

# Pacific Journal of Mathematics

**SOME CONGRUENCES FOR THE BELL POLYNOMIALS**

L. CARLITZ

# SOME CONGRUENCES FOR THE BELL POLYNOMIALS

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1. Let  $\alpha_1, \alpha_2, \alpha_3, \dots$  denote indeterminates. The Bell polynomial  $\phi_n(\alpha_1, \alpha_2, \alpha_3, \dots)$  may be defined by  $\phi_0 = 1$  and

$$(1.1) \quad \phi_n = \phi_n(\alpha_1, \alpha_2, \alpha_3, \dots) = \sum \frac{n!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2} \dots} \alpha_1^{k_1} \alpha_2^{k_2} \dots,$$

where the summation is over all nonnegative integers  $k_j$  such that

$$k_1 + 2k_2 + 3k_3 + \dots = n.$$

For references see Bell [2] and Riordan [5, p. 36]. The general coefficient

$$(1.2) \quad A_n(k_1, k_2, k_3, \dots) = \frac{n!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2} \dots}$$

is integral; this is evident from the representation

$$A_n(k_1, k_2, k_3, \dots) = \frac{n!}{k_1!(2k_2)!(3k_3)! \dots} \cdot \frac{(2k_2)!}{k_2!(2!)^{k_2}} \frac{(3k_3)!}{k_3!(3!)^{k_3}} \dots$$

and the fact that the quotient

$$\frac{(rk)!}{k!(r!)^k}$$

is integral [1, p. 57].

The coefficient  $A_n(k_1, k_2, k_3, \dots)$  resembles the multinomial coefficient

$$M(k_1, k_2, k_3, \dots) = \frac{(k_1 + k_2 + k_3 + \dots)!}{k_1!k_2!k_3 \dots}.$$

If  $p$  is a fixed prime it is known [3] that  $M(k_1, k_2, k_3, \dots)$  is prime to  $p$  if and only if

$$\begin{aligned} k_i &= \sum_j a_{ij} p^j & (0 \leq a_{ij} < p), \\ k_1 + k_2 + k_3 + \dots &= \sum_j a_j p^j & (0 \leq a_j < p) \end{aligned}$$

and

$$\sum_i a_{ij} = a_j \quad (j = 0, 1, 2, \dots).$$

It does not seem easy to find an analogous result for  $A_n(k_1, k_2, k_3, \dots)$ . For some special results see § 3 below.

Bell [2] showed that

$$(1.3) \quad \phi_p \equiv \alpha_1^p + \alpha_p \pmod{p}$$

and also determined the residues (mod  $p$ ) of  $\phi_{p+1}, \phi_{p+2}, \phi_{p+3}$ . He also obtained an expression for the residue of  $\phi_{p+r}$  as a determinant of order  $r + 1$ . Generalizing (1.3) we shall show first that

$$(1.4) \quad \phi_{p^r} \equiv \alpha_1^{p^r} + \alpha_p^{p^{r-1}} + \dots + \alpha_{p^r} \pmod{p}$$

and that

$$(1.5) \quad \phi_{pn}(\alpha_1, \alpha_2, \alpha_3, \dots) \equiv \phi_n(\phi_p, \alpha_{2p}, \alpha_{3p}, \dots) \pmod{p}$$

for all  $n \geq 1$ . Note that on the right the first argument in  $\phi_n$  is  $\phi_p$  and not  $\alpha_p$ .

2. From (1.1) we get the generating function

$$(2.1) \quad \sum_{n=0}^{\infty} \phi_n \frac{t^n}{n!} = \exp\left(\alpha_1 t + \alpha_2 \frac{t^2}{2!} + \alpha_3 \frac{t^3}{3!} + \dots\right).$$

