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THE WAVE EQUATION FOR DIFFERENTIAL FORMS

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1. The Problem. Let M be a compact C^∞ Riemannian manifold of dimension N , having a positive definite metric. The operator $\Delta = d\delta + \delta d$ (see [13] for notation) maps p -forms ($0 \leq p \leq N$) into p -forms and it reduces, when $p = 0$, to minus the Laplace-Beltrami operator. Let $c(P)$ be a C^∞ function which is nonpositive for $P \in M$, and consider the Cauchy problem of solving the system

$$(1.1) \quad \left(L + \frac{\partial^2}{\partial t^2} \right) v \equiv \left(\Delta + c + \frac{\partial^2}{\partial t^2} \right) v = f(P, t)$$

$$(1.2) \quad v(P, 0) = g(P), \quad \frac{\partial}{\partial t} v(P, 0) = h(P),$$

where f, g, h are C^∞ forms of degree p . The main purpose of the present paper is to solve the system (1.1), (1.2) by the method of Fourier.

The Cauchy problem for second order self-adjoint hyperbolic equations was solved by Fourier's method by Ladyzhenskaya [8] and more recently (with some improvements) by V. A. Il'in [6]. In [8], other methods are also described, namely: finite differences, Laplace transforms, and analytic approximations using a priori inequalities. Higher order hyperbolic equations were treated by Petrowski [12], Leray [9] and Garding [5].

The Fourier method can be based on the fact that the series

$$(1.3) \quad \sum_{\lambda_n > 0} \frac{|\varphi_n(x)|^2}{\lambda_n^\alpha}, \quad \sum_{\lambda_n > 0} \frac{|\partial \varphi_n(x) / \partial x|^2}{\lambda_n^{\alpha+1}}, \quad \sum_{\lambda_n > 0} \frac{|\partial^2 \varphi_n(x) / \partial x^2|^2}{\lambda_n^{\alpha+2}}$$

are uniformly convergent. Here $\{\varphi_n\}$ and $\{\lambda_n\}$ are the sequences of eigenfunctions and eigenvalues of the elliptic operator appearing in the hyperbolic equation. In [6] the convergence of (1.3) is proved for $\alpha = [N/2] + 1$. Our proof of the analogous result for eigenforms is different from that of [6] and yields a better (and sharp) value for α , namely, $\alpha = N/2 + \varepsilon$ for any $\varepsilon > 0$. It is based on asymptotic formulas which we derive for $\sum_{\lambda_n \leq \lambda} |\partial^j \varphi_n(x) / \partial x^j|^2$ as $\lambda \rightarrow \infty$.

In §2 we recall various definitions and introduce the fundamental solution for $L + \partial/\partial t$ which was constructed by Gaffney [4] in the case $c(P) \equiv 0$. In §3 we derive some properties of the fundamental solution. These properties are used in §4 to derive the asymptotic formulas for $\sum_{\lambda_n \leq \lambda} |\partial^j \varphi_n(x) / \partial x^j|^2$, by which the convergence of the series in (1.3) for any $\alpha > N/2$ follows. In §5 we solve the problem (1.1), (1.2); first for f, g, h

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infinitely differentiable and then under much weaker differentiability assumptions with regard to M, c, f, g, h . In § 6 we briefly treat the Cauchy problem for the parabolic system

$$(1.4) \quad Lu + \frac{\partial u}{\partial t} = f(P, t)$$

$$(1.5) \quad u(P, 0) = g(P).$$

2. Preliminaries. The first one to use fundamental solutions of the heat equation in the study of the asymptotic distributions of eigenvalues and eigenfunctions was Minakshisundaram [11]. Gaffney [4] extended his method to derive asymptotic formulas for eigenvalues and eigenforms. We shall describe here some well known facts and some of the results of [4] which we will need later on. Slight modifications will be made due to the fact that in [4] $c \equiv 0$.

