

Pacific Journal of Mathematics

MEAN CROSS-SECTION MEASURES OF HARMONIC MEANS OF CONVEX BODIES

WILLIAM JAMES FIREY

MEAN CROSS-SECTION MEASURES OF HARMONIC MEANS OF CONVEX BODIES

WILLIAM J. FIREY

1. In [2] the notion of p -dot means of two convex bodies in Euclidean n -space was introduced and certain properties of these means investigated. For $p = 1$, the mean is more appropriately called the harmonic mean; here we restrict the discussion to this case. The harmonic mean of two convex bodies K_0 and K_1 , which will always be assumed to share a common interior point Q , is defined as follows. Let \hat{K} denote the polar reciprocal of K with respect to the unit sphere E centred at Q ; let $(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1$, with $0 \leq \vartheta \leq 1$, be the usual arithmetic or Minkowski mean of \hat{K}_0 and \hat{K}_1 . The harmonic mean of K_0, K_1 is the convex body $[(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge$. In more analytic terms, if $F_i(x)$ are the distance functions with respect to Q of K_i , for $i = 0, 1$, then the body whose distance function with respect to Q is $(1 - \vartheta)F_0(x) + \vartheta F_1(x)$ is the harmonic mean of K_0 and K_1 .

In the paper mentioned, a dual Brunn-Minkowski theorem was established, namely

$$(1) \quad V^{1/n}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \leq 1 / \left[\frac{(1 - \vartheta)}{V^{1/n}(K_0)} + \frac{\vartheta}{V^{1/n}(K_1)} \right]$$

where $V(K)$ means the volume of K . There is equality if and only if K_0 and K_1 are homothetic with the centre of magnification at Q .

Here we develop a more inclusive theorem regarding the behaviour of each mean cross-section measure, ("Quermassintegral") $W_\nu(K)$, $\nu = 0, 1, \dots, n - 1$, cf. [1]. The result is

$$(2) \quad W_\nu^{1/(n-\nu)}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \leq 1 / \left[\frac{(1 - \vartheta)}{W_\nu^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_\nu^{1/(n-\nu)}(K_1)} \right].$$

The cases of equality are just those of the dual Brunn-Minkowski theorem, ($\nu = 0$).

2. We first list some preliminary items used in the proof of (2). We shall use Minkowski's inequality in the form

$$(3) \quad \int [(1 - \vartheta)f_0^p + \vartheta f_1^p]^{1/p} dx \leq \left[(1 - \vartheta) \left(\int f_0 dx \right)^p + \vartheta \left(\int f_1 dx \right)^p \right]^{1/p}.$$

Here the functions f_i are assumed to be positive and continuous over the closed and bounded domain of integration common to all the integrals,

Received September 29, 1960.

and, for our puposes, p satisfies $-1 \leq p < 0$. There is equality if and only if $f_0(x) \equiv \lambda f_1(x)$ for some constant λ . See [3], Theorem 201, coupled with the remark preceding Theorem 200.

Our second tool, which we shall refer to as the projection lemma, was established in [2]. Let K^* denote the projection of K onto a fixed, m -dimensional, linear subspace E_m through Q for $1 \leq m < n$. We have

$$(4) \quad [(1 - \vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*]^\wedge \cong \{[(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge\}^* .$$

Since E_m contains Q and the polar reciprocation is with respect to sphere E centred at Q , in forming \hat{K}^* the order of operations is immaterial. This result is proved by a polar reciprocation argument from

$$(1 - \vartheta)(\hat{K}_0 \cap E_m) + \vartheta(\hat{K}_1 \cap E_m) \subseteq [(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1] \cap E_m .$$

There is equality in either inclusion if K_0 and K_1 are homothetic with centre of magnification at Q .

The dual Brunn-Minkowski theorem (1) will be used.

Finally we shall make use of Kubota's formula and some of its consequences. This material is covered in [1]. An $(n - \nu)$ dimensional cross-section measure ("Quermass") of K is the $(n - \nu)$ dimensional volume of that convex body which is the vertical projection of K onto an $E_{n-\nu}$. The mean cross-section measures are usually defined as the coefficients in Steiner's polynomial which describes $V(K + \lambda E)$, that is

$$(5) \quad V(K + \lambda E) = \sum_{\nu=0}^n \binom{n}{\nu} W_\nu(K) \lambda^\nu .$$

If we denote the $(\nu - 1)^{\text{th}}$ mean cross-section measure of the projection of K onto that E_{n-1} through Q which is orthogonal to the vector u_1 by $W'_{\nu-1}(K, u_1)$, then Kubota's formula is

$$W_\nu(K) = \frac{1}{\kappa_{n-1}} \int_{\Omega_n} W'_{\nu-1}(K, u_1) d\omega_n , \quad \nu = 1, 2, \dots, \nu - 1 .$$

Here the integration with respect to the direction u_1 is extended over the surface Ω_n of E , $d\omega_n$ is the element of surface area on Ω_n and κ_{n-1} is the volume of the $n - 1$ dimensional unit sphere.