Indeed this may be taken as the definition of  $\phi_n$ . Differentiating with respect to  $t$  we get

$$\sum_{n=0}^{\infty} \phi_{n+1} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \phi_n \frac{t^n}{n!} \sum_{r=0}^{\infty} \alpha_{r+1} \frac{t^r}{r!},$$

so that

$$(2.2) \quad \phi_{n+1} = \sum_{r=0}^n \binom{n}{r} \phi_{n-r} \alpha_{r+1}.$$

Since the binomial coefficient

$$\binom{pn}{r} \equiv 0 \pmod{p}$$

unless  $p|r$  and

$$\binom{pn}{pr} \equiv \binom{n}{r} \pmod{p}$$

it follows from (2.2) that

$$(2.3) \quad \phi_{pn+1} \equiv \sum_{r=0}^n \binom{n}{r} \phi_{p(n-r)} \alpha_{pr+1} \pmod{p}.$$

If for brevity we put

$$A(t) = \sum_{r=1}^{\infty} \alpha_r t^r / r!,$$

so that

$$\sum_{n=0}^{\infty} \phi_n \frac{t^n}{n!} = \exp A(t) ,$$

it is easily seen by repeated differentiation and by (1.3) that

$$(2.4) \quad \sum_{n=0}^{\infty} \phi_{n+p} \frac{t^n}{n!} \equiv \{(A'(t))^p + A^{(p)}(t)\}e^{A(t)} \pmod{p} .$$

(By the statement

$$\sum_{n=0}^{\infty} A_n \frac{t^n}{n!} \equiv \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \pmod{m} ,$$

where  $A_n, B_n$  are polynomials with integral coefficients, is meant the system of congruences

$$A_n \equiv B_n \pmod{m} \quad (n = 0, 1, 2, \dots) .$$

Hurwitz [4, p. 345] has proved the lemma that if  $a_1, a_2, a_3, \dots$  are arbitrary integers then

$$\left( \sum_{n=1}^{\infty} a_n \frac{x^n}{n!} \right)^k \equiv 0 \pmod{k!} .$$

The proof holds without change when the  $a_n$  are indeterminates. Since

$$A'(t) = \sum_{n=0}^{\infty} \alpha_{n+1} \frac{t^n}{n!} ,$$

it follows easily from Hurwitz's lemma that

$$(A'(t))^p = \left( \alpha_1 + \sum_{n=1}^{\infty} \alpha_{n+1} \frac{t^n}{n!} \right)^p \equiv \alpha_1^p \pmod{p} .$$

Thus (2.4) becomes

$$\sum_{n=0}^{\infty} \phi_{n+p} \frac{t^n}{n!} \equiv \left( \alpha_1^p + \sum_{r=0}^{\infty} \alpha_{r+p} \frac{t^r}{r!} \right) \sum_{n=0}^{\infty} \phi_n \frac{t^n}{n!} ,$$

which yields

$$(2.5) \quad \phi_{n+p} \equiv (\alpha_1^p + \alpha_p) \phi_n + \sum_{r=1}^n \binom{n}{r} \alpha_{r+p} \phi_{n-r} \pmod{p} .$$

In particular, for  $n = 0$ , (2.5) reduces to Bell's congruence (1.3). Similarly

$$\begin{aligned} \phi_{p+1} &\equiv (\alpha_1^p + \alpha_p) \alpha_1 + \alpha_{p+1} \equiv \phi_p \alpha_1 + \alpha_{p+1} , \\ \phi_{p+2} &\equiv \phi_p \phi_2 + 2 \alpha_{p+1} \alpha_1 + \alpha_{p+2} , \end{aligned}$$

and so on.

We remark that (2.5) is equivalent to Bell's congruence involving a determinant [2, p. 267, formula (6.5)]. Also for  $s = \alpha_1 = \alpha_2 = \dots$ , (2.5) reduces to

$$(2.5)' \quad \begin{aligned} a_{n+p}(s) &\equiv (s^p + s)a_n(s) + s \sum_{r=1}^n \binom{n}{r} a_{n-r}(s) \\ &\equiv a_{n+1}(s) + s^p a_n(s) \end{aligned} \pmod{p},$$

where [5, p. 76]

$$a_n(s) = \phi_n(s, s, \dots) = \sum_k S(n, k) s^k$$

and  $S(n, k)$  denotes the Stirling number of the second kind. The congruence (2.5)' is due to Touchard [6].

