

# Pacific Journal of Mathematics

**SELF-INTERSECTION OF A SPHERE ON A COMPLEX  
QUADRIC**

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# SELF-INTERSECTION OF A SPHERE ON A COMPLEX QUADRIC

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1. The real part  $S^n$  of a quadric  $V$  in complex, affine  $(n + 1)$ -space is a sphere. The self-intersection of  $S^n$  in  $V$  is the same as the self-intersection of a "vanishing cycle," introduced by Lefschetz, and plays a certain role in [4], [5]. We will compute here this self-intersection number, using elementary tools.

Let us introduce some notations.  $P_{n+1}$  denotes the complex projective space of algebraic dimension  $n + 1$ , hence of topological dimension

$$\dim P_{n+1} = 2n + 2 .$$

To each projective sub-space  $P_k$  of  $P_{n+1}$  a *positive* orientation can be given, thus it can be considered as a *cycle*  $p_{2k}$ . Then we agree that

$$(1) \quad \text{if } k + l = n + 1, \text{ then } (p_{2k}, p_{2l}) = 1 \text{ in } P_{n+1}$$

be true for the *intersection numbers* of cycles. This is the usual convention, the one in [1], for example; in [7] another convention is adopted.

Let  $x_1, \dots, x_{n+2}$  be a fixed system of projective coordinates in  $P_{n+1}$ . Then

$$(2) \quad Q_n : x_1^2 + \dots + x_{n+2}^2 = 0$$

is a *non-singular quadric*;  $\dim Q_n = 2n$ . The points of  $P_{n+1}$  whose last coordinate is non-zero form a complex affine space  $C_{n+1}$ , and

$$V = Q_n \cap C_{n+1} = [x : x \in Q_n, x_{n+2} \neq 0]$$

is a *non-singular affine quadric*. If  $z \in C_{n+1}$ , we denote by  $z_1, \dots, z_{n+2}$  those coordinates for which  $z_{n+2} = i$  where  $i^2 = -1$ ; thus  $z_1, \dots, z_{n+1}$  are affine coordinates in  $C_{n+1}$ . Then

$$V : z_1^2 + \dots + z_{n+1}^2 = 1 \quad (z \in C_{n+1})$$

$$S^n : z_1^2 + \dots + z_{n+1}^2 = 1, z_1, \dots, z_{n+1} \text{ reals}$$

are the equations of an affine quadric and its real part respectively; *this real part*  $S^n$  is, of course, a sphere. We consider  $S^n$  with an arbitrarily chosen and fixed orientation as a cycle  $s$ . It is well known (see, for example, [2], p. 35, (g)) that

(3) *the homology class  $s$ , of the cycle whose carrier is  $S^n$ , generates  $H_n(V; Z)$ ,*

where  $Z$  denotes the ring of integers.

As  $\dim V = 2 \dim S^n$ , the self intersection number

$$(4) \quad (s, s) = (S^n S^n), \quad (\text{in } V),$$

of  $s$  in  $V$ , is well defined; we may write  $(S^n, S^n)$  for this self intersection number, because  $(s, s)$  does not depend on the orientation of  $S^n$ , used in (3).

2. M. F. Atiyah communicated to me his computation of the intersection number (4) for  $n = 2$ , showing that the sign in [2], p. 35 (10) is not the right one.<sup>1</sup> The determination of the sign of (4) given below is a generalization to  $n$  dimensions of the construction of Atiyah. In [2] we used only the fact that (4) is not zero, if  $n$  is even, hence other results of that paper are not invalidated by the false sign in (10), p. 35. The mistaken sign is "classical." Wrong sign appears in [4], p. 93, Théorème sur les  $\Gamma_{d-1}$  de  $C_u$ , I, [5] on top of p. 16, [8], p. 102, (3), and [7], p. 104, Theorem 45 (although in [7] not the convention (1) is used, the alternation of the sign in question is independent of any convention). After the completion of the present paper [6] appeared, where the classical mistake in sign is corrected (see (11.3) on p. 161). The results of [1] are in agreement with the sign (5) below.

