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AXIOMS FOR NON-RELATIVISTIC QUANTUM MECHANICS

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Introduction. In the approach to the axiomatization of quantum mechanics of George W. Mackey [7], a series of plausible axioms is completed by a final axiom that is more or less *ad hoc*. This axiom states that a certain partially ordered set—the set P of all two-valued observables—is isomorphic to the lattice of all closed subspaces of Hilbert space. The question arises as to whether this axiom can be deduced from others of a more *a priori* nature, or, more generally, whether the lattice of closed subspaces of Hilbert space can be characterized in a physically meaningful way. Our central result is a characterization of this lattice which may serve as a step in the indicated direction, although there is not now a precise sense in which our axioms are more plausible than his. Its principal features may be described as follows.

Suppose that P is an atomic lattice, define an element to be *finite* if it is the join of a finite number of points, and suppose that the unit element is not finite, but is the join of a countable set of points. Suppose for the moment that

(F) The lattice under every finite element of P is a real (or complex) projective geometry.

Then one additional axiom, which appears to be particularly mild from an operational viewpoint, is sufficient and necessary for us to show that P is isomorphic to the lattice of closed subspaces of a separable, infinite dimensional real (or complex) Hilbert space.

Of course, (F) is not taken as an axiom, but is deduced from more primitive assumptions. This part of the development follows well-known lines, but the structure of P (and its set S of states) permits us to give it a rather simple form. For example, in order to conclude that the lattice under every finite element of P is a projective geometry, we need make, in addition to the atomicity of P, only the following three assumptions: P is not a Boolean algebra; the lattices under any pair of finite elements of the same dimension are isomorphic; a certain weak (and rather intuitive) form of the modular law holds under finite elements (Theorem 2.1).

In a preliminary chapter we examine the interrelation of a number of regularity properties which a pair P, S satisfying a slight refinement of Mackey's basic axioms might have, and show that a few of the more

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plausible properties imply all the others (Theorem 1.1).

This work is a modification of part of a thesis submitted to the Department of Mathematics of Harvard University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

1. Events and states: preliminaries. Let P be a partly ordered set with least and greatest elements 0 and 1 respectively. If the greatest lower bound or least upper bound of elements a and b of P exists in P it is denoted ab or $a \vee b$ respectively. Let $a \to a'$ be an orthocomplementation in P; that is, for each $a \in P$, $a' \in P$ and

(1) $(a')' = a$,

(2) $a < b$ if and only if $b' < a'$,

(3) a' is a complement of a; i.e., $a'a$ and $a \vee a'$ exist and equal 0 and 1 respectively.

Two elements a and b of P are said to be *orthogonal*, $a \perp b$, if and only if $a \leqq b'$. Clearly $a \perp b$ is equivalent to $b \perp a$. If Q is a set of pairwise orthogonal elements of P we shall say, for short, that Q is orthogonal. It is easy to see that De Morgan's law holds in $P: (ab)' = a' \vee b'$ in the sense that if either ab or $a' \vee b'$ exists, so does the other and the equality holds.

We assume that P satisfies

(L1) If $\{a_1, a_2, \cdots\}$ is orthogonal, then $\bigvee a_i$ exists in P.

It follows readily that a variety of sups and infs exists in P: e.g., $b'c'$, ba' and $ba' \vee a$ if $b \perp c$ and $a \leqq b$; if $b_1 \leqq b_2 \leqq \cdots$ then $\bigvee b_i = b_1 \vee b_2 b_1' \vee b_3 b_2' \vee \cdots$.

Consider the following three properties for P.

(W) $a \leqq b$ implies $b = ba' \vee a$,

(W1) $a \leqq b$ and $ba' = 0$ imply $a = b$,

(W2) $a \leqq c$ and $b \perp c$ imply $(a \vee b)c = a$.[1]

LEMMA 1.1. *If* (W) *holds then* $a \perp b$ *implies* $b = (a \vee b)a'$.

Proof. $a \leqq b'$ so $b' = b'a' \vee a$ by (W) and $b = (b'a' \vee a)' = (a \vee b)a'$

LEMMA 1.2. *If* (W) *holds and* a, b *and* c *are pairwise orthogonal, then* $(a \vee b)(a \vee c) = a$ *and* $(a \vee b)(a \vee c)' = b$.

Proof. $b \leqq a'$, $b \leqq c'$ imply $b \leqq a'c'$ so $a'c' = a'c'b' \vee b$. Then $a = a(a \vee c \vee b) = (a \vee b)b'(a \vee c \vee b)$ by Lemma 1.1

$$= (a \vee b)(a'c'b' \vee b)' = (a \vee b)(a'c')' = (a \vee b)(a \vee c).$$

[1] That is, $(a \vee b)c$ exists and is equal to a. In general, when x exists a priori but y may not, the assertion $y = x$ is understood to include the assertion that y exists.

Since $b = (a \vee b)a'$ by Lemma 1.1 and $b \leqq c'$, $b = bc' = (a \vee b)a'c' = (a \vee b)(a \vee c)'$.

LEMMA 1.3. (W), (W1) *and* (W2) *are equivalent.*

(W1) implies (W). Suppose $a \leqq b$. Then $a \vee ba' \leqq b$ holds trivially and $b(a \vee ba')' = b(a'(ba')') = ba'(ba')' = 0$ so $b = a \vee ba'$ bv (W1).

(W) implies ($W2$). If $a \leqq c$ and $b \perp c$, then ca', a and b are orthogonal so $a = (a \vee b)(a \vee ca')$ Lemma 1.2 (since (W) holds) $= (a \vee b)c$ by (W).

(W2) implies (W1). Suppose $a \leqq b$ and $ba' = 0$ Then $b \perp b'$ so, by (W2), $a = (a \vee b')b = (ba')'b = 0'b = 1b = b$.

P is said to be *weakly modular* (relative to the given orthocomplementation) if it satisfies any and hence all of (W), (W1) and (W2). We assume now that P is weakly modular and, borrowing a traditional term from the theory of probability, we call its members *events*.

Two events a_1, a_2 are said to *commute* or to be *simultaneously measurable* if there exist pairwise orthogonal events b_1, b_2 and c such that $a_i = b_i \vee c$. The set of all events which commute with all other events is called the center $\mathscr{C}$ of P. If $\mathscr{C} = P$, P is said to be *commutative* or *deterministic*. It is an easy consequence of Lemma 1.2 that a and b commute if and only if ab, ab' and $a'b$ exist, $a = ab \vee ab'$ and $b = ab \vee a'b$, and hence that P is deterministic if and only if it is a Boolean algebra.

LEMMA 1.4. *Suppose ab and ab' exist and $a = ab \vee ab'$. Then a and b commute.*

Proof. $a' = (ab \vee ab')' = (ab)'(ab')' = (ab)'(b \vee a') \geqq (ab)'b$ while $b \geqq (ab)'b$ holds trivially. On the other hand, if $a' \geqq c$ and $b \geqq c$ then $(ab)' \geqq a' \geqq c$ so $(ab)'b \geqq c$. Hence $(ab)'b = a'b$ and so $b = (ab)'b \vee ab = a'b \vee ab$.

COROLLARY. *If a and b commute, so do a and b'.*

Proof. The statement of the lemma is symmetric in b and b'.

LEMMA 1.5. *Suppose P is a lattice. Then P is a Boolean algebra if and only if $ab = 0$ implies $a \perp b$.*

Proof. If P is a Boolean algebra and $ab = 0$, then $a1 = a = a(b \vee b') = ab \vee ab' = ab'$ so $a \leqq b'$. Conversely, for any a and b, $a(ab)'b = 0$ so $a(ab)' \leqq b'$ by hypotheses. Then $a = ab \vee a(ab)' = ab \vee a(ab)'b' = ab \vee ab'$ since $b' \leqq (ab)'$.

If we interpret the weakly modular lattice P as the logic of an abstract physical system,[2] $a \leqq b$ means "a implies b" and a' is the event

[2] Cf. [2].

"not a". If $a \perp b$, it is natural to say that a and b are "mutually exclusive"—a implies not b and b implies not a—and in this case the question of the simultaneous occurrence of a and b is completely settled. If, however, $ab = 0$ but a and b are not orthogonal, no experiment exists for the system whose outcome can indicate that a and b have both occurred even though a and b are not mutually exclusive. According to Lemma 1.5, the absence of this uncertainty is equivalent to the commutativity of P.

Digression. It may be shown that the notion of determinacy is further characterized in the following three ways (the statements depend on definitions which appear below). We suppose given a system of states and events $\mathscr{S}$, P.

(i) Let X denote the real linear space of signed measures on P generated by $\mathscr{S}$. P is deterministic if and only if X is a pre-L-space in a certain natural sense (see [4]).