Kubota's formula can be applied to the mean cross-section measure $W'_{\nu-1}(K, u_1)$ for fixed u_1 :

$$W'_{\nu-1}(K, u_1) = \frac{1}{\kappa_{n-2}} \int_{\Omega_{n-1}} W''_{\nu-2}(K, u_1, u_2) d\omega_{n-1}$$

where $W''_{\nu-2}$ is the $(\nu - 2)$ th mean cross-section measure of the projection of κ onto the E_{n-2} through Q orthogonal to u_1 and u_2 with u_2 orthogonal to u_1 . After ν such steps we have as the extended form of Kubota's formula:

$W_\nu(K)$

$$= \frac{1}{\kappa_{n-1}\kappa_{n-2}\cdots\kappa_{n-\nu}} \int_{\Omega_n} \int_{\Omega_{n-1}} \cdots \int_{\Omega_{n-\nu}} W_0^{(\nu)}(K, u_1, u_2, \dots, u_\nu) d\omega_{n-\nu} \cdots d\omega_{n-1} d\omega_n .$$

Each vector u_p is orthogonal to u_q for $q < p$ and $W_0^{(\nu)}(K, u_1, u_2, \dots, u_\nu)$ is the 0th mean cross-section measure of the projection of K onto that $E_{n-\nu}$ through Q which is the orthogonal complement of the subspace spanned by u_1, u_2, \dots, u_ν .

Steiner's formula (5) with $\lambda = 0$ shows that $W_0(K)$ is the volume of K and so $W_0^{(\nu)}$ is an $(n - \nu)$ dimensional cross-section measure of K . Thus, to within a numerical factor depending on n and ν , $W_\nu(K)$ is the arithmetic mean of the $(n - \nu)$ dimensional cross-section measures.

In § 3 we shall use the following abbreviations: for $d\omega_{n-\nu} \cdots d\omega_{n-1} d\omega_n$ we write $d\bar{\omega}$ with sign of integration and omit reference to the domains of integration; for one $1/\kappa_{n-1}\kappa_{n-2}\cdots\kappa_{n-\nu}$ we write k ; finally for $W_0^{(\nu)}(K, u_1, u_2, \dots, u_\nu)$ we write $\sigma(K^*)$. In this notation the extended Kubota formula reads

$$W(K) = k \int \sigma(K^*) d\bar{\omega} .$$

3. We now prove (2). By the extended form of Kubota's formula

$$(6) \quad \begin{aligned} W_\nu^{1/(n-\nu)}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) &= \left[k \int \sigma([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^*) d\bar{\omega} \right]^{1/(n-\nu)} \\ &\leq \left[k \int \sigma([(1 - \vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*]^\wedge) d\bar{\omega} \right]^{1/(n-\nu)} \end{aligned}$$

in virtue of the projection lemma and the set monotonicity of σ i.e., $\sigma(K^*) \leq \sigma(\bar{K}^*)$ if $K^* \subseteq \bar{K}^*$ with equality in the latter relation implying that in the former. We now apply (1), in $E_{n-\nu}$, to the integrand to obtain

$$\sigma([(1 - \vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*]^\wedge) \leq \left\{ 1 / \left[\frac{(1 - \vartheta)}{\sigma^{1/(n-\nu)}(K_0^*)} + \frac{\vartheta}{\sigma^{1/(n-\nu)}(K_1^*)} \right] \right\}^{(n-\nu)} .$$

Here we take advantage of the fact that

$$(\hat{K})^* = (K^*)^\wedge .$$

This gives

$$(7) \quad \begin{aligned} W_\nu^{1/(n-\nu)}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \\ \leq \left[k \int \left\{ 1 / \left[\frac{(1 - \vartheta)}{\sigma^{1/(n-\nu)}(K_0^*)} + \frac{\vartheta}{\sigma^{1/(n-\nu)}(K_1^*)} \right] \right\}^{(n-\nu)} d\bar{\omega} \right]^{1/(n-\nu)} . \end{aligned}$$

There is equality if and only if all the projections K_0^* and K_1^* are homothetic with the centre of magnification at Q . This condition is

sufficient for equality in (6); it is necessary and sufficient for (7).