As is well known, there exists a sequence of eigenvalues $\{\lambda_n\}$ ($0 \leq \lambda_1 \leq \dots \leq \lambda_k \rightarrow \infty$ as $k \rightarrow \infty$) and a sequence of the corresponding eigenforms $\{\omega_n\}$ of degree p ($0 \leq p \leq N$, p is fixed throughout the paper) of L , that is, $L\omega_n = \lambda_n\omega_n$, such that the eigenforms form a complete orthonormal set in $L_p^2(M)$ (square integrable p -forms on M). The $\omega_i(p)$ are C^∞ forms. The fundamental solution $\theta(P, Q, t)$ of

$$(2.1) \quad \left(L + \frac{\partial}{\partial t}\right)\omega = 0$$

is a double p -form which is twice differentiable in Q , once differentiable in t , satisfies (2.1) in (Q, t) , $Q \in M$, $t > 0$, (for any fixed P) and, for any $P \in M$,

$$(2.2) \quad \lim_{t \rightarrow 0} \int_M \theta(P, Q, t) * \alpha(Q) = \alpha(P)$$

for any L^2 p -form α which is continuous at P . As in [4] one easily derives the expansion (provided θ is known to exist)

$$(2.3) \quad \theta(P, Q, t) = \sum_{i=1}^{\infty} \omega_i(P)\omega_i(Q)e^{-\lambda_i t}$$

where the series on the right is pointwise convergent for all $P, Q \in M$, $t > 0$ (that is, the series of each component is pointwise convergent).

A p -form α can be written locally as

$$\alpha = \sum_{i_1 < \dots < i_p} A_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p} = \Sigma' A_I dx^I$$

where Σ' indicates summation on $I = (i_1, \dots, i_p)$ with $i_1 < \dots < i_p$. The absolute value of α at P is given by

$$|\alpha(P)| = [\Sigma' A_I(x)A^I(x)]^{1/2}$$

where x is the local coordinate of P . Similarly, for a double p -form having local representation $\alpha(P, Q) = \Sigma' A_{IJ}(x, y)dx^I dy^J$ where y is the local coordinate of Q , we define the absolute value by

$$|\alpha(P, Q)| = [\Sigma'_{I,J} A_{IJ}(x, y)A^{IJ}(x, y)]^{1/2}.$$

The right "half-norm" is defined by

$$|\alpha|||(P) = \left[\int_M |\alpha(P, Q)|^2 dV_Q \right]^{1/2}.$$

Given two double p -forms α and β , a new double p -form is defined by

$$[\alpha, \beta] = [\alpha, \beta](P, Q) = \int_M \alpha(P, W) * \beta(Q, W).$$

One then verifies:

$$(2.4) \quad |[\alpha, \beta](P, Q)| \leq |\alpha|||(P) |\beta|||(Q).$$

The following inequalities are immediate:

$$(2.5) \quad |\alpha + \beta| \leq |\alpha| + |\beta|, |\alpha + \beta|| \leq |\alpha|| + |\beta||,$$

where α, β are any double p -forms.

In order to construct θ , one first constructs a parametrix. Gaffney [4] constructs a parametrix by generalizing the method of Minakshisandaram [11], making use of some calculation of Kodaira [7]. Given a point P , let $y = (y^i)$ be normal coordinates about P (with coordinates x^i). A p -form can be written as a vector X with $\binom{n}{p}$ components and then

$$(2.6) \quad \Delta X = -\Sigma g^{ij}\partial_i\partial_j X + \Sigma A^i\partial_i X + BX$$

where (g_{ij}) is the metric tensor, (g^{ij}) is the inverse matrix, $\partial_i = \partial/\partial x^i$, and A^i, B are matrices depending on the g_{ij} and their first two derivatives. If $X = f(r^2)W(x, y)$ where r is the geodesic distance from x to y (each component of X is now a vector so that W is a square matrix), then

$$(2.7) \quad \Delta_v[f(r^2)W] = f(r^2)\Delta_v W - f'(r^2)\left\{2N - 4K + 4r \frac{\partial}{\partial r}\right\}W - 4r^2 f''(r^2)W,$$

where $K = K(x, y)$ is a C^∞ matrix which vanishes for $y = x$.