(ii) Define an *observable*, as in [7], to be a function A from the Borel subsets of the real line R to P such that $A_\phi = 0$, $A_R = 1$, $A_E \perp A_F$ if $E \cap F = \phi$ and $A_{\cup E_i} = \sum A_{E_i}$ if $E_i \cap E_j = \phi$ for $i \neq j$; A is *bounded* if $A_E = 1$ for some bounded Borel set E. Given $x \in X$ (see (i) above) and a bounded observable A, let $\mu_{x;A}$ denote the Borel measure on the line: $\mu_{x,A}(E) = x(A_E)$ and let L_A denote the functional on X: $L_A(x) = \int_{-\infty}^{\infty} \lambda d\mu_{x,A}(\lambda)$. The set Y of all such L_A is partially ordered as a subset of the dual of the partially ordered linear space X. P is deteministic if and only if Y is a lattice.

(iii) Suppose P has a unit. Then P is deterministic if and only if every pair A, B of observables is simultaneously measurable in the following intuitive sense: there exist an observable C and Borel functions α and β from R to R (depending on A and B) such that $A = \alpha(C)$ and $B = \beta(C)$ (where, by definition, $\alpha(C)_E = C_{\alpha^{-1}(E)}$).

A function f from the weakly modular partially ordered set P to the closed real unit interval is said to be a *state* for P if $f(1) = 1$ and f is countably additive in the sense that whenever $\{a_i\}$ is orthogonal, $f(\bigvee a_i) = \sum f(a_i)$. It is easy to see that if f is a state and $\{b_i\}$ is an increasing (decreasing) sequence of events with sup (inf) b, then $f(b_i) \to f(b)$.

Now suppose there exists a set $\mathscr{S}$ of states such that

(D) $a \leqq b$ if and only if $f(a) \leqq f(b)$ for all f in $\mathscr{S}$.

Of course, if $a \leqq b$ and f is any state, $f(a) = f(b) - f(ba') \leqq f(b)$. We observe that

E1. If $f(a) = f(b)$ for all f in $\mathscr{S}$, $a = b$,

E2. For each $a \in P$ there exists $b \in P$ such that $f(b) = 1 - f(a)$ for all f in $\mathscr{S}$; there exists $c \in P$ such that $f(c) = 0$ for all f in $\mathscr{S}$,

E3. Let $\{a_1, a_2, \cdots\}$ be a sequence of elements of P such that $i \neq j$

and $f \in \mathscr{S}$ imply $f(a_i) + f(a_j) \leqq 1$. Then there exists $a \in P$ such that $f(a) = \sum f(a_i)$ for all $f \in \mathscr{S}$.

Indeed, E1 is immediate from (D) and in E2 we need only set $b = a'$, $c = 0$. In E3, $f(a_i) \leqq 1 - f(a_j) = f(a'_j)$ implies $a_i \leqq a'_j$ by (D) so the $\{a_i\}$ are mutually orthogonal and we may set $a = \bigvee a_i$.

Suppose, on the other hand, that we are given a set P (without any *a priori* structure) and a set $\mathscr{S}$ of functions from P to the closed unit interval satisfying E1–E3. The elements b and c of E2 are unique by E1 and are denoted a' and 0 respectively and we let $1 = 0'$; the element a of E3 is also unique by E1 and is denoted $\sum a_i$. Let a partial ordering be defined in P by (D); evidently 0 and 1 are the least and greatest elements of P and $a \rightarrow a'$ is an orthocomplementation. We shall show that the orthocomplemented partly ordered set P is weakly modular and $\mathscr{S}$ is a collection of states for P (which trivially satisfies (D)).

Let $\{a_i\}$ be orthogonal, let $I_1, I_2, \cdots$ be a partition of the positive integers and let $b_i = \sum_{j \in I_i} a_j$ where $\sum$ denotes the sum of the a_j in the sense of E3 . It follows at once from the fact that the sum of a convergent series of nonnegative numbers is unaffected by a rearrange ment of its terms that the b_i are pairwise orthogonal and $\sum b_i = \sum a_i$. As a particular case we have,

LEMMA 1.6. *If $a_1, a_2, \cdots$ are pairwise orthogonal and $b \perp a_i$ for every i, then $b \perp \sum a_i$.*

LEMMA 1.7. *If $a_1, a_2, \cdots$ are pairwise orthogonal, then $\Sigma a_i = \bigvee a_i$.*

Proof. Clearly $a = \sum a_i \geqq a_j$ for all j. If $b \geqq a_j$ for all j, $a_j \perp b'$ so $a \perp b'$ by Lemma 1.6; i.e., $b \geqq a$.

Now suppose $a \leqq b$. Then $a \perp b'$ so $a + b'$ exists by E3 and equals $a \vee b'$ by Lemma 1.7; hence $ba' = (a \vee b')'$ exists. Since $ba' \perp a$, $ba' \vee a$ exists by E3 and Lemma 1.7. Then if $f \in \mathscr{S}$,

$$\begin{aligned} f(ba' \vee a) &= f(ba') + f(a) = 1 - f((ba')') + f(a) = 1 - f(b' \vee a) + f(a) \\ &= 1 - f(b') - f(a) + f(a) = 1 - f(b') = f(b) \end{aligned}$$

and it follows from E1 that $b = ba' \vee a$, i.e., (W) holds and P is weakly modular. If $\{a_i\}$ is orthogonal, $f(\bigvee a_i) = f(\sum a_i)$ by Lemma 1.7 $= \sum f(a_i)$ and so f is a state for P.

Let P be a weakly modular partially ordered set and let $\mathscr{S}$ be a family of states for P which determines the order relation in P (as in (D)). The pair $\mathscr{S}$, P will be called a *system of state and events*, or simply a *system*, if it has the following five properties.

E4. (Axiom of separability) Every orthogonal subset of P contains at most countably many non-zero elements.

E5. P is a lattice.

S1. $\mathscr{S}$ is closed under countable convex combination; i.e., if $f_1, f_2, \cdots$ are in $\mathscr{S}$ and $\lambda_1, \lambda_2, \cdots$ are nonnegative real numbers with $\sum \lambda_i = 1$, then $\sum \lambda_i f_i \in \mathscr{S}$.

S2. If a is a non-zero event, there exists $f \in \mathscr{S}$ such that $f(a) = 1$.

S3. If $f(a) = 0$ and $f(b) = 0$, then $f(a \vee b) = 0$.

The following series of lemmas, culminating in Theorem 1.1, develop a number of regularity properties that systems enjoy; interrelations among the properties are exhibited in accompanying remarks.

LEMMA 1.8. *Suppose P is separable* (i.e., satisfies E4). *Then if Q is a nonempty chain in P, a sequence $\{a_1, a_2, \cdots\}$ of elements of Q may be found such that $\bigvee a_i =$* sup Q; *in particular,* sup Q *exists in P.*

Proof. Let Q be a nonempty chain in P and let T be the set of all events of the form ab' where $a \in Q$ and $b < a$. Let $\{c_i\}$ be a maximal set of pairwise orthogonal non-zero elements of T which exists by Zorn's lemma and is countable by E4. Say $c_i = a_i b_i'$ where $a_i \in Q$, $a_1 < a_2 < \cdots$ and $b_i < a_i$ and let $a = \bigvee a_i$. Suppose there exists $b \in Q$ such that $b \not\leqq a$. Then since $b \nless a_i$, $a_i < b$ holds for all i since Q is a chain. Hence $a < b$ and the non-zero event ba' belongs to T and is orthogonal to all the c_i contrary to the maximality of $\{c_i\}$.

A *cut* in P is a subset of P which contains all lower bounds of the set of its upper bounds. If $Q \subseteqq P$, we denote by $\bar{Q}$ the smallest cut containing Q. Thus, for $a \in P$, $\bar{a} = \{b \in P : b \leqq a\}$ and for $Q \subseteqq P$, $\bar{Q} = \cap \, \bar{a} : Q \subseteqq \bar{a}$. The mapping $Q \to \bar{Q}$ is evidently a closure operation in the power class $\mathscr{B}(P)$ of P (see [1]); hence the set $\bar{P}$ of all cuts in P is a complete lattice under inclusion.

LEMMA 1.9. *If P is a lattice and every chain in P has a sup in P, then P is a complete lattice.*

REMARK. If P is a lattice and $\{b_i\} \subseteqq P$, $\bigvee b_i = \bigvee (b_1 \vee \cdots \vee b_i)$, i.e., P is σ-complete.