If in (2.5) we replace  $n$  by  $pn$  we get

$$(2.6) \quad \phi_{p(n+1)} \equiv \phi_p \phi_{np} + \sum_{r=1}^n \binom{n}{r} \alpha_{p(r+1)} \phi_{p(n-r)} \pmod{p}$$

for all  $n = 0, 1, 2, \dots$ . Thus  $\phi_{pn}$  is congruent to a polynomial in  $\phi_p, \alpha_{2p}, \alpha_{3p}, \dots$  alone. Moreover, comparing (2.6) with (2.2), it is clear that

$$(2.7) \quad \phi_{pn} \equiv \phi_n(\phi_p, \alpha_{2p}, \alpha_{3p}, \dots) \pmod{p},$$

so that we have proved (1.5).

Replacing  $n$  by  $pn$  in (2.7) we get

$$\phi_{p^2n} \equiv \phi_{pn}(\phi_p, \alpha_{2p}, \alpha_{3p}, \dots) \equiv \phi_n(\phi_p^p + \alpha_{p^2}, \alpha_{2p^2}, \alpha_{3p^2}, \dots).$$

In particular for  $n = 1$

$$\phi_{p^2} \equiv \phi_p^p + \alpha_{p^2} \equiv \alpha_1^{p^2} + \alpha_p^p + \alpha_{p^2}.$$

Again replacing  $n$  by  $pn$  we get

$$\phi_{p^3n} \equiv \phi_n(\phi_p^2 + \alpha_{p^3}, \alpha_{2p^3}, \alpha_{3p^3}, \dots),$$

so that in particular

$$\phi_{p^3} \equiv \phi_p^2 + \alpha_{p^3} \equiv \alpha_1^{p^3} + \alpha_p^{p^2} + \alpha_{p^2}^p + \alpha_{p^3}.$$

Continuing in this way we see that

$$(2.8) \quad \phi_{p^r n} \equiv \phi_n(\phi_{p^r}, \alpha_{2p^r}, \alpha_{3p^r}, \dots) \pmod{p}$$

and

$$(2.9) \quad \phi_{p^r} \equiv \phi_{p^{r-1}}^p + \alpha_{p^r} \equiv \alpha_1^{p^r} + \alpha_p^{p^{r-1}} + \dots + \alpha_{p^r} \pmod{p}.$$

We have therefore proved (1.4) as well as the more general congruence (2.8).

Since

$$\begin{aligned} \phi_2 &= \alpha_1^2 + \alpha_2, \\ \phi_3 &= \alpha_1^3 + 3\alpha_1\alpha_2 + \alpha_3, \\ \phi_4 &= \alpha_1^4 + 6\alpha_1^2\alpha_2 + 4\alpha_1\alpha_3 + 3\alpha_2^2 + \alpha_4, \end{aligned}$$

it follows from (2.8) that

$$(2.10) \quad \begin{cases} \phi_{2p^r} \equiv \phi_{p^r}^2 + \alpha_{2p^r}, \\ \phi_{3p^r} \equiv \phi_{p^r}^3 + 3\phi_{p^r}\alpha_{2p^r} + \alpha_{3p^r}, \\ \phi_{4p^r} \equiv \phi_{p^r}^4 + 6\phi_{p^r}^2\alpha_{2p^r} + 4\phi_{p^r}\alpha_{3p^r} + 3\alpha_{2p^r}^2 + \alpha_{4p^r}, \end{cases}$$

and so on.