3. Using the notations and conventions introduced above, we will prove the following theorem.

**THEOREM.** *Let  $s$  be the homology class of the oriented sphere  $S^n$  in  $H_n(Q_n; Z)$  where  $n = 2h$  is even. Let us denote by  $(s, s)$  the self-intersection number of  $s$  computed with the convention (1). Then*

$$(5) \quad (s, s) = \begin{cases} -2, & \text{if } h = \frac{n}{2} \text{ is odd;} \\ +2, & \text{if } h = \frac{n}{2} \text{ is even;} \end{cases}$$

*holds true.*

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<sup>1</sup> I take the opportunity to correct another mistake in [2], also noticed by Atiyah. In Proposition 2, p. 27, we have to suppose that the singularity in question is *conical*. In [2], Proposition 2 is stated without proof; Atiyah gave an example showing that the statement does not hold true, if the singularity is *not conical*, and gave a proof with the correct hypothesis. Proposition 2 is used in [2] only in connection with conical singularities; thus other results of [2] are not affected by the incomplete formulation of that Proposition.

4. We prepare the proof of this theorem; for the first part of the proof, see [1]. (See also [3], pp. 230-232.) In order to describe easily linear sub-spaces of  $Q_n$ , we introduce new projective coordinates in  $P_{n+1}$ :

$$\begin{aligned} u_j &= x_{2j-1} + ix_{2j} \\ v_j &= x_{2j-1} - ix_{2j} \end{aligned} \quad j = 1, \dots, h + 1 \quad (i^2 = -1).$$

Let us notice that

$$(6) \quad u_j = v_j = 0 \text{ if and only if } x_{2j-1} = x_{2j} = 0.$$

The equation of  $Q_n$  is

$$u_1v_1 + \dots + u_{h+1}v_{h+1} = 0,$$

in the new coordinates.

We consider the following linear sub-spaces of  $Q_n$ :

$$(7) \quad A : u_j = 0, \quad j = 1, \dots, h, h + 1;$$

$$(8) \quad B : u_j = 0, \quad j = 1, \dots, h; \quad v_{h+1} = 0;$$

$$(9) \quad C : v_j = 0, \quad j = 1, \dots, h + 1.$$

Let us remark that,

$$(10) \quad A \cap C = \phi, \quad B \cap C \text{ is just one point,}$$

by (6).

LEMMA 1. *Let  $X$  be one of the projective spaces  $A, B, C$ . If, in the system of equations defining  $X$ , we replace an even number of equations  $u_j = 0$  by the corresponding  $v_j = 0$ , or vice versa, we define a new linear sub-space of  $Q_n$  belonging to the same continuous system as  $X$ . Similarly, without leaving the continuous system containing  $B$ , we may replace  $u_h = 0, v_{h+1} = 0$  in (8) by  $v_h = 0$  and  $u_{h+1} = 0$ .*

*Proof.* Let us suppose that we want to replace  $v_1 = 0, v_2 = 0$  in (9) by  $u_1 = 0, u_2 = 0$ . Let us consider the linear space

$$\begin{aligned} \alpha v_2 + \beta u_1 &= 0, & v_3 &= 0, \dots, v_{h+1} = 0, \\ -\alpha v_1 + \beta u_2 &= 0, \end{aligned}$$

defined for every  $(\alpha, \beta) \neq (0, 0)$ . This projective space is clearly contained in  $Q_n$ . For  $(1, 0)$  we have  $C$  and for  $(0, 1)$  the desired replacement. The last statement of the lemma is proved similarly using the system

$$\begin{aligned} \alpha u_h + \beta u_{h+1} &= 0, \\ -\beta v_h + \alpha v_{h+1} &= 0. \end{aligned}$$

Let us consider now  $A, B, C$  as cycles of  $Q_n$ , and let us denote by  $a, b, c$  their respective homology classes in  $H_n(Q_n; Z)$ .