Proof. Suppose $Q \subseteqq P$, let Q_1 be a chain in $\bar{Q}$ and let $b = \sup Q_1$. If $a \in P$ such that $Q \subseteqq \bar{a}$, then $Q_1 \subseteqq \bar{a}$ so $b \leqq a$, i.e., $b \in \bar{Q}$. It follows now from Zorn's lemma that $\bar{Q}$ contains a maximal element b. The assumption that P is a lattice clearly implies that $\bar{Q}$ is a sublattice of P and so if $a \in Q$, $a \vee b \in \bar{Q}$. Then by the maximality of b in $\bar{Q}$, $a \vee b = b$ so $a \leqq b$; thus, $Q \subseteqq \bar{b}$ and $b = \sup Q$. Dually, $\inf Q = (\sup a' : a \in Q)'$.

For $Q \subseteqq P$ let $Q^\circ = \{f \in \mathscr{S} : f(a) = 0$ for all $a \in Q\}$ and if $T \subseteqq \mathscr{S}$ let $T^\circ = \{a \in \mathscr{S} : f(a) = 0$ for all $f \in T\}$. Clearly $Q = Q^{\circ\circ}$ and if $Q_1 \subseteqq Q_2$, then $Q_2^\circ \subseteqq Q_1^\circ$, and similarly for the subsets of $\mathscr{S}$. The first relation implies $Q^\circ \subseteqq Q^{\circ\circ\circ}$ and the second applied to the first yields $Q^{\circ\circ\circ} \subseteqq Q^\circ$; thus $Q^\circ = Q^{\circ\circ\circ}$ and similarly, $T^\circ = T^{\circ\circ\circ}$. A subset H of P or of $\mathscr{S}$ such that $H = H^{\circ\circ}$ is called an *annihilator*, and the mapping $H \to H^\circ$ is a one-to-one inclusion inverting correspondence between the annihilators in $\mathscr{B}(P)$ and those in $\mathscr{B}(\mathscr{S})$. In this notation, S3 is: $a^\circ \cap b^\circ = (a \vee b)^\circ$. It is easy to see that if $\mathscr{S}$, P has any one of the following three properties, it has the others.

(4) $a^{\circ\circ} \subseteqq b^{\circ\circ}$ implies $a \leqq b$,

(5) $b^\circ \subseteqq a^\circ$ implies $a \leqq b$,

(6) if $f(a) = 1$ whenever $f(b) = 1$, then $b \leqq a$.

LEMMA 1.10. *If* E5, S2 *and* S3 *hold for* $\mathscr{S}$, P, *so do* (4)–(6).

Proof. Suppose $b^\circ \subseteqq a^\circ$. Then $b^\circ = b^\circ \cap a^\circ = (a \vee b)^\circ$ by S3. If $a \vee b \neq b$, then $(a \vee b)b' \neq 0$ by (W1) so, by S2, there exists $f \in \mathscr{S}$ such that $f((a \vee b)b') = 1$. Then $f \in b^\circ$ so $f \in a^\circ$ and $f(a \vee b) = 0$ by S3. But $f(a \vee b) \geqq f((a \vee b)b') = 1$, so $a \vee b = b$, $a \leqq b$ must hold.

LEMMA 1.11. *Suppose P is a separable lattice. Then P is a complete lattice and $Q \subseteqq P$ implies there exists $Q_1 \subseteqq Q$ with at most countably many elements such that* $\sup Q_1 = \sup Q$.

Proof. P is a complete lattice by Lemmas 1.8 and 1.9. Let Q be a nonempty subset of P and let $a = \sup Q$. Let T denote the set of all joins of countable subsets of Q. If T_1 is a chain in T, its join is obtainable as the join of a countable subsets of T_1 by Lemma 1.8 and hence belongs to T. Hence, we may use Zorn's lemma to extract a maximal element a from T, and then, clearly, $a = \sup Q$.

REMARK. The converse is also true. Indeed, suppose $\{a_\alpha\}$ is orthogonal and a is its join; by hypothesis $a = \bigvee a_{\alpha_i}$ for appropriate α_i. If $\alpha \notin \{\alpha_i\}$, then $a_\alpha \perp a$ by Lemma 1.6. Since $a_\alpha \leqq a$ by definition of a, $a_\alpha = 0$.

We consider now the general form of S3:

(7) If a is the sup of the subset Q of P and $f(b) = 0$ for all $b \in Q$, then $f(a) = 0$ (equivalently: $Q^\circ = a^\circ$).

It is easy to see that if E5 and S3 hold, so does (7) whenever Q has countably many elements.

LEMMA 1.12. *If* E4, E5 *and* S3 *hold for* $\mathscr{S}$, P, *so does* (7).

Proof. Let $Q \subseteqq P$, $a = \sup Q$ and let $f \in Q^\circ$. By Lemma 1.11 we may choose a sequence $\{a_i\} \subseteqq Q$ such that $a = \bigvee a_i$; let $b_n = a_1 \vee \cdots \vee a_n$.

Then $b_1 \leqq b_2 \leqq \cdots, f(b_i) = 0$ imply $f(\bigvee b_i) = f(a) = 0$.

An event a is said to be a *carrier* of the state f on P if $f(b) = 0$ is equivalent to $b \perp a$; if a carrier exists for f it is clearly unique and is denoted a_f. Evidently, if f is a state with a carrier, $f(b) = 1$ if and only if $a_f \leqq b$, and $f^\circ = a_f'^{\circ\circ} = \overline{a_f'}$.

LEMMA 1.13. *Suppose P is a complete lattice. Then if $\mathscr{S}, P$ satisfies* (7), *it also satisfies*

(8) Every $f \in \mathscr{S}$ has a carrier in P.

Proof. $a_f = (\sup f^\circ)'$.

REMARK 1. Conversely, if P is a complete lattice and $\mathscr{S}, P$ satisfies (8), then (7) holds. Indeed, if $Q \subseteqq P$ and $a = \sup Q$, let $f \in Q^\circ$. Then $b \leqq a_f'$ for all $b \in Q$ so $a \leqq a_f'$ and hence $f \in a^\circ$, i.e., $Q^\circ \subseteqq a^\circ$.

REMARK 2. (8) is equivalent to the following: $Q \subseteqq P, f \in Q^\circ, f(a) = 1$ imply there exists $b \leqq a$ such that $f(b) = 1$ and $b \perp Q$. For if (8) holds, we may take $b = a_f$ while, conversely, given f, observe that since $f(1) = 1$, the hypothesis implies the existence of b such that $f(b) = 1$ and $b \perp f^\circ$; clearly $b = a_f$.

LEMMA 1.14. *Suppose $\mathscr{S}, P$ satisfies* (4)–(6) *and* (8). *Then it also satisfies*

(9) $\bar{Q} = Q^{\circ\circ}$ *for every subset Q of P.*

Proof. $Q^{\circ\circ} = \bigcap \overline{a_f'} : f \in Q^\circ$ by definition and (8). But $Q \subseteqq \overline{a_f'}$ for all $f \in Q^\circ$ so $\bar{Q} = \{\bigcap \bar{a} : Q \subseteqq \bar{a}) \subseteqq (\bigcap \overline{a_f'} : f \in Q^\circ\} = Q^{\circ\circ}$. On the other hand, $b \in Q^{\circ\circ}$ implies $Q^\circ = Q^{\circ\circ\circ} \subseteqq b^\circ$ while $Q \subseteqq \bar{a}$ implies $a^\circ \subseteqq Q^\circ$. Hence $a^\circ \subseteqq b^\circ$ so $b \leqq a$ by (5). Thus, $Q^{\circ\circ} \subseteqq \bar{Q}$ so $Q^{\circ\circ} = \bar{Q}$.

REMARK. If $\mathscr{S}, P$ satisfies (9), it also satisfies (4)–(6) and (7). Indeed, (4)–(6) are immediate. To prove (7), suppose $a \in P$ is the sup of the subset Q of P. Then $\bar{a} = \bar{Q} = Q^{\circ\circ}$ by (9) so $Q^{\circ\circ\circ} = Q^\circ = a^\circ$.

LEMMA 1.15. *Suppose $\mathscr{S}, P$ satisfies* E4, E5, S1, S2 *and* (8). *Then*

(10) *Every non-zero event is the carrier of some $f \in \mathscr{S}$.*

Proof. We may use the conclusion of Lemma 1.11. Assuming $a \neq 0$, it follows from S2 that $a'^\circ \neq \phi$; let $b = \bigvee a_f : f \in a'^\circ$. Since $a_f \leqq a$ for all $f \in a'^\circ$, $b \leqq a$. If $ab' \neq 0$, choose $g \in \mathscr{S}$ with $g(ab') = 1$; then $0 = g((ab')') = g(a' \vee b) \geqq g(a')$ so $g \in a'^\circ$ and $a_g \leqq b$ by definition of b. On the other hand, $g(b') \geqq g(ab') = 1$ implies $a_g \leqq b'$ so $a_g = 0$. Since 0 cannot be the carrier of a state, $ab' = 0$ must hold and so $a = b$ by (W1). Choose $\{f_1, f_2, \cdots\} \subseteqq a'^\circ$ such that $a = \bigvee a_{f_i}$; $f_0 = f_1/2 + f_2/2^2 + \cdots$ belongs to $\mathscr{S}$ by S1. Then $f_0(a) = 1$ so $a_{f_0} \leqq a$; but, clearly, $f_i(a_{f_0}) = 1$ so $a_{f_i} \leqq a_{f_0}$ for $i = 1, 2, \cdots$ and $a = a_{f_i} \leqq a_{f_0}$. Hence $a = a_{f_0}$ and the proof is complete.

