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**1. Introduction.** Let $A$ be a closed operator from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{H}'$. The main purpose of this present paper is to develop a spectral theory for an operator $A$ of this type. This theory is analogous to the given in the self-adjoint case and reduces to the standard theory when $A$ is self-adjoint. The spectral theory here given is based on generalization of the concept of self-adjointness. Let $A^*$ denote the adjoint of $A$. An operator $T$ on $\mathfrak{H}$ to $\mathfrak{H}'$ will be said to be an elementary operator if $TT^*T = T$. If $T$ is elementary, the operator $TA^*T$ can be considered to be an adjoint of $A$ relative to $T$. If $A = TA^*T$, then $A$ will be said to be self-adjoint relative to $T$. The polar decomposition theorem for $A$ implies the existence of a unique elementary operator $R$ relative to which $A$ is self-adjoint and having the further property that $R$ has the same null space as $A$ and that $A^*R$ is a nonnegative self-adjoint operator in the usual sense. Every elementary operator $T$ relative to which $A$ is self-adjoint is of the form $T = T_0 + R_1 - R_2$, where $R = R_1 + R_2$ and $T_0, R_1, R_2$ are $*$-orthogonal. Two operators $B$ and $C$ are said to be $*$-orthogonal if $B^*C = 0$ and $BC^* = 0$ on dense sets in $\mathfrak{H}$ and $\mathfrak{H}'$ respectively.

An operator $B$ will be called a section of an operator $A$ if there is an operator $C$ $*$-orthogonal to $B$ such that $A = B + C$. If $R$ is the elementary operator associated with $A$, there exists a one parameter family $A_\lambda, R_\lambda$ $(0 < \lambda < \infty)$ of sections of $A, R$ respectively such that $R_\lambda$ is the elementary operator belonging to $A_\lambda$, $\| A_\lambda \| \leqq \lambda$, $A_\mu$ $(\mu < \lambda)$ is a section of $A_\lambda$ and $A = \int_0^\infty \lambda dR_\lambda$. From this result it is seen that $A$ possesses a spectral decomposition relative to any elementary operator $T$ relative to which $A$ is self-adjoint. These results can be extended to the case in which $A$ is normal relative to T. When $\mathfrak{H}' = \mathfrak{H}$ and $T$ is the identity, these results give the usual spectral theory for self-adjoint operators. Examples are given in §§ 4 and 10 below. In particular spectral resolutions are given for the gradient of a function and its adjoint, the divergence of a vector. The finite dimensional case has been treated in a recent paper by the author[1].

The results given below are elementary in nature and are based

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[1] M. R. Hestenes, Relatively hermitian matrices, to be published in the Pacific Journal of Mathematics.

upon the fundamental ideas concerning Hilbert spaces. These ideas can be found in the standard treatises on Hilbert space. The concept of $*$-commutativity is introduced. This concept is used in the development of the spectral theory. It is shown that a reciprocally compact operator has a discrete principal spectrum. The concept of reciprocal compactness is connected with the concept of ellipticity of differential operators, as is indicated in the last section below.

**2. Preliminaries.** Let $\mathfrak{H}$ and $\mathfrak{H}'$ be two Hilbert Spaces over a scalar field $\mathfrak{C}$. The field $\mathfrak{C}$ will be taken to be either the field of real numbers or the field of complex numbers. The two case can be treated simultaneously by defining the conjugate $\bar{b}$ of $b$ to be $b$ itself in the field of reals. The spaces $\mathfrak{H}$ and $\mathfrak{H}'$ may coincide. The same notations will be used for the inner product in each of the two spaces. Thus, the symbol $(x_1, x_2)$ denotes the *inner product* of $x_1$ and $x_2$, whether $x_1$ and $x_2$ are in $\mathfrak{H}$ or in $\mathfrak{H}'$. The *norm* of $x$ will be denoted by $\|x\|$. *Strong convergence* of a sequence $\{x_n\}$ to $x_0$ will be denoted by $x_n \Rightarrow x_0$ and *weak convergence* by $x_n \rightarrow x_0$.

