Pacific Journal of Mathematics

SIMPLE PATHS ON CONVEX POLYHEDRA

THOMAS ANDREW BROWN

Vol. 11, No. 4 1961

SIMPLE PATHS ON CONVEX POLYHEDRA

THOMAS A BROWN

1. Introduction. In problems of linear programming, one sometimes wants to find all vertices of a given convex polyhedron. An algorithm for finding all such vertices will often define a path which passes from vertex to vertex along the edges of the polyhedron in question [1], and thus it is natural to ask, as Balinski does in [2], whether or not one can always find a path along the edges of a convex polyhedron which visits each vertex once and only once. This question has been answered in the negative independently by Grünbaum and Motzkin [5] and the author [3]. The purpose of the present paper is to present a modification of the results of [3], and answer certain questions which were asked by Grünbaum and Motzkin.

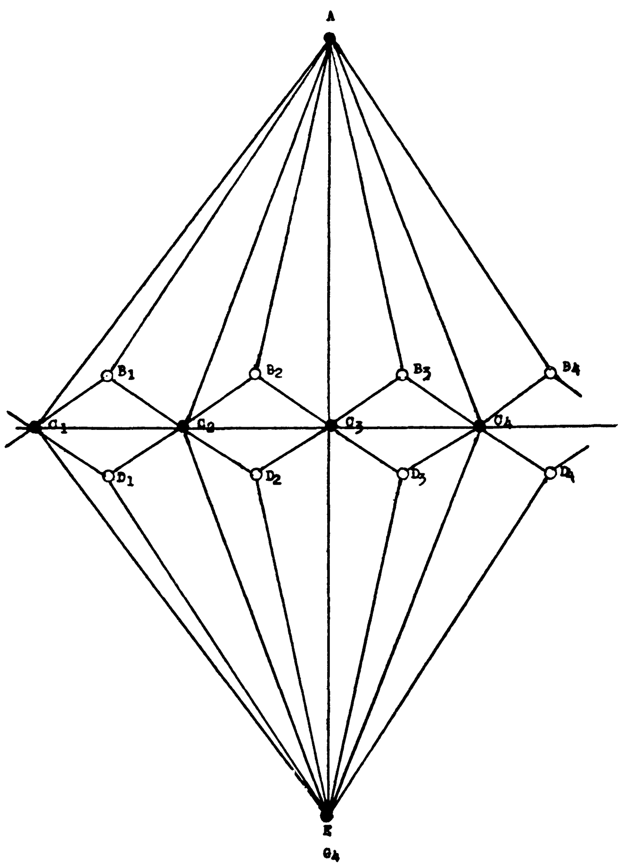

Figure 1.

Received September 5, 1960 in revised form October 20, 1960.

2. Path numbers and path lengths. For any graph G with $n(G)$ nodes we let $m(G)$ denote the number of disjoint simple paths required to cover all vertices of G, and let $p(G)$ denote the maximum number of nodes contained in a simple path on G. We call $m(G)$ the "path number" of G and $p(G)$ the "path length" of G. If G can be represented as the edges and vertices of a convex polyhedron in three-dimensional space, we say that G is "3-polyhedral". Now let

$$p(n) = \min\{p(G)\colon\ G \text{ is 3-polyhedral and } n(G) = n\}$$
$$m(n) = \max\{m(G)\colon G \text{ is 3-polyhedral and } n(G) = n\}\ .$$

We will show, by means of a class of examples, that $m(n) \geqq (n - 10)/3$ and $p(n) \leqq (2n + 13)/3$ for all n.

3. The graphs G_k. Let the graph $G_k (k \geqq 3)$ have $3k + 2$ vertices, which we will denote by a, b_i, c_i, d_i, and e(i ranging from 1 to k). Let the edges of G_k be (a, b_i), (a, c_i), (e, d_i), (e, c_i), (c_i, c_{i+1}), (c_i, b_i), (c_i, d_i), (d_i, c_{i+1}), and (b_i, c_{i+1}). Thus a and e are of valence $2k$, the c_i are of valence 8, and the b_i and d_i are of valence 3. See Figure 1 for a drawing of G_4. G_k can be represented as a triangulation of the plane, and it is easy to show by induction [4] that if $n(G) \geqq 4$ and G can be

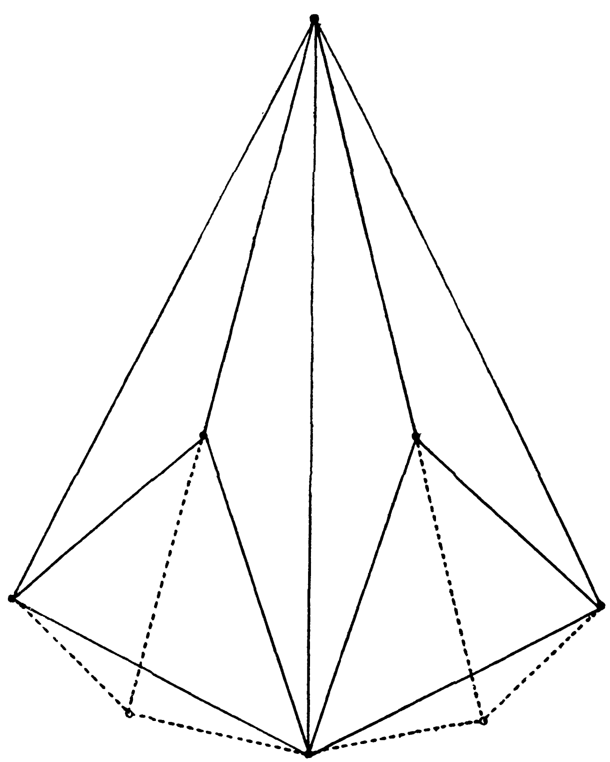

Figure 2.