There exists a C^∞ matrix M satisfying

$$(2.8) \quad r \frac{\partial}{\partial r} M = KM \quad (x \text{ fixed}), \quad M(x, x) = I$$

where I is the identity matrix. Using (2.8), (2.7) is simplified to

$$(2.9) \quad M^{-1}\Delta_y(fMW) = f(M^{-1}\Delta M)_y W - f' \left\{ 2N + 4r \frac{\partial}{\partial r} \right\} W - 4r^2 f'' W.$$

(2.9) will now be applied with

$$f(r^2, t) = \frac{1}{(4\pi t)^{N/2}} e^{-r^2/4t} \quad (t > 0 \text{ fixed}).$$

Setting

$$H_m = \sum_{j=0}^m f M U_j t^j, \quad U_0 = I$$

one then gets

$$\Delta H_\infty = f M \sum_{j=0}^\infty \left\{ (M^{-1}\Delta M) U_j t^j + \frac{1}{4t} \left(2N + 4r \frac{\partial}{\partial r} \right) U_j t^j - \frac{r^2}{4t^2} U_j t^j \right\}.$$

Calculating also $\partial H_\infty / \partial t$, one then obtains

$$\left(L_y + \frac{\partial}{\partial t} \right) H_\infty = f M \sum_{j=0}^\infty \left\{ (M^{-1}\Delta M + c) U_j + \left(r \frac{\partial}{\partial r} + j + 1 \right) U_{j+1} \right\} t^j$$

which leads to the successive definitions:

$$(2.10) \quad U_j = -\frac{1}{r^j} \int_0^r (M^{-1}\Delta M + c) U_{j-1} dr \quad (1 \leq j < \infty), \quad \text{where } U_0 = I.$$

We conclude that, for any $m \geq 0$,

$$(2.11) \quad \left(L_y + \frac{\partial}{\partial t} \right) H_m = \frac{1}{(4\pi)^{N/2}} e^{-r^2/4t} t^{m-N/2} L_y(MU_m).$$

H_m is a local parametrix. Note that when P, Q vary in a sufficiently small neighborhood V (contained in one coordinate patch), H_m is defined and is C^∞ in (P, Q, t) if $t > 0$. Let $\eta_\varepsilon(r)$ be a C^∞ function of r which is equal to 1 for $r < \varepsilon$ and is equal to 0 for $r > 2\varepsilon$. If ε is sufficiently small then the support of $\eta_\varepsilon(r)H_m(P, Q, t)$ (where r is the distance from P to Q) as a form in Q lies in V , provided $P \in W$, where W is a given open subset of V , $\bar{W} \subset V$. We can cover the manifold M by a finite number of sets W , call them W_i . Let the H_m corresponding to (the corresponding) V_i be denoted by H_m^i . If $\{\alpha_i\}$ is a C^∞ partition of unity subordinate to $\{W_i\}$, then the support of $\alpha_i(P)\eta_\varepsilon(r)H_m^i(P, Q, t)$ as a form of (P, Q) lies in $W_i \times V_i$ and hence this form is C^∞ in (P, Q, t) if $t > 0$.

The global parametrix is given by

$$(2.12) \quad \Theta_m(P, Q, t) = \sum \alpha_i(P)\eta_\varepsilon(r)H_m^i(P, Q, t).$$

The fundamental solution should then formally be

$$(2.13) \quad \theta(P, Q, t) = \theta_m(P, Q, t) + \int_0^t [\gamma_m(P, U, \tau), \theta_m(Q, U, t - \tau)] d\tau$$

where γ_m is defined by

$$(2.14) \quad \gamma_m(P, Q, t) = \sum_{i=1}^{\infty} (-1)^i \delta_m^i(P, Q, t)$$

$$(2.15) \quad \delta_m^i(P, Q, t) = \int_0^t [\delta_m^{i-1}(P, U, \tau), \delta_m^i(Q, U, t - \tau)] d\tau, \\ \delta_m^1 = \left(L_\nu + \frac{\partial}{\partial t} \right) \theta_m .$$

Using (2.4) and the inequality

$$(2.16) \quad \left| \int_0^t \alpha(P, Q, \tau) d\tau \right| \leq \binom{N}{p} \int_0^t |\alpha| d\tau ,$$

Gaffney establishes the uniform convergence of the right side of (2.14) and then proves that θ , as defined in (2.13), is a fundamental solution, for any $m \geq 0$, written in matrix form. We shall use the matrix notation of θ and the usual double form notation for θ interchangeably; the same for θ_m .