We now use Minkowski's inequality (3) with $p = -1/n - \nu$. This yields

$$\begin{aligned} & W_\nu^{1/(n-\nu)}((1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1) \\ & \leq 1 \left/ \left[\frac{(1-\vartheta)}{\left(k \int \sigma(K_0^*) d\bar{\omega}\right)^{1/(n-\nu)}} + \frac{\vartheta}{\left(k \int \sigma(K_1^*) d\bar{\omega}\right)^{1/(n-\nu)}} \right] \right. \\ & = 1 \left/ \left[\frac{(1-\vartheta)}{W_\nu^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_\nu^{1/(n-\nu)}(K_1)} \right] \right. . \end{aligned}$$

The necessary and sufficient conditions for equality in (7) are sufficient for equality in (3) since $K_0 = \lambda K_1$ implies $\sigma(K_0^*) = \lambda^{n-\nu} \sigma(K_1^*)$. This establishes (2).

REFERENCES

1. T. Bonnesen and W. Fenchel, *Konvexe Körper*, Berlin, 1934, reprint N. Y. (1948), 48-50.
2. W. J. Firey, *Polar Means of Convex Bodies and a Dual to the Brunn-Minkowski theorem*. Canadian Math. J., **13** (1961), 444-453.
3. G. Hardy, J. Littlewood, and G. Pólya, *Inequalities*, Cambridge, (1934), 148.

WASHINGTON STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS
Stanford University
Stanford, California

F. H. BROWNELL
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

L. J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY

D. DERRY
M. OHTSUKA

H. L. ROYDEN
E. SPANIER

E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Reprinted 1966 in the United States of America

A. V. Balakrishnan, <i>Prediction theory for Markoff processes</i>	1171
Dallas O. Banks, <i>Upper bounds for the eigenvalues of some vibrating systems</i>	1183
A. Białyński-Birula, <i>On the field of rational functions of algebraic groups</i>	1205
Thomas Andrew Brown, <i>Simple paths on convex polyhedra</i>	1211
L. Carlitz, <i>Some congruences for the Bell polynomials</i>	1215
Paul Civin, <i>Extensions of homomorphisms</i>	1223
Paul Joseph Cohen and Milton Lees, <i>Asymptotic decay of solutions of differential inequalities</i>	1235
István Fáry, <i>Self-intersection of a sphere on a complex quadric</i>	1251
Walter Feit and John Griggs Thompson, <i>Groups which have a faithful representation of degree less than $(p - 1/2)$</i>	1257
William James Firey, <i>Mean cross-section measures of harmonic means of convex bodies</i>	1263
Avner Friedman, <i>The wave equation for differential forms</i>	1267
Bernard Russel Gelbaum and Jesus Gil De Lamadrid, <i>Bases of tensor products of Banach spaces</i>	1281
Ronald Kay Getoor, <i>Infinitely divisible probabilities on the hyperbolic plane</i>	1287
Basil Gordon, <i>Sequences in groups with distinct partial products</i>	1309
Magnus R. Hestenes, <i>Relative self-adjoint operators in Hilbert space</i>	1315
Fu Cheng Hsiang, <i>On a theorem of Fejér</i>	1359
John McCormick Irwin and Elbert A. Walker, <i>On N-high subgroups of Abelian groups</i>	1363
John McCormick Irwin, <i>High subgroups of Abelian torsion groups</i>	1375
R. E. Johnson, <i>Quotient rings of rings with zero singular ideal</i>	1385
David G. Kendall and John Leonard Mott, <i>The asymptotic distribution of the time-to-escape for comets strongly bound to the solar system</i>	1393
Kurt Kreith, <i>The spectrum of singular self-adjoint elliptic operators</i>	1401
Lionello Lombardi, <i>The semicontinuity of the most general integral of the calculus of variations in non-parametric form</i>	1407
Albert W. Marshall and Ingram Olkin, <i>Game theoretic proof that Chebyshev inequalities are sharp</i>	1421
Wallace Smith Martindale, III, <i>Primitive algebras with involution</i>	1431
William H. Mills, <i>Decomposition of holomorphs</i>	1443
James Donald Monk, <i>On the representation theory for cylindric algebras</i>	1447
Shu-Teh Chen Moy, <i>A note on generalizations of Shannon-McMillan theorem</i>	1459
Donald Earl Myers, <i>An imbedding space for Schwartz distributions</i>	1467
John R. Myhill, <i>Category methods in recursion theory</i>	1479
Paul Adrian Nickel, <i>On extremal properties for annular radial and circular slit mappings of bordered Riemann surfaces</i>	1487
Edward Scott O'Keefe, <i>Primal clusters of two-element algebras</i>	1505
Nelson Onuchic, <i>Applications of the topological method of Ważewski to certain problems of asymptotic behavior in ordinary differential equations</i>	1511
Peter Perkins, <i>A theorem on regular matrices</i>	1529
Clinton M. Petty, <i>Centroid surfaces</i>	1535
Charles Andrew Swanson, <i>Asymptotic estimates for limit circle problems</i>	1549
Robert James Thompson, <i>On essential absolute continuity</i>	1561
Harold H. Johnson, <i>Correction to "Terminating prolongation procedures"</i>	1571