REMARK. If $\mathscr{S}, P$ satisfies (10), it also satisfies (4)–(6), for suppose $a^{\circ\circ} \subseteqq b^{\circ\circ}$. Now $a \leqq b$ holds trivially if $b = 1$ so suppose $b \neq 1$ and choose $f \in \mathscr{S}$ in accordance with (10) such that $a_f = b'$. Then $\bar{a} \subseteqq a^{\circ\circ} \subseteqq b^{\circ\circ} = a_f'^{\circ\circ} = \overline{a_f'} = \bar{b}$ so $a \leqq b$.

A state f on P such that $f(a) = 0$ implies $a = 0$ is said to be a *unit* for P. It is easy to see that if P has a unit, it is separable.

LEMMA 1.16. *If $\mathscr{S}, P$ satisfies* (10), *$\mathscr{S}$ contains a unit.*

Proof. $f \in \mathscr{S}$ such that $a_f = 1$ is a unit.

We have proved, in particular:

THEOREM 1.1. *Let $\mathscr{S}, P$ be a system of states and events. Then P is a complete lattice and the sup of any infinite family of its elements is obtainable as the sup of a countable subfamily. Furthermore, $\mathscr{S}$ contains a unit for P, and the pair $\mathscr{S}, P$ has the following properties.*

(6) *If $f(a) = 1$ whenever $f(b) = 1$, then $b \leqq a$.*

(7) *If $Q \subseteqq P$ and $f(b) = 0$ for all $b \in Q$, then $f(\sup Q) = 0$.*

(8) *Every $f \in \mathscr{S}$ has a carrier in P.*

(9) *$\bar{Q} = Q^{\circ\circ}$ for every $Q \subseteqq P$.*

(10) *Every non-zero event is the carrier of some $f \in \mathscr{S}$.*

2. The model for non-relativistic quantum mechanics. We shall show that certain further constraints on a system $\mathscr{S}, P$ imply that P is isomorphic to the lattice of closed subspaces of a separable infinite dimensional Hilbert space.

We recall that *a covers b* means that $a > b$ and $a \geqq c > b$ implies $a = c$. A *point* is an element which covers 0 and P is *atomic* if each of its elements is the join of points. We shall call an event *finite* if it is the join of a finite number of points and let P_f denote the set of all finite events. Suppose now that $\mathscr{S}, P$ is a system satisfying

(A). P is atomic; $1 \notin P_f$.

Let (a) denote the lattice under a; clearly (a) is weakly modular relative to the orthocomplementation $b \rightarrow ab'$. We assume

(M). Let $a \in P_f$ and suppose b, c and d are elements of (a) with $d \leqq c$ and $bc = 0$. Then $(d \vee b)c = d$.[3]

[3] If $d \leqq c$ and $b \perp c$, $(d \vee b)c = d$ by weak modularity (cf. Lemma 1.3); thus, (M) asserts that, under finite elements, $bc = 0$ bears a certain resemblance to $b \perp c$.

LEMMA 2.1. *If a is finite, (a) is modular.*

Proof. Let d, b and c be elements of (a) with $d \leqq c$. Then $d \vee bc \leqq c$ and $b(bc)'c = 0$ so writing $b = bc \vee b(bc)'$ (by weak modularity) and letting $d \vee bc$, $b(bc)'$ and c play the roles of d, b and c of (M) respectively in the last of the following equalities, $(d \vee b)c = (d \vee (bc \vee b(bc)'))c = ((d \vee bc) \vee b(bc)')c = d \vee bc$.

REMARK. This result is valid for an arbitrary orthocomplemented lattice L; that is, if L has the property attributed to (a) in (M), it obviously satisfies (W2) of § 1, hence is weakly modular (see Lemma 1.3), so the proof applies, and L is modular.

LEMMA 2.2. *Suppse $a > b$. Then a covers b if and only if ab' is a point.*

Proof. Suppose ab' is a point and $a \geqq c > b$. Then $0 < cb'$ by (W1) so $cb' \leqq ab'$ implies $cb' = ab'$. Hence $c = cb' \vee b = ab' \vee b = a$, i.e., a covers b. If ab' is not a point, $ab' > c > 0$ for some $c \in P$ and then $a = b \vee ab' = b \vee ab'c' \vee c > b \vee c > b$ so a does not cover b.

COROLLARY. *Let $a \in P$. The chain $0 = a_0 < a_1 < a_2 < \cdots$ is maximal in (a) if and only if $a_i a'_{i-1}$ is a point for $i = 1, 2, \cdots$ and $\bigvee a_i = a$.*

LEMMA 2.3. ([1, pp. 66, 67]) *Let $a \in P_f$ and suppose every orthogonal set of points in (a) is finite. Then if $b \leqq a$ and $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_m\}$ are two maximal orthogonal sets of points in (b), $m = n$.*

LEMMA 2.4. ([1, p. 66]) *Let $a \in P_f$ and suppose b, c and d are elements of (a) such that b covers d, b and c are not comparable and $d < c$. Then $b \vee c$ covers c.*

For $a \in P_f$ let $\dim a = -1 + \min\{n : a$ is the join of n points$\}$ and let $P_i = \{a \in P_f : \dim a = i\}$, $i = -1, 0, 1, \cdots$. Clearly, $P_{-1} = \{0\}$, P_0 is the set of points and $P_f = \bigcup P_i$.

Suppose there exists $a \in P_f$ such that (a) contains an infinite orthogonal set $\{b_i\}_{i=0}^{\infty}$ of points, and assume that $n = \dim a$ is a minimum for a with this property; clearly $n > 0$. Let $a_0, \cdots, a_n$ be points with join a. Since $\dim a_0 \vee \cdots \vee a_{n-1} = n - 1$, Lemma 2.3 implies the existence of orthogonal points $c_0, \cdots, c_{n-1}$ such that $c_0 \vee \cdots \vee c_{n-1} = a_0 \vee \cdots \vee a_{n-1}$. Then a_n covers 0 and is not comparable with $a_0 \vee \cdots \vee a_{n-1}$ so $a = a_0 \vee \cdots \vee a_{n-1} \vee a_n$ covers $a_0 \vee \cdots \vee a_{n-1}$ by Lemma 2.4 and hence $c_n = a(a_0 \vee \cdots \vee a_{n-1})'$ is a point by Lemma 2.2; clearly $a = c_0 \vee \cdots \vee c_n$ and the c_i are orthogonal. Now $c_0 \neq b_0$, for otherwise $c_1 \vee \cdots \vee c_n = b_1 \vee b_2 \vee \cdots$ contrary to the choice of a with minimum dimension. Hence

$c_0 \vee b_0$ covers c_0, so $d_0 = (c_0 \vee b_0)c_0'$ is a point. For $i = 1, 2, \cdots$ let $d_i = (c_0 \vee b_0 \vee \cdots \vee b_i)(c_0 \vee b_0 \vee \cdots \vee b_{i-1})'$. If $b_i \leqq c_0 \vee b_0 \vee \cdots \vee b_{i-1}$, $d_i = 0$ while if not, $c_0 \vee b_0 \vee \cdots \vee b_i$ covers $c_0 \vee b_0 \vee \cdots \vee b_{i-1}$ by Lemma 2.4 so d_i is a point. Since all the d_i are orthogonal and lie under ac_0', all but a finite number must be 0, since $\dim ac_0' = n - 1 < \dim a$. Since $\bigvee d_i = ac_0'$, exactly n of the d_i are points by Lemma 2.3 and we assume without essential loss of generality that $d_0, \cdots, d_{n-1}$ are points. But then $a = ac_0' \vee c_0 = d_0 \vee \cdots \vee d_{n-1} \vee c_0 = b_0 \vee \cdots \vee b_{n-1} \vee c_0$. Since c_0 is a point not comparable with $b_0 \vee \cdots \vee b_{n-1}$, a covers $b_0 \vee \cdots \vee b_{n-1}$ and so $e = a(b_0 \cdots b_{n-1})'$ is a point. But $b_i \leqq e$ for $i \geqq n$ and so all but one of these b_i must be zero. This contradiction completes the proof of

LEMMA 2.5. *If a is finite, every orthogonal set of points in (a) is finite.*

COROLLARY. *If a is finite and $\{a_i\}_{i=0}^n$ is an orthogonal set of points in (a) with join a then $n = \dim a$.*

We call the elements of P_1 *lines*, of P_2, *planes*, and use the following notation: if $a \in P$, $(a)_i = \{b \leqq a : \dim b = i\}$, $i = -1, 0, 1, \cdots$.