3. Properties of the fundamental solution. We denote by $\partial_p^h \theta(P, Q, t)$ an h th derivative of θ with respect to the coordinates of P , in a given coordinate system. If $h = (h_1, \dots, h_N)$, set $|h| = h_1 + \dots + h_N$. From the formulas defining θ it is clear that $\partial_p^h \theta(P, Q, t)$ exists and is continuous (in fact C^∞) in $P, Q \in M$ and $t > 0$. Let

$$(3.1) \quad \partial_p^h \theta(P, Q, t) \sim \sum_{i=1}^{\infty} B_i(P, t) \omega_i(Q)$$

be the Fourier expansion of $\partial_p^h \theta$, for (P, t) fixed. Then (recalling (2.3))

$$(3.2) \quad B_i(P, t) = \int_M \partial_p^h \theta(P, U, t) * \omega_i(U) = \partial_p^h \int_M \theta(P, U, t) * \omega_i(U) \\ = \partial^h \omega_i(P) e^{-\lambda_i t} ,$$

where ∂_p^h is abbreviated by ∂^h when there is no confusion.

By the (easily verified) Parseval's equality we get

$$(3.3) \quad \psi(P, Q, t) \equiv \left[\partial_p^h \theta \left(P, U, \frac{t}{2} \right), \partial_q^h \theta \left(Q, U, \frac{t}{2} \right) \right] \\ = \sum_{i=1}^{\infty} \partial_p^h \omega_i(P) \partial_q^h \omega_i(Q) e^{-\lambda_i t}$$

and the series is pointwise convergent for $P, Q \in M, t > 0$.

We need the following notations. Let α be a double p -form. If it is locally represented by $\Sigma' A_{I,r} dx^i dy^j$, then we set

$$[\alpha(P, P)] = \Sigma' A_i^i .$$

If β is also a double p -form, then we define $[[\alpha(P, U), \beta(P, U)]_\nu]$ to be $[\gamma(P, P)]$ where $\gamma(P, Q) = [\alpha(P, U), \beta(Q, U)]$.

Using (2.13) and the definition of ψ in (3.3) we have

$$\begin{aligned} (3.4) \quad & \sum_{i=1}^{\infty} |\partial^h \omega_i(P)|^2 e^{-\lambda_i t} = [\psi(P, P, t)] \\ & = \left[\left[\partial_P^h \theta_m \left(P, U, \frac{t}{2} \right), \partial_P^h \theta_m \left(P, U, \frac{t}{2} \right) \right]_\nu \right] \\ & \quad + 2 \left[\left[\int_0^{t/2} \left[\partial_P^h \gamma_m \left(P, W, \tau \right), \theta_m \left(U, W, \frac{t}{2} - \tau \right) \right] d\tau, \partial_P^h \theta_m \left(P, U, \frac{t}{2} \right) \right]_\nu \right] \\ & \quad + \left[\left[\int_0^{t/2} \left[\partial_P^h \gamma_m \left(P, W, \tau \right), \theta_m \left(U, W, \frac{t}{2} - \tau \right) \right] d\tau, \right. \right. \\ & \quad \quad \left. \left. \int_0^{t/2} \left[\partial_P^h \gamma_m \left(P, W, \tau \right), \theta_m \left(U, W, \frac{t}{2} - \tau \right) \right] d\tau \right]_\nu \right] \\ & \equiv J_1(P, t) + 2J_2(P, t) + J_3(P, t) . \end{aligned}$$

We proceed to estimate the J_i . We shall make use of the inequality [4]

$$(3.5) \quad [\alpha(P, P)] \leq \binom{N}{p} |\alpha(P, P)|^2 ,$$

and of the inequality [1]

$$\begin{aligned} (3.6) \quad & \int_0^t \int_{-\infty}^{\infty} \frac{\exp \{-\lambda |x - z|^2 / (t - \tau)\}}{(t - \tau)^\mu} \frac{\exp \{-\lambda |z - y|^2 / \tau\}}{\tau^\nu} dz d\tau \\ & \leq \text{const.} \frac{\exp \{-\lambda |x - y|^2 / t\}}{t^{\mu + \nu - 1 - N/2}} \end{aligned}$$

where $dz = dz^1 \dots dz^N$ and $\lambda > 0, \mu < N/2 + 1, \nu < N/2 + 1$. The following, easily verified, inequality will also be used:

$$\begin{aligned} (3.7) \quad & \int_{-\infty}^{\infty} \exp \{-\lambda |x - z|^2 / t\} \exp \{-\lambda |z - y|^2 / t\} dz \\ & \leq \text{const.} \exp \{-\mu |x - y|^2 / t\} t^{N/2} \end{aligned}$$

where $dz = dz^1 \dots dz^N$ and $\lambda > \mu > 0$. We shall denote by A_j constants which (unless otherwise stated) may depend only on h and on the manifold M .