The *closure* of a subclass $\mathscr{B}$ of $\mathfrak{H}$ will be denoted by $\overline{\mathscr{B}}$ and its *orthogonal complement* in $\mathfrak{H}$ by $\mathscr{B}^\perp$. Clearly $\mathscr{B}^\perp$ is a subspace of $\mathfrak{H}$. By the sum $\mathfrak{A} + \mathscr{B}$ of two linear subclasses $\mathfrak{A}$ and $\mathscr{B}$ will be meant the class of all elements of the form $x + y$ with $x$ in $\mathfrak{A}$ and $y$ in $\mathscr{B}$. It will be called a *direct sum* if $\mathfrak{A}$ and $\mathscr{B}$ have no nonnull elements in common.

A linear tranformation $A$ will be said to be *from* $\mathfrak{H}$ to $\mathfrak{H}'$ if its *domain* $\mathfrak{D}_A$ is in $\mathfrak{H}$ and its *range* $\mathscr{R}_A$ is in $\mathfrak{H}'$. If $\mathscr{D}_A = \mathfrak{H}$ the phrase "on $\mathfrak{H}$ to $\mathfrak{H}'$" will be used to emphasize this fact. The phrase "$A$ in $\mathfrak{H}$" will be used occassionally in case $\mathfrak{H}' = \mathfrak{H}$. A linear transformation $B$ from $\mathfrak{H}$ to $\mathfrak{H}'$ will be called an *extension* of $A$, written $A \leqq B$ or $B \geqq A$, in case $\mathscr{D}_B \supset \mathscr{D}_A$ and $B = A$ on $\mathscr{D}_A$. If $\overline{\mathscr{D}_A} = \mathfrak{H}$, then $A$ will be said to be *dense* in $\mathfrak{H}$. The transformation $A$ will be said to be *bounded* if it maps bounded subsets of $\mathscr{D}_A$ into bounded sets of $\mathfrak{H}'$. If $A$ is bounded, its *norm* $\|A\|$ is defined to be the least upper bound of $\|Ax\|$ for all $x$ in $\mathscr{D}_A$ having $\|x\| = 1$. If whenever $x_n \in D_A$, $x_n \Rightarrow x_0$, $Ax_n \Rightarrow y_0$ we also have $x_0 \in \mathscr{D}_A$ and $Ax_0 = y_0$, then $A$ will be said to be *closed*. If whenever $x_0, x_n \in \mathscr{D}_A$ and $x \Rightarrow x_0$, $Ax_n \Rightarrow y_0$ we have $Ax_0 = y_0$ then $A$ is said to be *preclosed*. A closed dense linear transformation is bounded if and only if $\mathscr{D}_A = \mathfrak{H}$. The minimal closed extension of $A$, if it exists, will be called the *closure* of $A$ and will be denoted by $\bar{A}$. If $A$ is preclosed, its closure exists. By the *null class* $\mathfrak{N}_A$ of $A$ will be meant all $x$ in $\mathscr{D}_A$ such that $Ax = 0$. There is a unique extension of $A$ whose domain is $\mathscr{D}_A + \overline{\mathfrak{N}}_A$ and whose null space is $\overline{\mathfrak{N}}_A$. If $A$ is closed then $\mathfrak{N}_A$ is closed.

Consider now a dense linear transformation $A$ from $\mathfrak{H}$ to $\mathfrak{H}'$ and let

$\mathscr{D}_A{}^*$ be the class of all vectors $y$ in $\mathfrak{H}'$ for which there exists a vector $A^*y$ in $\mathfrak{H}$ such that the relation.

$$(Ax, y) = (x, A^*y)$$

holds for all $x$ in $\mathscr{D}_A$. The transformation $A^*$ from $\mathfrak{H}'$ to $\mathfrak{H}$ so defined is a closed linear transformation whose domain is $\mathscr{D}_A{}^*$, whose null class $\mathfrak{N}_A{}^*$ is $\mathscr{R}_A^\perp$ and whose range $\mathscr{R}_A{}^*$ is a subclass of $\mathfrak{N}_A^\perp$.