We note also that (2.3) implies

$$(2.11) \quad \begin{cases} \phi_{p^{r+1}} \equiv \phi_{p^r}\alpha_1 + \alpha_{p^{r+1}}, \\ \phi_{2p^{r+1}} \equiv \phi_{2p^r}\alpha_1 + 2\phi_{p^r}\alpha_{p^{r+1}} + \alpha_{2p^{r+1}}, \\ \phi_{3p^{r+1}} \equiv \phi_{3p^r}\alpha_1 + 3\phi_{2p^r}\alpha_{p^{r+1}} + 3\phi_{p^r}\alpha_{2p^{r+1}} + \alpha_{3p^{r+1}}. \end{cases}$$

3. By means of (1.5) we can obtain certain congruences for the coefficient  $A(k_1, k_2, k_3, \dots)$ . Indeed by (1.1) and (1.3)

$$(3.1) \quad \begin{aligned} &\phi_n(\phi_p, \alpha_{2p}, \alpha_{3p}, \dots) \\ &\equiv \sum A_n(k_1, k_2, k_3, \dots)(\alpha_1^p + \alpha_p)^{k_1}\alpha_{2p}^{k_2}\alpha_{3p}^{k_3}\dots \pmod{p}, \end{aligned}$$

where the summation is over nonnegative  $k_j$  such that

$$k_1 + 2k_2 + 3k_3 + \dots = n.$$

The right member of (3.1) is equal to

$$(3.2) \quad \sum_{(k_j)} A_n(k_1, k_2, k_3, \dots) \sum_{r=0}^{k_1} \binom{k_1}{r} \alpha_1^{p(k_1-r)} \alpha_p^r \alpha_{2p}^{k_2} \alpha_{3p}^{k_3} \dots$$

On the other hand

$$(3.3) \quad \phi_{pn} = \sum A_{pn}(h_1, h_2, h_3, \dots) \alpha_1^{h_1} \alpha_2^{h_2} \alpha_3^{h_3} \dots,$$

summed over

$$(3.4) \quad h_1 + 2h_2 + 3h_3 + \dots = pn.$$

It follows from (1.5) that

$$A_{pn}(h_1, h_2, h_3, \dots) \equiv 0 \pmod{p}$$

except possibly when

$$(3.5) \quad h_j = 0 \quad (j > 1, p + j).$$

When this condition is satisfied (3.4) becomes

$$h_1 + p(h_p + 2h_{2p} + \dots) = pn ;$$

consequently  $h_1 = pk_1$  and (3.3) becomes

$$\phi_{pn} \equiv \sum A_{pn}(pk_1, 0, \dots, 0, h_p, \dots) \alpha_1^{pk_1} \alpha_h^{h_p} \alpha_{2p}^{h_{2p}} \dots .$$

We have therefore proved the following result:

**THEOREM 1.** *The coefficient  $A_{pn}(h_1, h_2, h_3, \dots)$  occurring in (3.3) is certainly divisible by  $p$  unless (3.5) is satisfied and  $h_1 = pk_1$ . If these conditions are satisfied then*

$$A_{pn}(h_1, h_2, h_3, \dots) \equiv \binom{k_1}{h_p} A_n(k_1 - h_p, h_p, h_{2p}, \dots) \pmod{p} .$$

If we make use of (1.4) we obtain the following simpler

**THEOREM 2.** *Let*

$$h_1 + 2h_2 + 3h_3 + \dots = p^r .$$

*Then the coefficient  $A_{p^r}(h_1, h_2, h_3, \dots)$  is divisible by  $p$  except when*

$$h_i = 0 \quad (i \neq j) , \quad h_j = p^s ,$$

*for some  $j$ , in which case*

$$A_{p^r}(h_1, h_2, h_3, \dots) \equiv 1 \pmod{p} .$$

Using (2.10) and (2.11) we can obtain additional results. For example take

$$h_1 + 2h_2 + 3h_3 + \dots = 2p^r .$$

Then  $A_{2p^r}(h_1, h_2, h_3, \dots)$  is divisible by  $p$  unless (i) all  $h_s = 0$  ( $s \neq j$ ),  $h_j = 1$  or 2; (ii) all  $h_s = 0$  ( $s \neq i, j$ ),  $h_i = h_j = 1$ . In case (i)  $A \equiv 1$ , in case (ii)  $A \equiv 2 \pmod{p}$ .

For  $n = 3p^r$  the corresponding results are more complicated.