Using (3.6) one can prove by induction on i that

$$(3.8) \quad |\partial_P^h \delta_m^i(P, U, t)| \leq \frac{A_i^{i+1}}{i!} t^{i(m+1-h/2)-1-N/2} e^{-r^2/5t} .$$

The case $i = 1$ follows by (2.11), (2.12). (In deriving (3.8) we also use the elementality inequality $\lambda e^{-\alpha\lambda} \leq \text{const. } e^{-\delta\lambda}$ for all $\lambda > 0$, where α, δ are constants and $\alpha > \delta \geq 0$.) In (3.8) it is understood that t° (if it occurs) must be replaced by $-\log t$. From now on we take m such that

$$m + 1 - \frac{|h|}{2} > 0 .$$

Using the definition (2.14) we then conclude from (3.8) that

$$(3.9) \quad |\partial_P^h \gamma_m(P, Q, t)| \leq A_2 e^{-r^2/5t} t^{m - (|h| + N)/2} .$$

Next, from the definition of Θ_m one derives

$$(3.10) \quad |\partial_P^h \Theta_m(P, Q, t)| \leq A_3 e^{-r^2/5t} t^{-(|h| + N)/2} .$$

Combining (3.9) and (3.10) ($h = 0$) and applying (3.6), we get

$$(3.11) \quad \left| \int_0^{t/2} \left[\partial_P^h \gamma_m(P, W, \tau), \Theta_m\left(U, W, \frac{t}{2} - \tau\right) \right] d\tau \right| \leq A_4 e^{-2r^2/5t} t^{m+1 - (|h| + N)/2} .$$

Using (3.10), (3.11) one easily derives, applying (3.7),

$$(3.12) \quad J_2(P, t) \leq A_5 t^{m+1 - |h| - N/2} .$$

Similarily one gets

$$(3.13) \quad J_3(P, t) \leq A_6 t^{2(m+1) - |h| - N/2} .$$

Evaluation of $J_1(P, t)$. From the construction of Θ_m it follows that for every sufficiently small neighborhood V we may take it to be of the form

$$(3.14) \quad \Theta_m(P, U, t) = H_m(P, U, t) + R_m(P, U, t) \quad \text{for all } P \in V$$

where H_m is constructed in § 2 and where, for some $\alpha' > 0$,

$$(3.15) \quad |\partial_P^h R_m(P, U, t)| \leq A_7 e^{-\alpha'/t} t^{|h| + N/2} \leq A_8 t^\zeta$$

for any $\zeta > 0$. A_8 depends also on ζ . Next,

$$(3.16) \quad \partial_P^h H_m(P, U, t) = \sum_{j=0}^m t^j \sum_{|\nu|=0}^{|h|} \binom{h}{\nu} \partial_P^\nu f \partial_P^{h-\nu}(MU_j)$$

where $\binom{h}{\nu} = \binom{h_1}{\nu_1} \cdots \binom{h_N}{\nu_N}$. It is easily seen that

$$(3.17) \quad \partial_P^\nu f(r^2, t) = \sum_{|\mu|=0}^{\nu_0} H_{\nu\mu} \left(\frac{y-x}{\sqrt{t}} \right) f(r^2, t) t^{|\nu|/2 + |\mu|/2}$$

where y^i, x^i are the coordinates of U, P respectively, and $H_{\nu\mu}(z)$ is a polynomial in $z = (z^1, \dots, z^N)$ with C^∞ coefficients which, for H_{ν_0} , are

functions of x only. Substituting (3.17) into (3.16) and recalling that $M(P, U)\nu_2$ becomes (δ_i^j) at $P = U$, we obtain

$$(3.18) \quad \partial_P^h H_m(P, U, t) = H_{h_0} \left(\frac{y-x}{\sqrt{t}} \right) f(r^2, t) t^{-|h|/2} Y + S_h(P, U, t)$$