A linear transformation A from $\mathfrak{H}$ to $\mathfrak{H}$ will be said to be *self-adjoint* if it is dense and if $A^* = A$. A self-adjoint linear transformation $A$ will be said to be *nonnegative*, written $A \geqq 0$, if the inequality $(Ax, x) \geqq 0$ holds for all $x$ in $\mathscr{D}_A$. By a *projection* $E$ in $\mathfrak{H}$ will be meant a self-adjoint operator such that $E^2 = E$.

It will be convenient to use the term "*operator*" to denote a closed dense linear transformation. We shall have occassion to use the following well known result.

THEOREM 2.1. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$. Then its adjoint $A^*$ is an operator from $\mathfrak{H}'$ to $\mathfrak{H}$. Moreover, $A^{**} = A$, $\mathfrak{N}_A{}^* = \mathscr{R}_A^\perp$, $\mathfrak{N}_A = \mathscr{R}_A^{\perp *}$. For each vector $x_0$ in $\mathfrak{H}$ and $y_0$ in $\mathfrak{H}'$ there is a unique vector $x$ in $\mathscr{D}_A$ and $y$ in $\mathscr{D}_A{}^*$ such that*

$$x_0 = x + A^*y\ , \quad y_0 = Ax - y\ . \tag{2.1}$$

*The transformation $A^*A$ is a nonnegative self-adjoint operator in $\mathfrak{H}$ whose null space is $\mathfrak{N}_A$. Similarly $AA^*$ is a nonnegative self-adjoint operator in $\mathfrak{H}'$ whose null space is $\mathfrak{N}_A{}^*$. The operator $A$ is bounded if and only if $A^*$ is bounded. In this event $\| A \| = \| A^* \|$.*

3. **The reciprocal and *-reciprocal of a closed operator.** Consider a linear transformation $A$ from $\mathfrak{H}$ to $\mathfrak{H}'$ whose domain $\mathfrak{D}_A$ is expressible as a direct sum $\mathscr{D}_A = \mathscr{C}_A + \mathfrak{N}_A$, where $\mathfrak{N}_A$ is the null space of $A$ and $\mathscr{C}_A$ is orthogonal to $\mathfrak{H}_A$. The class $\mathscr{C}_A$ will be called the *carrier* of $A$. If $\mathfrak{N}_A$ is closed, then $\mathscr{D}_A$ has such a representation. Consequently, the carrier of a closed linear transformation is well defined.

The transformation $A$ establishes a one-to-one correspondence between its carrier and its range. The inverse transformation on $\mathscr{R}_A$ onto $\mathscr{C}_A$, when extended linearly so as to have $\mathscr{R}_A^\perp$ as its null space and $\mathscr{R}_A + \mathscr{R}_A^\perp$ as its domain, defines a linear transformation $A^{-1}$ which will be called the *reciprocal*[2] of $A$. The carrier of $A^{-1}$ is the range of $A$ and the range of $A^{-1}$ is the carrier of $A$. It is clear that $A^{-1}$ is dense in $\mathfrak{H}'$ and that $\mathfrak{N}_{A-1}$ is closed.

The reciprocal of $A^{-1}$ for an arbitrary linear transformation $A$ will

[2] See E. H. Moore General analysis I, Memoirs, American Philosophical Society (1935). See also, J. von Neumann, *On regular rings*, Proc. Nat. Acad. Sci, **22**, (1936), 707-715.

be defined to be the reciprocal of the extension $A_0$ of $A$ whose domain is $\mathscr{D}_A + \overline{\mathfrak{N}}_A$ and whose null space is $\overline{\mathfrak{H}}_A$. The carrier of $A_0$ will be defined to be the carrier of $A$. The reciprocal of $A^{-1}$ is accordingly the extension of $A$ whose domain is $\mathscr{D}_A + \overline{\mathfrak{N}}_A + \mathscr{D}_A^{\perp}$ and whose null space is $\overline{\mathfrak{N}}_A + \mathscr{D}_A^{\perp}$. Hence $A$ is the reciprocal of $A^{-1}$ if and only if $A$ is dense in $\mathfrak{H}$ and its null space is closed. If $\mathfrak{N}_A$ is closed, then $A$ is closed if and only if $A^{-1}$ is closed. If $A$ possesses an inverse, then $A^{-1}$ is the inverse of $A$.