LEMMA 2. *If  $h$  is odd, then  $c = a$ . If  $h$  is even, then  $c = b$ .*

*Proof.* If  $h$  is odd, the  $h + 1$  equations of (9) can be replaced by the equations  $u_j = 0$ ,  $j = 1, \dots, h + 1$ . Hence  $A$  and  $C$  belong to the same continuous system. If  $h$  is even, we can replace the first  $h$  equations defining  $C$  by  $u_j = 0$ ,  $j = 1, \dots, h$ . Hence  $C$  and  $B$  belong to the same continuous system.

LEMMA 3. *As to the intersection numbers, we have*

$$(11) \quad \text{if } h \text{ is odd, then } (a, a) = 0, (b, b) = 0, (a, b) = 1,$$

$$(12) \quad \text{if } h \text{ is even, then } (a, a) = 1, (b, b) = 1, (a, b) = 0.$$

*Proof.* (1) Let  $h$  be odd. By Lemma 2 and the first equation of (10), we have  $(a, a) = 0$ . Similarly, the second equation of (10) and Lemma 2 prove  $(a, b) = 1$ . In order to prove  $(b, b) = 0$ , we consider the space

$$B' : v_j = 0, \quad j = 1, \dots, h, \quad u_{h+1} = 0.$$

We claim that  $B$  and  $B'$  are in the same continuous system. In order to prove this statement, we use Lemma 1 twice. First, we replace the last two equations of (8) by  $v_h = 0$ , and  $u_{h+1} = 0$ . Second, in the system obtained by the first step, we replace the first  $h - 1$  equations by  $v_j = 0$ . Now  $B \cap B' = \phi$ , and this proves  $(b, b) = 0$ .

(2) Let  $h$  be even. The proof of (12) is similar to the previous one. The last two equations of (12) are immediate from (10) and Lemma 2. Using Lemma 1, we can find presently a  $B''$ , such that  $B \cap B''$  be just one point.

LEMMA 4. *Using the previous notations  $s, a, b$ , for homology classes,*

$$(13) \quad s = \pm(a - b),$$

*the sign depending on the chosen orientation of  $S^n$ .*

*Proof.* Let us denote by  $I$  the hyperplane  $x_{n+2} = 0$ . Then, clearly,

$$A \cap I = B \cap I.$$

We denote by  $J$  this intersection ( $J = A \cap B$ ). Let us consider a pencil

of  $k$ -planes,  $2k + \dim A = 2n + 2$ , in general position. If  $N$  is a neighborhood of  $J$  in  $B$ , the  $k$ -planes of the pencil project  $N$  into a neighborhood  $M$  of  $J$  in  $A$ . Given now a Riemann metric of  $P_{n+1}$ , if  $N$  is a small enough neighborhood of  $J$ , the corresponding points of  $N$ ,  $M$  determine unique geodesic segments. We consider now  $B$  as a cycle, whose simplexes are so small that those intersecting  $J$  are contained in  $N$ . Using the geodesic segments introduced above which start at points of the simplexes of  $B$  intersecting  $J$ , it is easy to construct a chain  $E$  of  $Q_n$ , such that

$$(14) \quad A - B + \partial E$$

be a sum of simplexes of  $V = Q_n - I$ . Hence,  $s$  being a generator of  $H_n(V; Z)$ , (14) will be homologous to a multiple of  $s$ . Thus  $a - b = ms$  for some integer  $m$ . Now  $(a - b, a) = m(s, a)$  is  $\pm 1$  by Lemma 3, hence  $m = \pm 1$ .

*Proof of the Theorem.* (1) Let us suppose that  $h$  is odd. We use (13) and (11):  $(s, s) = (a - b, a - b) = (a, a) - (b, a) - (a, b) + (b, b) = -(b, a) - (a, b) = -2$ .

(2) Let us suppose that  $h$  is even. This time we use (12):  $(s, s) = (a, a) + (b, b) = +2$ . Hence the proof of (5) is complete.

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