We make the following assumption of homogeneity:

(H) If a and b are finite elements of the same dimension, then (a) and (b) are isomorphic.

LEMMA 2.6. *Suppose P contains a pair of distinct points a_0, b_0 such that the line $a_0 \vee b_0$ contains no third point. Then P is deterministic.*

Proof. $(a_0 \vee b_0)b_0'$ is a point distinct from b_0 so is equal to a_0 by hypothesis and hence $a_0 \perp b_0$. It follows now from (H) that if a_1 and b_1 are any two distinct points, then $a_1 \perp b_1$. Hence if a and b are events with $ab = 0$, $a = \bigvee a_1 : a_1 \in (a)_0 \leqq \bigwedge b_1' : b_1 \in (b)_0 = (\bigvee b_1 : b_1 \in (b)_0)' = b'$ so $a \perp b$ and P is deterministic by Lemma 1.5.

We assume

(ND) P is not deterministic.

COROLLARY. *Every line contains at least three distinct points.*

LEMMA 2.7. $\mathscr{C} = \{0, 1\}$.

Proof. Suppose $a \in \mathscr{C}$ with $0 < a < 1$. Then there exist points b_1 and b_2 such that $b_1 \leqq a$ and $b_2 \leqq a'$. Let c be a point in $b_1 \vee b_2$ distinct from b_1 and b_2. Then $c = ca \vee ca'$ so either $c \leqq a$ or $c \leqq a'$ since c is a point. But the former implies that $b_2 < a$ since then $b_1 \vee b_2 = b_1 \vee c \leqq a$

and similarly the latter implies that $b_1 < a'$; hence the assumption $0 < a < 1$ is untenable.

We have shown that for $a \in P_f$, (a) is an orthocomplemented, modular lattice of finite dimension with trivial center and at least three points on each line. Thus, we have (see e.g., [1, Theorem 6, p. 120]):

THEOREM 2.1. *Let $\mathscr{S}$, P be a system satisfying* (A), (M), (H) *and* (ND). *Then the lattice under every finite element of P is a projective geometry.*

It follows from (H) that there exists a division ring D such that a coordinatizing division ring[4] for any finite (a) is isomorphic to D. We shall make use of the natural metric ρ for P: $\rho(a,b) = \sup |f(a) - f(b)| : f \in \mathscr{S}$.

LEMMA 2.8. *Orthocomplementation is continuous in (a) for any $a \in P$. That is, if $\{b_n\} \subset (a)$, $b \in (a)$ and $b_n \to b$, then $ab'_n \to ab'$.*

Proof. Given $\varepsilon > 0$ choose N so that $n > N$ implies that $\rho(b_n, b) < \varepsilon$. Then if $f \in \mathscr{S}$ and $n > N$,

$$\begin{aligned}\varepsilon > |f(b_n) - f(b)| &= |(1 - f(b)) - (1 - f(b_n))| \\ &= |f(b') - f(b'_n)| = |f(b'a \vee a') - f(b'_n a \vee a')| \\ &= |f(b'a) + f(a') - f(b'_n a) - f(a')| = |f(b'a) - f(b'_n a)| \,.\end{aligned}$$

Thus, $\rho(ab'_n, ab') < \varepsilon$ and the result follows.

We assume now

(C′) If a is finite and $0 \leqq i \leqq \dim a$, $(a)_i$ is compact.

REMARK. It seems reasonable to suppose that there exists $\varepsilon > 0$ so small that if the probabilities of occurrence of two events b and c differ in every state by less than ε, then $b = c$, i.e., b and c are operationally identical. The completeness of $(a)_i$ is clearly weaker than this operational assumption. The assumption that $(a)_i$, in addition to being complete, is totally bounded, may be paraphrased as follows: for each $\varepsilon > 0$ there exists a finite set $\{b_1, \cdots, b_m\}$ of elements of $(a)_i$ such that given any b in $(a)_i$ and $f \in \mathscr{S}$ the probability of occurrence of the event b in the state f differs from the probability of occurrence of one of the b_j in f by an amount less than ε.

LEMMA 2.9. *Let a be a finite event of dimension at least two. Let $0 \leqq i$, $j < \dim a$, let $\{b_n\} \subset (a)$, $\{c_n\} \subset (a)$ with $\dim b_n = i$ and $\dim c_n = j$ for all n. Suppose that $b_n \to b$ and $c_n \to c$ where b and c are in "general position,"* i.e., $\dim b \vee c = \min(\dim a, i + j + 1)$. *Then $b_n \vee c_n \to b \vee c$ and, dually, $b_n c_n \to bc$.*

[4] [1, Theorem 15, p. 131].

Proof. $\{b_n \vee c_n\}$ clusters at some $d \leqq a$ by (C′); assume for convenience that $b_n \vee c_n \rightarrow d$. Let $\varepsilon > 0$ and choose N so that $n > N$ implies $\rho(b_n, b) < \varepsilon/2$ and $\rho(b_n \vee c_n, d) < \varepsilon/2$. Then if $f(b) = 1$ and $n > N$, $f(d) + \varepsilon/2 > f(b_n \vee c_n) \geqq f(b_n) > 1 - \varepsilon/2$ so $f(d) > 1 - \varepsilon$. Hence $f(d) = 1$ and so $b \leqq d$ by (6) of Theorem 1.1. Similarly $c \leqq d$ so $b \vee c \leqq d$. Since $\dim d \leqq \max_n \dim b_n \vee c_n \dim b \vee c$, $b \vee c = d$ must hold. The dual follows from Lemma 2.8.

COROLLARY. *Let $a \in P_f$. Then, in (a), the lattice operations are continuous in both variables simultaneously.*

We have therefore

LEMMA 2.10.[5] *D is a locally compact division ring.*

We now assume

(Co) For some $b \in P_f$ and real interval I there exists a continuous non-constant function $t \rightarrow a_t$ from I to (b).

REMARK. Postulate (Co) may be obtained from the following "intuitive" assumptions. There exist a one-parameter family L_t of mappings of $\mathscr{S}$ on $\mathscr{S}$ (describing how the states change with time (regarded as a real parameter)—corresponding to certain assumptions concerning the dynamics of the system (see [6, 7])) and a state f such that, letting a_t denote the carrier of $L_t(f)$, a_t is continuous, non-constant and remains in some finite (b) for all t in an interval I.

For convenience assume $I = [0, 1]$, let $n = \dim b$, $m = \dim a_0$. It follows at once from the continuity of a_t and the compactness of $(b)_m$ that $\dim a_t = m$ for all $t \in I$. Suppose $m > 0$. Without essential loss of generality we assume that $a_t \neq a_0$ for $t > 0$ and choose a point $c < a_0$ such that $c \nleqq a_t$ for (again, for convenience) $t > 0$. Let $d = c \vee a_0'$. Choose $\delta > 0$ such that $0 \leqq t < \delta$ implies $\rho(a_0, a_t) < 1/2$. Then for such t, $a_t a_0' = 0$, for otherwise there exists $f \in \mathscr{S}$ such that $f(a_t a_0') = 1$ so $f(a_t) = f(a_0') = 1$. But then $|f(a_0) - f(a_t)| = |f(a_0) - 1| < 1/2$ implies $f(a_0) > 1/2$, a contradiction. Hence, taking $\delta = 1$ for convenience, $da_t = d_t$ is a point for all t (for $d_t \neq 0$ by a count of dimension while $a_0' a_t = 0$ implies $\dim da_t \leqq 0$). Since $d_0 = c$ and $d_t \neq c$ for $t > 0$, d_t is not constant, while it follows from Lemma 2.9 that d_t is a continuous function of t; in case $m = 0$ we set $d_t = a_t$. Again by continuity and without essential loss of generality, we can find a point $e^{(1)}$ disjoint from $\{d_t\}_{t \in I}$ and hyperplane $h^{(1)}$ such that $(e^{(1)} \vee d_t)h^{(1)} = d_t^{(1)}$, which is automatically continuous, is not constant. Similarly, if $\dim h^{(1)} = n - 1 > 1$, we can find $e^{(2)} \varepsilon h^{(1)}$ disjoint from $\{d_t^{(1)}\}$ and $h^{(2)} = h^{(1)}$ with $\dim h^{(2)} = n - 2$ such that $d_t^{(2)} = (e^{(2)} \vee d_t^{(1)})h^{(2)}$ is non-constant in some subinterval of I. Continuing in this way, we arrive finally at a continuous non-constant function

[5] See Kolmogorov [5].