THEOREM 3.1. *The adjoint of the reciprocal of an operator $A$ is the reciprocal of its adjoint, that is, $(A^{-1})^* = (A^*)^{-1}$. The operators $A$ and $A^{-1*}$ have the same null spaces.*

Clearly $\mathfrak{N}_{A^{-1*}} = \mathscr{R}_{A^{-1}}^{\perp} = \mathfrak{N}_A$. Let $x$ be a vector in $\mathscr{C}_{A^{-1*}}$. Then $(A^{-1}y_0, x) = (y_0, A^{-1*}x)$ for every $y_0$ in $\mathscr{C}_{A^{-1}} = \mathscr{R}_A$. Hence $(x_0, x) = (Ax_0, A^{-1*}x)$ if $x_0 \in \mathscr{C}_A$ and hence if $x_0 \in \mathscr{D}_A$. It follows that $A^{-1*}x$ is in $\mathscr{D}_A$ and that $x = A^*A^{-1}x$. Consequently $\mathscr{C}_{A^{-1*}} \subset \mathscr{R}_{A^*} = \mathscr{C}_{A^{*-1}}$. Conversely if $x \in \mathscr{C}_{A^{*-1}}$, then $(x_0, x) = (Ax_0, A^{*-1}x)$ holds for all $x_0$ in $\mathscr{C}_A$ or equivalently $(A^{-1}y_0, x) = (y_0, A^{*-1}x)$ holds for all $y_0$ in $\mathscr{C}_{A^{-1*}}$. It follows that $x$ is in $\mathscr{C}_{A^{-1*}}$ and that $A^{-1*}x = A^{*-1}x$. It follows that $A^{-1*}$, $A^{*-1}$ coincide on their carriers, as well as their null spaces and hence are identical.

The element $A^{*-1}$ plays an important role in the results given below and will be called the *$*$-reciprocal* of $A$.

As an immediate consequence of the last theorem we have

THEOREM 3.2. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$. Then $A^{-1}$, $A^*$, $A^{*-1} = A^{-1*}$ are operators. The products $A^*A$, $A^{-1}A^{*-1}$ are nonnegative self-adjoint operators, are reciprocals of each other and have the same null space as $A$. Similarly, the products $AA^*$, $A^{*-1}A^{-1}$ are nonnegative self-adjoint operators, are reciprocals of each other and have the same null space as $A^*$.*

A linear transformation A will be said to be *reciprocally bounded* if its reciprocal is bounded, or, equivalently if there is a positive number $m > 0$ such that $\| Ax \| \geqq m \| x \|$ on the carrier of $A$. The following theorem is self-evident.

THEOREM 3.3. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$. Then $A$ is reciprocally bounded if and only if its range is closed. Hence $A$ is reciprocally bounded if and only if the equation $Ax = y$ has a solution $x$ in $\mathscr{D}_A$ whenever $y$ is orthogonal to every solution $z$ of $A^*z = 0$. The operator $A$ is reciprocally bounded if and only if $A^*$ is reciprocally bounded. Finally, $A$ is reciprocally bounded if and only if $A^*A$ (or $AA^*$) is reciprocally bounded.*

The concept of reciprocal boundedness is the basis for a large class of existence theorems for ordinary and partial differential equations. In view of the last conclusion in the theorem existence theorems for non-self-adjoint problems follows from those for self-adjoint problems.

THEOREM 3.4. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$. If $\alpha$ and $\beta$ are positive numbers, then*

$$\alpha A + \beta A^{*-1} , \quad \alpha A^* + \beta A^{-1}$$

*are reciprocally bounded operators and are adjoints of each other.*

In order to prove the theorem it is sufficient to consider a transformation of the form $B = \lambda A + (1/\lambda)A^{*-1}$, where $\lambda$ is a positive number. Let $y_0$ be a vector in $\overline{\mathscr{R}_A}$. By Theorem 2:1, with $A$ replaced by $\lambda A$, there is a unique vector $x$ in $\mathscr{D}_A$ and $y$ in $\mathscr{D}_{A^*}$ such that

$$0 = x + \lambda A^* y , \quad y_0 = \lambda A x - y .$$

The vector $y$ is therefore in $\overline{\mathscr{R}_A} = \mathfrak{N}_{A^*}^{\perp}$ and in the carrier of $A^*$. Consequently, $y = (1/\lambda)A^{*-1}x$ and

$$y_0 = \left(\lambda A + \frac{1}{\lambda} A^{*-1}\right)x = Bx .$$

The range of $B$ is therefore closed. It follows that $B$ is reciprocally bounded and closed. Similarly $C = \lambda A^* + (1/\lambda)A^{-1}$ is reciprocally bounded and closed. Clearly $C = B^*$. This proves the theorem.