4. We turn now to the polynomial  $C_n(\alpha_1, \alpha_2, \alpha_3, \dots)$ , the cycle indicator of the symmetric group [5, p. 68]:

$$(4.1) \quad C_n = C_n(\alpha_1, \alpha_2, \alpha_3, \dots) = \phi_n(\alpha_1, \alpha_2, 2!\alpha_3, \dots) \\ = \sum \frac{n!}{k_1!k_2!k_3 \dots} \left(\frac{\alpha_1}{1}\right)^{k_1} \left(\frac{\alpha_2}{2}\right)^{k_2} \left(\frac{\alpha_3}{3}\right)^{k_3} \dots ,$$

where the summation is over all nonnegative  $k_j$  such that

$$k_1 + 2k_2 + 3k_3 + \dots = n .$$

It is convenient to define  $C_0 = 1$ .

Put

$$(4.2) \quad c_n(k_1, k_2, k_3, \dots) = \frac{n!}{k_1!k_2!k_3 \dots 1^{k_1}2^{k_2}3^{k_3} \dots},$$

the general coefficient of  $C_n$ . Clearly  $c_n(k_1, k_2, k_3, \dots)$  is integral and indeed a multiple of  $A_n(k_1, k_2, k_3, \dots)$ .

From (4.1) we get the generating function

$$(4.3) \quad G(t) = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \exp\left(\alpha_1 t + \frac{1}{2}\alpha_2 t^2 + \frac{1}{3}\alpha_3 t^3 + \dots\right).$$

For brevity put

$$C(t) = \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n t^n.$$

Differentiating (4.3) with respect to  $t$  we get

$$G'(t) = C'(t)G(t),$$

that is

$$\sum_{n=0}^{\infty} C_{n+1} \frac{t^n}{n!} = \sum_{r=0}^{\infty} \alpha_{r+1} t^r \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}.$$

This implies

$$(4.4) \quad C_{n+1} = \sum_{r=0}^n \frac{n!}{r!} \alpha_{n-r+1} C_r,$$

so that

$$(4.5) \quad C_{n+1} \equiv \alpha_1 C_n \pmod{n}.$$

By repeated differentiation of (4.3) we get (compare (2.4))

$$(4.6) \quad \frac{d^p}{dt^p} G(t) \equiv \{(C'(t))^p + C^{(p)}(t)\}G(t) \pmod{p}.$$

Now since

$$C'(t) = \sum_{n=0}^{\infty} \alpha_{n+1} t^n, \quad C^{(p)}(t) = \sum_{n=0}^{\infty} (n+p-1)! \alpha_{n+1} \frac{t^n}{n!},$$

it is clear that

$$(C'(t))^p \equiv \alpha_1^p, \quad C^{(p)}(t) \equiv -\alpha_p \pmod{p};$$

at the last step we have used Wilson's theorem. Thus (4.6) becomes



$$\sum_{n=0}^{\infty} C_{n+p} \frac{t^n}{n!} \equiv (\alpha_1^p - \alpha_p) \sum_{n=0}^{\infty} C_n \frac{t^n}{n!},$$

so that

$$(4.7) \quad C_{n+p} \equiv (\alpha_1^p - \alpha_p) C_n \pmod{p}.$$

In particular we have

$$(4.8) \quad C_p \equiv \alpha_1^p - \alpha_p \pmod{p}$$

and

$$(4.9) \quad C_{n+rp} \equiv (\alpha_1^p - \alpha_p)^r C_n \pmod{p}$$

We remark that for  $p = 3, 5, 7$ , (4.8) is in agreement with the explicit values of  $C_n$  given in [5, p. 69].

By (4.9) with  $n = 0$  we find that the coefficient

$$c_{rp}(k_1, k_2, k_3, \dots) \equiv 0 \pmod{p}$$

unless all  $k_j$  except  $k_1$  and  $k_p$  vanish and  $k_1$  is a multiple of  $p$ ; in this case we have

$$(4.10) \quad c_{rp}(pk, 0, \dots, 0, k_p, \dots) \equiv (-1)^{k_p} \binom{r}{k} \pmod{p}.$$

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