$d_t^{(n)}$ from some subinterval of I to a line $h^{(n)}$ in (b). Then for a subinterval J of I, $\{d_t^{(n)}\}_{t\in J}$ omits a point p of $h^{(n)}$. But D is homomorphic to $h^{(n)}$ with p removed and hence contains a connected set, the image of $\{d_t^{(n)}\}_{t\in J}$ under such a homomorphism. Since a locally compact division ring is readily seen to be either connected or totally disconnected we have

LEMMA 2.11. *D is connected.*

It follows now from Pontrjagin's theorem that D is the real, complex or quaternion division ring.[6] We assume henceforward that the real or complex case has been singled out, e.g., by the assumption of simple ordering on the one hand or algebraic closure on the other, the quaternions having been set aside by postulating commutativity for D, i.e., that Pappus's theorem holds under finite elements. Turning now to the representation of P itself, we shall need the final postulate

(C) For each $i = 0, 1, \cdots, P_i$ is complete.[7]

LEMMA 2.12. *Let L and Λ be complete weakly modular lattices and let $L_0(\Lambda_0)$ be a subset of $L(\Lambda)$ such that every element of $L(\Lambda)$ is a join of elements of $L_0(\Lambda_0)$. Suppose further that φ is a mapping of L_0 onto Λ_0 such that*

(1) *$a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$.*

Then φ can be extended to an isomorphism of L onto Λ.

Define θ: $L \rightarrow \Lambda$ by $\theta(a) = \bigvee \varphi(c) : c \in [a]$ where $[a] = \{b \leqq a : b \in L_0\}$. Clearly θ preserves order and $\theta \mid L_0 = \varphi$. The lemma is proved in the following steps:

(2) $\theta(a') \leqq \theta(a)'$.

(3) $a < b$ implies $\theta(a) < \theta(b)$.

(4) Let A be a subset of $[a]$ such that $a = \sup A$. Then $\theta(a) = \sup \varphi(b) : b \in A$.

(5) $\theta(a \vee b) = \theta(a) \vee \theta(b)$.

(6) θ is one-to-one.

(7) θ^{-1} preserves order.

(8) θ is onto.

The proofs are as follows.

(2) If $b \in [a']$ and $c \in [a]$, $\varphi(b) \leqq \varphi(c)'$ by (1) so $\theta(a') = \bigvee \varphi(b) : b \in [a'] \leqq \bigwedge \varphi(c)' : c \in [a] = (\bigvee \varphi(c))' : c \in [a] = \theta(a)'$.

(3) If $a < b$, there exists $c \neq 0 \varepsilon [ba']$. Then $\varphi(c) \perp \varphi(a_1)$ for all $a_1 \in [a]$ by (1) so $\varphi(c) \perp \theta(a)$. Clearly $\varphi(c) \leqq \theta(b)$ and $\theta(a) \leqq \theta(b)$ so $\theta(a) < \theta(b)$.

(4) Let $\alpha = \sup \varphi(b) : b \in A$; clearly $\alpha \leqq \theta(a)$. If $c \in L_0$ with $\varphi(c) \in [\theta(a)\alpha']$ then $c \perp b$ for every $b \in A$ by (1) so $c \perp a$. Hence $\varphi(c) \leqq$

[6] See [8] for a unified derivation of the classification of locally compact division rings.

[7] Cf. the remark following the statement of postulate (C′).

$\theta(a') \leqq \theta(a)'$ by (2). Since $\varphi(c) \leqq \theta(a)$, $c = 0$ and hence $\theta(a) = \alpha$ by weak modularity.

(5) Let $A = [a] \cup [b]$. Then $\sup A = a \vee b$ so $\theta(a \vee b) = \sup \varphi(c) : c \in A$ by (4). Now if $c \in A$, $\varphi(c) \leqq \theta(a)$ or $\varphi(c) \leqq \theta(b)$ so $\varphi(c) \leqq \theta(a) \vee \theta(b)$; the opposite inequality is immediate.

(6) and (7). If $a \nleqq b$ then $b < a \vee b$ so $\theta(b) < \theta(a \vee b)$ by (3) $= \theta(a) \vee \theta(b)$ by (5) and hence $\theta(a) \nleqq \theta(b)$.

(8) Let $\alpha \in \Lambda$ and let $A = \{\varphi^{-1}(\beta) : \beta \in [\alpha]\}$. Then, by (4), $\theta(\sup A) = \bigvee \beta : \beta \in [\alpha] = \alpha$.

For each $a \in P_f$ we choose a distinct Euclidean space H_a over D of dimension $1 + \dim a$ and an isomorphism φ_a of (a) onto L_a, the lattice of subspaces of H_a. Assuming $n = \dim a > 0$, we wish to choose a scalar product for H_a so that the orthocomplementation $\varphi_a b \to \varphi_a a b'$ which is induced in L_a by that in (a) coincides with the one induced by the scalar product. First of all, there exists an involution σ of D and non-zero numbers (i.e., elements of D) $\gamma_0, \cdots, \gamma_n$ such that $\gamma_i^\sigma = \gamma_i$, $\sum_{i=0}^n x_i \gamma_i x_i^\sigma = 0$ implies all $x_i = 0$, and if $b \in (a)_0$ and $\varphi_a b = [(x_0, \cdots, x_n)]$ then $\varphi_a a b' = \{(y_0, \cdots, y_n) : \sum y_i \gamma_i x_i^\sigma = 0\}$.[8] In the real case, $\sigma = 1$ is the only automorphism; we shall show that σ is continuous, hence is either 1 or conjugation in the complex case, and the value 1 is excluded, for otherwise $(\gamma_0^{-1/2}, (-\gamma_1^{-1/2}), 0, \cdots, 0)$ would be self-orthogonal. Then all the γ_i must be positive real numbers and the desired scalar product is $(y, z) = \sum y_i \gamma_i \bar{z}_i$.

Let b and c be orthogonal points in (a), and choose x, y in H_a such that $\varphi_a b = [x]$, $\varphi_a c = [y]$. Let λ_m be a sequence of numbers with $\lambda_m \to 0$ and let $b_m = \varphi_a^{-1}[x + \lambda_m y]$. Then $b_m \to b$ so $(b \vee c)b'_m \to (b \vee c)b' = c$ by Lemma 2.8 and we may assume that $(b \vee c)b'_m \neq b$ holds for all m. Then a sequence μ_m of numbers with $\mu_m \to 0$ is determined by: $\varphi_a(b \vee c)b'_m = [\mu_m x + y]$. Since $b \perp c$, $\sum(\mu_m x_i + y_i)\gamma_i(x_i + \lambda_m y_i)^\sigma = 0$ so $0 = \mu_m \sum x_i \gamma_i x_i^\sigma + \lambda_m^\sigma \sum y_i \gamma_i y_i y_i^\sigma$ and it follows from the fact that $\mu_m \to 0$ and $\sum y_i \gamma_i y_i^\sigma \neq 0$ that $\lambda_m^\sigma \to 0$. Thus, σ is continuous at 0 and hence, by its additivity, is continuous everywhere, and the proof is complete.

We assume now, in accordance with the foregoing, that each H_a has been provided with a scalar product such that $\varphi_a b \perp \varphi_a c$ for b, c in (a) if and only if $b \perp c$. If $a \leqq b \in P_f$, $\varphi_{ba} = \varphi_b \varphi_a^{-1}$ is clearly an orthogonality preserving isomorphism of L_a in L_b. It is well known that there then exists an isometric transformation ψ_{ba} of H_a in H_b, unique up to multiplication by a number of absolute value one, such that if $v \in H_a$, $\varphi_{ba}[v] = [\psi_{ba} v]$. We shall show that the ψ's may be chosen consistently, i.e., so that

(15) $a \leqq b \leqq c$ implies $\psi_{ca} = \psi_{cb}\psi_{ba}$.