where Y is the matrix (δ_i^j) and

$$(3.19) \quad |S_h(P, U, t)| \leq A_9 e^{-r^2/2t} t^{(1-|h|-N)/2}.$$

Combining (3.14), (3.15), (3.18), (3.19) we conclude that

$$(3.20) \quad \partial_P^h \Theta_m(P, U, t) = H_{h_0} \left(\frac{y-x}{\sqrt{t}} \right) f(r^2, t) t^{-|h|/2} Y + T_h(P, U, t)$$

and

$$|T_h(P, U, t)| \leq A_{10} t^{(1-|h|-N)/2}.$$

Using the definition of J_1 , and substituting (3.20) in the part of the integral $[\partial_P^h \Theta_m(P, U, t/2), \partial_P^h \Theta_m(P, U, t/2)]_{\sigma}$ taken over a coordinate patch V_0 containing \bar{V} : $y^i - x^i = \xi^i \sqrt{t}$, we find that

$$(3.21) \quad J_1(P, t) = (C_h(P) + B_0(P, t)) t^{-|h|-N/2}$$

where $C_h(P)$ is a continuous function of P , and $|B_0(P, t)| \leq A_{11} \sqrt{t}$ for $P \in V, 0 < t \leq b$, for any $b > 0$. A_{11} depends on b .

Combining the evaluation of J_1 with (3.12), (3.13), we obtain from (3.4),

$$(3.22) \quad \sum_{i=1}^{\infty} |\partial^h \omega_i(P)|^2 e^{-\lambda_i t} = C_h(P) t^{-|h|-N/2} + D_h(P, t) t^{-|h|-(N-1)/2}$$

where $D_h(P, t)$ is a uniformly continuous function of $(P, t), P \in V$ and $0 < t \leq b$ for any $b > 0$. Thus

$$(3.23) \quad |D_h(P, t)| \leq A_{12}$$

where A_{12} depends on b .

Note that the A_i , in particular A_{12} , are independent of P which varies in V .

4. Asymptotic formulas. To derive asymptotic formulas from the equation (3.22) we use a Tauberian theorem due to Karamata, specialized to Dirichlet series [14; p. 192]. It states:

Let $a_k \geq 0$ and $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$, and assume that the Dirichlet series $f(t) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k t}$ converges for $t > 0$ and satisfies

$$f(t) \sim \frac{A}{t^\gamma} \text{ as } t \searrow 0 \quad (\gamma \geq 0).$$

Then the function $\alpha(x) = \sum_{\lambda_k \leq x} a_k$ satisfies

$$\alpha(x) \sim \frac{Ax^\gamma}{\Gamma(\gamma + 1)} \text{ as } x \rightarrow \infty.$$

Applying it to (3.22) (using (3.23)), we get

$$(4.1) \quad \sum_{\lambda_i \leq \lambda} |\partial^h \omega_i(P)|^2 = \frac{C_h(P)}{\Gamma(|h| + 1 + N/2)} \lambda^{h_1 + N/2} [1 + o(1)] \quad (\lambda \rightarrow \infty)$$

and $o(1) \rightarrow 0$ as $\lambda \rightarrow \infty$, uniformly in $P \in V$.

Let $\lambda_1 = \dots = \lambda_{q-1} = 0, \lambda_q > 0$. Using the asymptotic formula (4.1) we shall prove:

THEOREM 1. For any h and for any $\varepsilon > 0$, the series

$$(4.2) \quad \sum_{i=q}^{\infty} \frac{|\partial^h \omega_i(P)|^2}{\lambda_i^{N/2 + |h| + \varepsilon}}$$

is uniformly convergent in $P \in M$.

Proof. We introduce the function

$$B(P, \lambda) \equiv \sum_{\lambda_q \leq \lambda_i \leq \lambda} |\partial^h \omega_i(P)|^2.$$

Then, we can write the series (4.2) in the form

$$\int_{\lambda'}^{\infty} \frac{dB(P, \lambda)}{\lambda^{N/2 + |h| + \varepsilon}} \text{ for any } 0 < \lambda' < \lambda_q.$$

Integrating by parts we get

$$(4.3) \quad \lim_{\mu \rightarrow \infty} \left[\frac{B(P, \lambda)}{\lambda^{N/2 + |h| + \varepsilon}} \right]_{\lambda=\lambda'}^{\lambda=\mu} - \left(\frac{N}{2} + |h| + \varepsilon \right) \int_{\lambda'}^{\infty} \frac{B(P, \lambda)}{\lambda^{N/2 + |h| + \varepsilon + 1}} d\lambda.$$

Since, by (4.1), $B(P, \lambda) \leq A_{13} \lambda^{h_1 + N/2}$ and since $B(P, \lambda') = 0$, the first term in (4.3) vanishes. The integral in (4.3) converges uniformly in P in view of the bound on $B(P, \lambda)$ just given. The proof of Theorem 1 is thereby completed.