COROLLARY 1. *If $A$ is self-adjoint operator, and $\alpha$, $\beta$ are positive numbers, then $\alpha A + \beta A^{-1}$ is a reciprocally bounded self-adjoint operator, Moreover the reciprocal of $A^2$ is $A^{-2} = (A^{-1})^2$.*

THEOREM 3.5. *Let $C = BA$, $D = A^{-1}B^{-1}$, where $A$ is an operator from $\mathfrak{H}$ to $\mathfrak{H}'$ and $B$ is an operator from $\mathfrak{H}'$ to a Hilbert space $\mathfrak{H}''$. Suppose that $\mathfrak{N}_{A^*} = \mathfrak{N}_B$. Then $\mathfrak{N}_C = \mathfrak{N}_A$, $\mathfrak{N}_D = \mathfrak{N}_{B^*}$. If $D$ is dense, then $D = C^{-1}$. If either $A$ or $B^{-1}$ is bounded then $C$ and $D$ are closed.*

Suppose that $x_n \in \mathscr{D}_C$, $x_n \Rightarrow x_0$, $Cx_n \Rightarrow z_0$. Set $y_n = Ax_n$, $z_n = By_n = Cx_n$. If $A$ is bounded, then $y_n \Rightarrow Ax_0$. Since $By_n = Cx_n \Rightarrow z_0$ it follows that $z_0 = BAx_0 = Cx_0$. Consequently $C$ is closed. Observe that this conclusion is valid even if $\mathfrak{N}_{A^*} \neq \mathfrak{N}_B$. Since $y_n \in \mathscr{C}_B$ we have $y_n = Ax_n = B^{-1}z_n$. If $B^{-1}$ is bounded, then $y_n = Ax_n \Rightarrow B^{-1}z_0$. Hence $B^{-1}z_0 = Ax_0$, that is $z_0 = BAx_0 = Cx_0$. Consequently $C$ is closed in this event also. The remaining statements in the theorem are readily verified.

COROLLARY. *If $A$ is bounded and reciprocally bounded and $\mathfrak{N}_{A^*} \supset \mathfrak{N}_B$, then the products $C$ and $D$ described in Theorem 3.5 are operators and are reciprocals of each other.*

This follows readily from Theorem 3.5 because we can replace $A$ by $F'A$ where $F'$ is the projection in $\mathfrak{H}'$ whose null class is $\mathfrak{N}_B$. We then have $\mathfrak{N}_{A^*} = \mathfrak{N}_B$.

4. **Examples.** The results here given were motivated in part by certain applications to differential equations. It will be convenient to explain in part two of these applications at this time.

EXAMPLE 1. Let $\mathfrak{H}$ be the class of all real valued Lebesgue square integrable functions $x$ in the interval $0 \leqq t \leqq \pi$. This class with

$$(x, y) = \int_0^\pi x(t)y(t)dt$$

as its inner product and the real numbers as scalars from a Hilbert space. Let $\mathfrak{A}$ be the class of all absolutely continuous functions $x(t)$ $(0 \leqq t \cong \pi)$ whose derivatives $\dot{x}$ are in $\mathfrak{H}$. Let $A$ be the differential operator $d/dt$ having as its domain the class $\mathscr{D}_A$ of all functions in $\mathfrak{A}$ having $x(0) = x(\pi) = 0$. The carrier of $A$ is $\mathscr{D}_A$ itself. Then range $\mathscr{R}_A$ consists of all functions $y$ in $\mathfrak{H}$ satisfying the condition

$$\int_0^\pi y(t)dt = 0 \; . \tag{4:1}$$

Since $\mathscr{R}_A$ is closed it follows that $A$ is reciprocally bounded. The reciprocal of $A$ is