We establish a one-to-one correspondence $\alpha \leftrightarrow a_\alpha$ between the elements

[8] [**2**, Appendix]. $[(x_0, \cdots, x_n)]$ denotes the 1-dimensional subspace of H_ι generated by the element $(x_0, \cdots, x_n)$.

of P_f and the ordinal numbers less than an ordinal ζ such that $\alpha < \beta$ implies $\dim a_\alpha \leqq \dim a_\beta$. Thus, in particular, $a_0 = 0$; it is understood that all ordinals $\alpha, \beta, \cdots$ which occur lie under ζ and, where no confusion can result, we shall write "α" for "a_α". In particular, we let $\alpha\beta$ represent (the index of) $a_\alpha a_\beta$. Let $\gamma < \zeta$ and suppose that ψ has already been defined so that (15) holds for $c = a_\alpha$ with $\alpha < \gamma$. Now choose α such that $a_\alpha < a_\gamma$ and $\dim a_\gamma - \dim a_\alpha = 1$; we call such an α "maximal". Fix $\psi_{\gamma,\alpha}$ arbitrarily; then if $a_\eta < a_\alpha$, $\psi_{\gamma,\eta}$ is defined as $\psi_{\gamma,\alpha}\psi_{\alpha,\eta}$. If β is a second maximal element, and assuming $\dim a_\gamma > 1$ (i.e., $\dim H_\gamma > 2$), for otherwise there is nothing to prove, $\alpha\beta \neq 0$ and we define $\psi_{\gamma,\beta}$ by $\psi_{\gamma,\alpha\beta} = \psi_{\gamma,\beta}\psi_{\beta,\alpha\beta}$. Now let η be any ordinal with $a_\eta < a_\gamma$ and let β, ε both be maximal such that $a_\eta \leqq a_\beta$ and $a_\eta \leqq a_\varepsilon$. Assuming that $\dim a_\gamma \geqq 3$, we shall show

(16) $\psi_{\gamma,\beta}\psi_{\beta,\eta} = \psi_{\gamma,\varepsilon}\psi_{\varepsilon,\eta}$ (β, ε maximal, $a_\eta \leqq a_{\beta\varepsilon}$, $\dim a_\gamma \geqq 3$).

But if (16) holds for $\eta = \beta\varepsilon$ then, by the inductive hypothesis, it will hold for arbitrary η, for then $\psi_{\gamma,\beta}\psi_{\beta,\eta} = \psi_{\gamma,\beta}\psi_{\beta,\beta\varepsilon}\psi_{\beta\varepsilon,\eta} = \psi_{\gamma,\varepsilon}\psi_{\varepsilon,\beta\varepsilon}\psi_{\beta\varepsilon,\eta} = \psi_{\gamma,\varepsilon}\psi_{\varepsilon,\eta}$. To prove (16) in the case $\eta = \beta\varepsilon$ observe that $\psi_{\gamma,\alpha}\psi_{\alpha,\alpha\beta\varepsilon} = \psi_{\gamma,\alpha}\psi_{\alpha,\alpha\beta}\psi_{\alpha\beta,\alpha\beta\varepsilon} = \psi_{\gamma,\alpha\beta}\psi_{\alpha\beta,\alpha\beta\varepsilon} = \psi_{\gamma,\beta}\psi_{\beta,\alpha\beta}\psi_{\alpha\beta,\alpha\beta\varepsilon} = \psi_{\gamma,\beta}\psi_{\beta,\alpha\beta\varepsilon} = \psi_{\gamma,\beta}\psi_{\beta,\beta\varepsilon}\psi_{\beta\varepsilon,\alpha\beta\varepsilon}$. Similarly—interchanging β and ε—$\psi_{\gamma,\alpha}\psi_{\alpha,\alpha\beta\varepsilon} = \psi_{\gamma,\varepsilon}\psi_{\varepsilon,\beta\varepsilon}\psi_{\beta\varepsilon,\alpha\beta\varepsilon}$. In other words, $\psi_{\gamma,\varepsilon}\psi_{\varepsilon,\beta\varepsilon} = \psi_{\gamma,\beta}\psi_{\beta,\beta\varepsilon}$ on $\psi_{\beta\varepsilon,\alpha\beta\varepsilon}H_{\alpha\beta\varepsilon}$ and since $\alpha\beta\varepsilon \neq 0$ (by our assumption that $\dim a_\gamma \geqq 3$), this equality holds on all of $H_{\beta\varepsilon}$ and (16) is proved. Thus, if β is maximal and $a_\eta < a_\beta$, $\psi_{\gamma,\eta}$ is unambiguously defined by: $\psi_{\gamma,\eta} = \psi_{\gamma,\beta}\psi_{\beta,\eta}$. If $a_\eta < a_\delta < a_\gamma$, choose β maximal with $a_\delta < a_\beta$ and then $\psi_{\gamma,\eta} = \psi_{\gamma,\beta}\psi_{\beta,\eta} = \psi_{\gamma,\beta}\psi_{\beta,\delta}\psi_{\delta,\eta} = \psi_{\gamma,\delta}\psi_{\delta,\eta}$, completing the proof that ψ as extended to all γ, η with $a_\eta < a_\gamma$ satisfies (15) providing that $\dim a_\gamma \geqq 3$. We begin the induction and complete the proof by "constructing" all $\psi_{c,b}$ with $\dim c \leqq 2$ in the following way. Let A_i denote the set of all $\alpha < \zeta$ for which $\dim a_\alpha = i$, $i = 0, 1, \cdots$ Let $\beta_1 \varepsilon A_1$, let $B_1 = \{\beta \in A_1 : \beta < \beta_1\}$ and make the inductive assumption that $\psi_{\gamma,\beta}$ and $\psi_{\beta,\alpha}$ (and consequently $\psi_{\gamma,\alpha}$) have already been consistently defined whenever $\beta \in B_1$, $\gamma \in A_2$, $\alpha \in A_0$ and $a_\alpha < a_\beta < a_\gamma$. For all $\gamma \in A_2$ such that $a_{\beta_1} < a_\gamma$, define ψ_{γ,β_1} arbitrarily and then, choosing $\alpha \in A_0$ with $a_\alpha < a_{\beta_1}$, define $\psi_{\beta_1,\alpha}$ by $\psi_{\gamma,\alpha} = \psi_{\gamma,\beta_1}\psi_{\beta_1,\alpha}$ if $\psi_{\gamma,\alpha}$ has already been defined for some $\gamma \in A_2$ with $a_\alpha < a_\gamma$—i.e., if $a_\alpha < a_\beta < a_\gamma$ for some $\beta \in B_1$; otherwise define $\psi_{\beta_1,\alpha}$ arbitrarily and set $\psi_{\gamma,\alpha} = \psi_{\gamma,\beta_1}\psi_{\beta_1,\alpha}$ for all $\gamma \in A_2$ with $a_{\beta_1} < a_\gamma$. This procedure evidently extends ψ consistently to all γ, β_1; β_1, α and γ, α such that $\gamma \in A_2$, $\alpha \in A_0$ and $a_\alpha < a_{\beta_1} < a_\gamma$. It then follows inductively—beginning with $B_1 = \phi$—that ψ may be consistently defined for all $\psi_{c,a}$ such that $a \leqq c$ and $\dim c \leqq 2$.

Now let H be a separable, infinite dimensional Hilbert space over D, let L be its lattice of closed subspaces and let $\{v_i\}$ be a complete orthonormal set in H. Let $\{a_i\}$ be a maximal orthogonal subset of P_0 which exists by Zorn's lemma and is countable by E4 and (A), and for each i let u_i be a fixed unit vector in H_{a_i}. Let $a \in P_0$, let $u \in H_a$ and define

$$\lambda_i(u) = (\psi_{a \vee a_i, a} u, \psi_{a \vee a_i, a_i} u_i) ,$$
$$\xi u = \sum \lambda_i(u) v_i,$$
$$\theta(a) = \{\xi u : u \in H_a\} .$$

Thus, the domain of λ_i and ξ is $\bigcup_{a \in P_0} H_a$, that of θ is P_0 and their ranges are in D, H, and the set L_0 of one dimensional subspaced of H respectively.[9] We shall show that θ is one-to-one, onto and that θ and θ^{-1} preserve orthogonality. Hence by Lemma 2.12, θ can be extended to an isomorphism $\bar{\theta}$ of P on L. Then $f\bar{\theta}^{-1}$ will be a state for L and the characterization of $\mathscr{S}$ is given by the[10]

Theorem of Gleason. ([5]) Let μ be a state on the lattice L of closed subspaces of the separable real or complex Hilbert space H of dimension at least three. Then there exists an orthonormal basis $\{x_i\}$ for H and nonnegative real numbers λ_i with $\sum \lambda_i = 1$ such that if Q is the projection on $M \in L$, $\mu(M) = \sum \lambda_i (Qx_i, x_i)$.

Each L_a for $a \in P_f$ becomes a metric space under the definition: distance $(M_1, M_2) = \sup \{| \omega(M_1) - \omega(M_2) | : \omega$ a state for $L_a\}$. An immediate consequence of Gleason's theorem is that φ_a is an isometry of (a) on L_a.