5. Solution of the system (1.1), (1.2). We first derive the formal solution. Substituting

$$(5.1) \quad g(P) = \sum_{n=1}^{\infty} g_n \omega_n(P), h(P) = \sum_{n=1}^{\infty} h_n \omega_n(P), f(P, t) = \sum_{n=1}^{\infty} f_n(t) \omega_n(P)$$

$$(5.2) \quad v(P, t) = \sum_{n=1}^{\infty} v_n(t)\omega_n(P)$$

into (1.1), (1.2) we arrive at the equations

$$(5.3) \quad v_n''(t) + \lambda_n v_n(t) = f_n(t)$$

$$(5.4) \quad v_n(0) = g_n, v_n'(0) = h_n.$$

If $\lambda_n = 0$ the solution is

$$v_n(t) = g_n + h_n t + \int_0^t f(\tau)(t - \tau)d\tau.$$

If $\lambda_n > 0$ the solution is

$$v_n(t) = g_n \cos \sqrt{\lambda_n} t + \frac{h_n}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} t + \frac{1}{\sqrt{\lambda_n}} \int_0^t f_n(\tau) \sin \sqrt{\lambda_n} (t - \tau)d\tau.$$

Hence, the formal solution of (1.1), (1.2) is

$$(5.5) \quad v(P, t) = \sum_{n=1}^{\infty} g_n \omega_n(P) \cos \sqrt{\lambda_n} t + \sum_{n=1}^{q-1} h_n \omega_n(P) t + \sum_{n=q}^{\infty} \frac{h_n}{\sqrt{\lambda_n}} \omega_n(P) \sin \sqrt{\lambda_n} t + \sum_{n=1}^{q-1} \omega_n(P) \int_0^t f_n(\tau)(t - \tau)d\tau + \sum_{n=q}^{\infty} \frac{1}{\sqrt{\lambda_n}} \omega_n(P) \int_0^t f_n(\tau) \sin \sqrt{\lambda_n} (t - \tau)d\tau.$$

To prove that the formal solution is a genuine one we observe that if $\lambda_n > 0$

$$(5.6) \quad g_n = \int_M g(Q) * \omega_n(Q) = \frac{1}{\lambda_n^m} \int_M L^m g(Q) * \omega_n(Q)$$

for any positive integer m . Applying Bessel's inequality, we get

$$(5.7) \quad \sum_{n=1}^{\infty} \lambda_n^{2m} g_n^2 \leq \int_M L^m g(Q) * L^m g(Q) = \|L^m g\|^2.$$

Similarly,

$$(5.8) \quad \sum_{n=1}^{\infty} \lambda_n^{2m} h_n^2 \leq \|L^m h\|^2, \sum_{n=1}^{\infty} \lambda_n^{2m} (f_n(t))^2 \leq \|L^m f(\cdot, t)\|^2.$$

It will be enough to show that the part of the first series on the right side of (5.5), where summation is on $\lambda_n > 0$, when differentiated term-by-term twice with respect to P is uniformly convergent in $P \in M$, $0 \leq t \leq b$, for any $b > 0$. Now the series obtained is majorized by

$$\Sigma |g_n| |\partial^2 \omega_n(P)| \leq \Sigma \lambda_n^k |g_n| \frac{|\partial^2 \omega_n(P)|}{\lambda_n^k} \leq \Sigma \lambda_n^{2k} g_n^2 \Sigma \frac{|\partial^2 \omega_n(P)|^2}{\lambda_n^{2k}}.$$

Hence that series is uniformly convergent if $k > N/2 + 1$.

It is clear that each series in (5.5) can actually be differentiated term-by-term any number of times and the resulting series is uniformly convergent.

