient  $k(s): \partial u/\partial n + k(s)u = 0$  along  $\Gamma$ .

Analogously to § 3.1, we verify immediately: The homogeneous membrane covering D, with specific mass  $\equiv \lambda_1$  and "interior springs" W(x, y), vibrates with the ground eigenfrequency 1.

Let us now consider another vibrating system: Given in D an admissible vector field  $\vec{p}$  with  $\vec{p} \cdot \vec{n} \leq k(s)$ , we study the system formed by:

(a) A nonhomogeneous membrane covering a copy  $D_a$  of D, with specific mass = (div  $\vec{p} - \vec{p}^2$ ) and elastic coefficient =  $\vec{p} \cdot \vec{n}$  along  $\Gamma$ ;

(b) Another copy  $D_b$  of D, without any "transversal elasticity", where every area element dxdy contains a mass W(x, y)dxdy vibrating independently under the action of a spring with elastic coefficient W(x, y)dxdy.

According to §3, the nonhomogeneous membrane (a) has ground eigenfrequency  $\geq 1$ ; each infinitesimal mass of the system (b) vibrates

with the exact frequency  $\omega = 1$ , as this mass is equal to the spring coefficient.—Therefore 1 is the ground eigenfrequency of the system (a) + (b).

By superposing  $D_a$  and  $D_b$  and welding, in each point (x, y), the two masses there placed, we synthesize a nonhomogeneous membrane with specific mass  $W(x, y) + \operatorname{div} \vec{p} - \vec{p}^2$ , elastic coefficient  $= \vec{p} \cdot \vec{n}$  along  $\Gamma$ , and "interior springs" W(x, y).—As the addition of supplementary constraints (welding!) can only make the ground eigenfrequency higher ([1], p. 354), our "synthetic" membrane vibrates with a ground frequency  $\geq 1$ .

Consider now the homogeneous membrane with specific mass  $\equiv \inf_{D} [W(x, y) + \operatorname{div} \vec{p} - \vec{p}^2]$ , elastic coefficient k(s) along  $\Gamma$ , and the same "interior springs" W(x, y); this membrane has smaller masses and greater constraints: therefore ([1], pp. 354 and 357), its ground frequency is a fortiori  $\geq 1$ .

As our initial membrane [specific mass  $\equiv \lambda_1$ ; elastic coefficient = k(s); interior springs W(x, y)] has ground eigenfrequency 1, its specific mass  $\lambda_1$  must be  $\geq \inf_{D}[W(x, y) + \operatorname{div} \vec{p} - \vec{p}^2]$ , which explains (11).

7.4. (Analogous to § 5): Let 
$$\vec{p} = \frac{\vec{t}}{b^2} - \frac{\operatorname{grad} b}{b}$$
; we get

(12) 
$$\lambda_1 \geq \inf_{D} \left\{ W(x,y,z) + \frac{\operatorname{div} \vec{t}}{b^2} - \frac{\vec{t}^2}{b^4} - \frac{\varDelta b}{b} \right\}$$

where adequate boundary restrictions must be imposed on the concurrent vector fields  $\vec{t}(x, y, z)$  and scalar fields b(x, y, z).

7.5. An application.—Small variation of the potential; boundary conditions on the surface  $\Gamma$  of  $D: \partial u/\partial n + k(X)u = 0$   $(X \in \Gamma)$ .

Boundary conditions to be satisfied by  $\vec{t}$  and b:

(5') 
$$\frac{\vec{t} \cdot \vec{n}}{b^2} - \frac{1}{b} \frac{\partial b}{\partial n} \leq k(X) \text{ on } \Gamma.$$

First case:

$$W(x, y, z)$$
,  $k(X)$ ;  $\lambda_1$ ,  $u_1(x, y, z)$ .

Second case:

$$\widetilde{W}(x, y, z) = W(x, y, z) + \varepsilon w(x, y, z)$$
,  $\widetilde{k}(X) = k(X)$ ;  $\widetilde{\lambda}_1$ ,  $\widetilde{u}_1(x, y, z)$ .  
By Rayleigh's principle (10),

(13) 
$$\widetilde{\lambda}_1 \leq \frac{D(u_1) + \iiint \widetilde{W} u_1^2 d\tau}{\iiint u_1^2 d\tau} = \lambda_1 + \varepsilon U , \text{ where } U = \frac{\iiint u_1^2 d\tau}{\iiint u_1^2 d\tau} .$$

Now let  $b = u_1(x, y, z)$  into (12):  $\tilde{\lambda}_1 \ge \lambda_1 + \inf_D \left[ \varepsilon w + \frac{\operatorname{div} \vec{t}}{u_1^2} - \frac{\vec{t}^2}{u_1^4} \right]$  under the condition  $\vec{t} \cdot \vec{n} \le 0$  on  $\Gamma$ . We want to use a vector field  $\vec{t}$  such that  $\vec{t} \cdot \vec{n} = 0$  and  $\frac{\operatorname{div} \vec{t}}{u_1^2} + \varepsilon w = c = \operatorname{const}$ ,  $\operatorname{div} \vec{t} = u_1^2(c - \varepsilon w)$ ; the constant c is determined by the condition  $0 = \oint \oint \vec{t} \cdot \vec{n} dS = \iiint \operatorname{div} \vec{t} d\tau = c \iiint u_1^2 d\tau - \varepsilon \iiint u_1^2 d\tau$ , where dS is the surface element; hence,  $c = \varepsilon U$ ;  $\operatorname{div} \vec{t} = \varepsilon u_1^2(U - w)$ ; there exists such a vector field  $\vec{t}$ : we can even impose that it be a gradient field;  $\vec{t}$  is proportional to  $\varepsilon$ .

(13') 
$$\widetilde{\lambda}_1 \geq \lambda_1 + \varepsilon U - \sup_D (\vec{t}^2/u_1^4) = \lambda_1 + \varepsilon U - O(\varepsilon^2);$$

(13) and (13') give

(13") 
$$\widetilde{\lambda}_1 = \lambda_1 + \varepsilon U - O(\varepsilon^2).$$

The first approximation  $\tilde{\lambda}_1 = \lambda_1 + \varepsilon U$  of the perturbation calculus is, as we see, the tangent to the exact curve  $\tilde{\lambda}_1 = \tilde{\lambda}_1(\varepsilon)$ .

Post-scriptum. For the case  $k \equiv \infty$  and  $\rho \equiv 1$ , the inequality (2), written for the components  $\vec{p} = \{\varphi(x, y), \psi(x, y)\}$  instead of vectorially, was known (except for the allowed discontinuities) to E. Picard as early as 1893: Traité d'Analyse, t. II, p. 25-26, and to T. Boggio: Sull'equazione del moto vibratorio delle membrane elastiche, Atti Accad. Lincei, ser. 5, vol. 16 (2° sem., 1907), 386-393, especially p. 390.—They also chose  $\varphi$  and  $\psi$  to be continuous in the domain, which is criticized here and in [5] as an unnecessary restriction.—In contrast with M. H. Protter, both Picard and Boggio seem to have under-estimated the importance of inequality (2): it just incidentally appears (in the quoted places) in the course of demonstrations for very simple monotony properties.

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