represented as a triangulation of the plane, then G can be represented as the edges and vertices of a convex polyhedron in 3-space. Alternatively, one could apply the "Fundamentalsatz der Konvexen Typen" of E. Steinitz [6]. But in the case of G_k it is really unnecessary to use any such general results, for G_k is clearly the graph of the polyhedron obtained by appropriately truncating a bipyramid whose base is a regular $2k$-gon (Figure 2 illustrates how the top half of a bipyramid should be truncated in obtaining G_4).

If we color a, c_i, and e black and let b_i and d_i be white (where i ranges from 1 to k), then G_k consists of $n + 2$ black nodes and $2n$ white ones. Since each white node has only black neighbors, each simple path in G_k must contain at most one more white node than black. Thus at least $2k - (k + 2) = k - 2$ disjoint simple paths are required to visit every node of G_k. The following set of paths shows that the path-number of G_k is, in fact, exactly $k - 2$:

$$b_1 \to c_1 \to d_1 \to e \to d_2 \to c_2 \to b_2 \to a \to b_3 \to c_3 \to d_3$$
$$b_i \to c_i \to d_i \quad (i = 4, \cdots, k) .$$

Similarly, since no simple path can contain more than $k + 2$ black vertices, it follows that no simple path can contain more than

$$(k + 2) + (k + 3) = 2k + 5$$

vertices. It is easy to construct simple paths containing exactly this many vertices, and thus the path-length of G_k is $2k + 5$. Since $n(G_k) = 3k + 2$, it follows that if $n \equiv 2(\bmod 3)$,

$$p(n) \leqq \frac{2n + 11}{3}$$
$$m(n) \geqq \frac{n - 8}{3} .$$

To get bounds for $n \equiv 1 \pmod 3$, consider the graph G_k^- obtained by omitting one white vertex from G_k. For $n \equiv 0 \pmod 3$, consider the graph G_k^+ obtained by adjoining to G_k a vertex connected to c_1, d_1, and e. It follows that

$$\left.\begin{aligned} p(n) &\leqq \frac{2n + 13}{3} \\ m(n) &\geqq \frac{n - 10}{3} \end{aligned}\right\} n \equiv 1 \pmod 3 \qquad \left.\begin{aligned} p(n) &\leqq \frac{2n + 13}{3} \\ m(n) &\geqq \frac{n - 9}{3} \end{aligned}\right\} n \equiv 0 \pmod 3 .$$

Grünbaum and Motzkin asked if $n(G) = p(G)$ provided all of the faces of the polyhedron representing G were triangles, and our examples show that this is not the case. They further conjectured that

$$\max_{n(G)=n} m(G)\cdot p(G) \geqq n^{1+\gamma} \qquad \text{for some } \gamma > 0 .$$

Our examples show that

$$\max_{n(G)=n} m(G)\cdot p(G) \geqq \frac{2n^2 - 7n\,130}{9} .$$

Thus for any $\gamma < 1$ we can find an N_γ such that

$$\max_{n(G)=n} m(G)\cdot p(G) > n^{1+\gamma} \qquad \text{for all } n \geqq N_\gamma .$$

Furthermore, this result is the best possible in a sense; for since $m(G) < n$ and $p(G) \leqq n$, it follows that

$$\max_{n(G)=n} m(G)\cdot p(G) < n^2 \qquad \text{for all } n .$$

I want to thank Dr. Michel Balinski for drawing this subject to my attention, and the referee for making me aware of the paper by Grünbaum and Motzkin.

Bibliography

1. Michel L. Balinski, *An Algorithm for Finding All Vertices of Convex Polyhedral Sets*, Doctoral Dissertation, Princeton University, June 1959.
2. ———, *On the graph structure of convex polyhedra in n-space*. Pacific J. Math., (to appear).
3. T. A. Brown, *Hamiltonian Paths on Convex Polyhedra*, unpublished note, the RAND Corporation, August, 1960.
4. ———, *The Representation of Planar Graphs by Convex Polyhedra*, unpublished note, the RAND Corporation, August, 1960.
5. B. Grünbaum, and T. S. Motzkin, *Longest Simple Paths in Polyhedral Graphs*, (to appear).
6. E. Steinitz, and H. Rademacher, *Vorlesungen über die Theorie des Polyeder*, Springer, Berlin. 1934.

THE RAND CORPORATION AND HARVARD UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Reprinted 1966 in the United States of America

Pacific Journal of Mathematics
Vol. 11, No. 4 , 1961