By a solution of (1.1), (1.2) we mean a p -form which is (a) twice continuously differentiable in (P, t) for $P \in M, t > 0$ (b) once continuously differentiable in t for $P \in M, t \geq 0$ and (c) satisfies (1.1), (1.2).

The uniqueness of the solution can be proved as for the classical wave equation. Assuming $g \equiv 0, h \equiv 0, f \equiv 0$ and using the rule $\int du^* \omega = \int u^* \delta \omega$ one finds that if u is a solution then

$$\frac{\partial}{\partial t} \int_M [u_i^* u_i + \delta u^* \delta u + du^* du - cu^* u] = 0.$$

Since the integral vanishes for $t = 0$, it vanishes for all $t > 0$. Since the integrand is nonnegative, $u_i^* u_i \equiv 0$, which implies $u_i \equiv 0$ and hence, $u \equiv 0$.

We have thus completed the proof of the following theorem.

THEOREM 2. *Let g, h be C^∞ p -forms and let f be a C^∞ p -form such that $\partial_p^\lambda f$ is continuous in (P, t) , for any λ . Then the Cauchy problem (1.1), (1.2) has one and only one solution. The solution is a C^∞ p -form and is given by (5.5).*

The assumption that the manifold M is C^∞ can be weakened. Indeed, the theory of differential forms used above remains valid under the assumption that the metric tensor is C^5 (Gaffney [3]; see also Friedrichs [2]). The assumptions on f, g, h can also be weakened without any modification of the preceding proof of Theorem 2.

We need the assumptions:

- (A) The metric tensor g_{ij} belongs to $C^{[N/2]+2}$ and to C^5 , and c belongs to $C^{[N/2]+1}$ (recall that $c \leq 0$).
- (B) The form g belongs to $C^{[N/2]+3}$ and $L^{[(N+4)/4]}g$ belongs to C^1 .
- (C) The form h belongs to $C^{[N/2]+2}$ and $L^{[(N+2)/2]}h$ belongs to C^1 .
- (D) The form f and its first $[N/2] + 2$ p -derivatives are continuous for $P \in M, 0 \leq t \leq b$ (for any $b > 0$); $L^{[(N+2)/2]}f$ and its first p -derivatives are continuous for $P \in M, 0 \leq t \leq b$.

THEOREM 2'. *Under the assumptions (A) – (D), there exists one and only one solution of the Cauchy problem (1.1), (1.2). It is given by (5.5).*

The assertion of Theorem 2' remains valid if we further weaken the assumptions (A) – (D) by replacing the classes of continuous deriva-

tives C^q by classes of "strong" derivatives W_2^q (see [6]), assuming that $g_{ij} \in C^5$.

6. The heat equation. The method of § 5 can easily be extended to solve the system (1.4), (1.5). The formal solution is

$$(6.1) \quad u(P, t) = \sum_{n=1}^{\infty} g_n \omega_n(P) e^{-\lambda_n t} + \sum_{n=1}^{\infty} \omega_n(P) \int_0^t f_n(\tau) e^{-\lambda_n(t-\tau)} d\tau.$$

We shall need the assumptions:

(A') g_{ij} belong to $C^{[N/2]+1}$ and to C^5 , and e belongs to $C^{[N/2]}$.

(B') The form g belongs to $C^{[N/2]+1}$ and $L^{[N/4]}g$ belongs to C^1 .

THEOREM 3. *Under the assumption (A'), (B'), (D) there exists a unique solution of the system (1.4), (1.5). It is given by (6.1).*

REMARK 1. The assumption $c \leq 0$ is not needed for the validity of Theorem 3 since it can be achieved by a transformation $u = e^{\alpha t} v$ for any constant $\alpha \geq c$.

REMARK 2. Assuming $c \leq 0$, $f \equiv 0$, we can rewrite (6.1) as an operator equation

$$(6.2) \quad T_t = H + \sum_{k=1}^{\infty} e^{-\mu_k t} H_k$$

where $\{\mu_k\}$ is the sequence $\{\lambda_j\}$ taken without multiplicities, H_k is the projection into the space of eigenforms corresponding to μ_k , H corresponds to $\mu_0 = 0$, and T_t is the operator which maps g into the solution u , that is, $u(P, t) = T_t g(P)$. Formula (6.2) was derived, in a different way (for $c \equiv 0$) by Milgram and Rosenbloom [10].

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