LEMMA 2.13. *Let $a \in P_0$, $u \in H_a$. Then $\| \xi u \| = \| u \|$.*

Proof. For $n = 1, 2, \cdots$ let $b_n = a \vee a_1 \vee \cdots \vee a_n$. Then if $1 \leqq i \leqq n$,

$$\begin{aligned} \lambda_i(u) &= (\psi_{a \vee a_i, a} u, \psi_{a \vee a_i, a_i} u_i) \\ &= (\psi_{b_n, a \vee a_i} \psi_{a \vee a_i, a} u, \psi_{b_n, a \vee a_i} \psi_{a \vee a_i, a_i} u_i) \\ &= (\psi_{b_n, a} u, \psi_{b_n, a_i} u_i) \text{ so } \sum_{i=1}^{n} | \lambda_i(u) |^2 \leqq \| \psi_{b_n, a} u \|^2 \\ &= \| u \|^2 \text{ since the } \psi_{b_n, a_i} u_i \text{ are orthonormal in } H_{b_n} \text{ and } \psi_{b_n, a} \text{ is an isometry.} \end{aligned}$$

Since ξ is linear, we assume without essential loss of generality that $\| u \| = 1$ and suppose that, contrary to the assertion of the lemma, $\| \xi u \| = (\sum_{i=1}^{\infty} | \lambda_i(u) |^2)^{1/2} = \delta < 1$. Then, in particular, $(\psi_{b_n, a_i} u_i)_{i=1}^{n}$ must fail to be a basis in all but a finite number of the H_{b_n}, so, for convenience, we assume $b_n > a_1 \vee \cdots \vee a_n$ for all n and let $c_n = b_n a_1' \cdots a_n'$; evidently, $c_n \in (b_n)_0$. Let $w_n = \sum_{i=1}^{n} \lambda_i(u) \psi_{b_n, a_i}$, let $\alpha_n = \| \psi_{b_n, a} u - w_n \|$ and let $y_n = (\psi_{b_n, a} u - w_n) \alpha_n^{-1}$. Clearly $\alpha_n \to \sqrt{1 - \delta^2}$ and $y_n \in \varphi_{b_n c_n}$. Then if $n > m$,

$$(y_n, \psi_{b_n, b_m} y_m) = \alpha_n^{-1} \alpha_m^{-1} (\psi_{b_n, a} u - w_n, \psi_{b_n, a} u - \psi_{b_n, b_m} w_m)$$

[9] For the convergence of ξu, see the proof of Lemma 2.13.

[10] It follows then from S1 and S2 that $\mathscr{S}$ contains *all* states for P.

$$= \alpha_n^{-1}\alpha_m^{-1}(\|\psi_{b_n,a}u\|^2 - (w_n, \psi_{b_n,a}u) - (\psi_{b_n,a}u, \psi_{b_n,b_m}w_m) + (w_n, \psi_{b_n,b_m}w_m))$$
$$= \alpha_n^{-1}\alpha_m^{-1}\Big(1 - \sum_{i=1}^{n}|\lambda_i|^2\Big) - \sum_{i=1}^{m}|\lambda_i|^2 + \sum_{i=1}^{m}|\lambda_i|^2\Big)$$
$$= \alpha_n^{-1}\alpha_m^{-1}\Big(1 - \sum_{i=1}^{n}|\lambda_i|^2\Big) \to \frac{1-\delta^2}{1-\delta^2} = 1 .$$

Thus, given $\varepsilon > 0$, we may choose N so that $n > m > N$ implies $\varepsilon > 1 - (y_n, \psi_{b_n,b_m}y_m) = \text{distance}\,([y_n], [\psi_{b_n,b_m}y_m]) = \rho(c_n, c_m)$. Then, in virture of (C), there exists a point c in P such that $c_n \to c$. We shall complete the proof by showing that $c \perp a_i$ for all i contrary to the maximality of $\{a_i\}$. Indeed, if $f \in \mathscr{S}$ with $f(a_i) = 1$ and $n > i$, then $c_n \perp a_i$, and if n is chosen so large that $\rho(c_n, c)$ is less than a preassigned $\varepsilon > 0$, $f(c) < f(c_n) + \varepsilon = \varepsilon$, i.e., $f(c) = 0$ so $c \perp a_i$ by (6) of Theorem 1.1 and the proof of Lemma 2.13 is complete.

COROLLARY 1. *Let a and b be points. Then $\theta a \perp \theta b$ if and only if $a \perp b$.*

Proof. For $u \in H_{a\vee b}$ let $\eta u = \sum(\psi_{a\vee b\vee a_i, a\vee b}u, \psi_{a\vee b\vee a_i, a_i}u_i)v_i$. Clearly η is linear and if $c = \varphi_{a\vee b}^{-1}[u]$ and

$$\begin{aligned} w = \psi_{c\vee a_i,c}^{-1}u,\ \eta u &= \sum(\psi_{a\vee b\vee a_i, c\vee a_i}\psi_{c\vee a_i,c}w, \psi_{a\vee b\vee a_i, c\vee a_i}\psi_{c\vee a_i,a_i}u_i)v_i \\ &= \sum(\psi_{c\vee a_i,c}w, \psi_{c\vee a_i,a_i}u_i)\, v_i = \xi w\ ; \end{aligned}$$

clearly $\theta c = [\eta u]$. Hence $\|\eta u\| = \|\xi w\| = \|w\| = \|u\|$ so η is an isometry and then letting $0 \neq u \in \varphi_{a\vee b}a$, $0 \neq v \in \varphi_{a\vee b}b$, $a \perp b$ if and only if $u \perp v$ if and only if $\eta u \perp \eta v$ if and only if $\theta a \perp \theta b$.

COROLLARY 2. *θ is one-to-one.*

Proof. If $\theta a = \theta b$ and $c \in (b')_0$, $c \perp b$ so $\theta c \perp \theta b$ by Corollary 1, $\theta c \perp \theta a$ by our assumption and then $a \leqq c'$ by Corollary 1. Hence $a \leqq \bigwedge c' : c \in (b')_0 = (\bigvee c : c \in (b')_0)' = b$ by postulate (A). Similarly $b \leqq a$, so $a = b$.

COROLLAY 3. *Let $b_1 < b_2 < \cdots$ be a chain of finite elements and suppose $y_n \in H_{b_n}$ with $\|y_n\| = 1$ such that given $\varepsilon > 0$ there exists N such that $n > m > N$ implies $\|y_n - \psi_{b_n,b_m}y_m\| < \varepsilon$. Let $c_n = \varphi_{b_n}^{-1}[y_n]$. Then the sequence of points $\{c_n\}$ converges to a point c in P.*

LEMMA 2.14. *θ is onto.*

Proof. Let $M \in L_0$, $v = \Sigma\mu_i v_i$ a unit vector in M. Let $b_n = a_1 \vee \cdots \vee a_n$, $w_n = \sum_{i=1}^n \mu_i\psi_{b_n,a_i}u_i$, $y_n = w_n/\|w_n\|$ when $w_n \neq 0$ and $c_n = \varphi_{b_n}^{-1}[y_n]$. It follows at once from Corollary 3 above that there exists $c \in P_0$ with $c_n \rightarrow c$. Let $d_n = c \vee b_n$ and let y be a unit vector in H_c. Now $(\psi_{a_n, b_n} y_n, \psi_{a_n, c} y)$ tends to a limit η with $|\eta| = 1$ and $\|\psi_{a_n, b_n} y_n - \eta \psi_{a_n, c} y\| \rightarrow 0$. Hence, by Lemma 2.13, $\xi y_n \rightarrow \xi \eta y$. Since $\xi y_n \rightarrow v$ is obvious, $\xi \eta y = v$, $\theta c = M$ and the proof is complete.

θ is one-to-one from P_0 onto L_0 by Corollary 2 of Lemma 2.13 and the preceding lemma. Furthermore, $\theta a \perp \theta b$ if and only if $a \perp b$ by Corollary 1 of Lemma 2.13 and so we may apply Lemma 2.12 to obtain

THEOREM 2.2. *Suppose the system $\mathscr{S}$, P satisfies the following eight postulates:*

(A) *P is atomic; $1 \notin P_f$.*

(M) *If a is finite and b, c and d are elements of (a) such that $d \leqq c$ and $bc = 0$, then $(d \vee b)c = d$.*

(H) *If a and b are finite elements of the same dimension, then (a) and (b) are isomorphic.*

(ND) *P is not deterministic.*

(C′) *If a is finite and $0 \leqq i \leqq \dim a$, $(a)_i$ is compact.*

(Co) *There exists a continuous, non-constant function from an interval of the real line to the lattice under a finite event.*

(P) *If a is finite, Pappus's theorem holds in (a).*

(C) *For each $i = 0, 1, \cdots, P_i$ is complete.*

Then P is isomorphic to the lattice L of closed subspaces of a separable, infinite dimensional Hilbert space over either the real or the complex field in such a way that the orthocomplementations in P and L correspond.

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