$$A^{-1}y = \int_0^t y(s)ds - \frac{t}{\pi}\int_0^\pi y(s)ds \; .$$

The adjoint $A^*$ of $A$ is the operator $-d/dt$ with $\mathscr{D}_{A^*} = \mathfrak{A}$ as its domain and $\mathfrak{A} \cap \mathscr{R}_A$ as its carrier. Since $A$ is reciprocally bounded so also $A^*$. Moreover $\mathscr{R}_{A^*} = \mathfrak{H}$. The reciprocal of $A^*$ is

$$A^{*-1}x = -\int_0^t x(s)ds + \frac{1}{\pi}\int_0^\pi\int_0^r x(s)dsdr$$

by virtue of the relation (4.1). Let $\mathscr{B}$ be all functions in $\mathfrak{A}$ whose derivatives are also in $\mathfrak{A}$. The operator $A^*A$ is the operator $-d^2/dt^2$ having as its domain all functions in $\mathscr{B}$ such that $x(0) = x(\pi) = 0$. The range of $A^*A$ is $\mathfrak{H}$. The operator $AA^*$ is the operator $-d^2/dt^2$ having as its domain all functions $x$ in $\mathscr{B}$ whose derivative $\dot{x}$ satisfies the conditions $\dot{x}(0) = \dot{x}(\pi) = 0$. The range of $AA^*$ coincides with that of $A$.

The operator $AA^*$ is also reciprocally bounded.

A preview of the theory to be presented below can be given for this example by recalling certain known facts. Let

$$x_n(t) = \sqrt{\frac{2}{\pi}}\sin nt\ , \quad y_0(t) = \sqrt{\frac{1}{\pi}}\ , \ y_n(t) = \sqrt{\frac{2}{\pi}}\cos nt \quad (n = 1, 2, 3, \cdots)\ .$$

The function $x_n$ form a complete orthonormal system in $\mathfrak{H}$. A function $x$ in $\mathfrak{H}$ is accordingly given by the fourier sine series.

$$x = \sum_{n=1}^{\infty} a_n x_n\ , \quad a_n = (x, x_n)$$

where convergence is taken to be convergence in the mean of order 2. Similarly a function $y$ in $\mathfrak{H}$ is expressible in the form

$$y = b_0 y_0 + \sum_{n=1}^{\infty} b_n y_n\ , \qquad b_j = (y, y_j)\ (j = 1, 2, \cdots)\ .$$

If $x$ and $y$ are in the appropriate domains we have

$$\begin{aligned} Ax &= \sum_{n=1}^{\infty} n a_n y_n\ , \quad A^{-1}y = \sum_{1=n}^{\infty} \frac{1}{n} b_n x_n\ , \\ A^{*-1}x &= \sum_{n=1}^{\infty} \frac{1}{n} a_n y_n\ , \quad A^* y = \sum_{n=1}^{\infty} n b_n x_n\ , \end{aligned} \tag{4.2}$$

as one readily verifies. These formulas can be put in another form by defining the operators $R$ and $R_i$ $(i = 1, 2, 3, \cdots)$ by the formulas

$$Rx = \sum_{n=1}^{\infty} a_n y_n\ , \quad R_i x = a_i y_i \qquad (i = 1, 2, 3, \cdots)\ .$$

Observe that

$$R^* y = \sum_{n=1}^{\infty} b_n x_n\ , \quad R_i^* y = b_i x_i \qquad (i = 1, 2, 3, \cdots)\ .$$

The operator $R$ maps $\mathfrak{H}$ isometrically onto $\mathscr{R}_A$. Its adjoint $R^*$ maps $\mathscr{R}_A$ isometrically onto $\mathfrak{H}$ and annihilates $\mathscr{R}_A^{\perp}$. We have the relations

$$\begin{aligned} &R = \sum_{n=1}^{\infty} R_n\ , \quad R^* R_n = R_n^* R_n\ , \quad RR_n^* = R_n R_n^* \\ &R_i^* R_j = 0\ , \quad R_i R_j^* = 0 \quad (i \neq j)\ . \end{aligned} \tag{4.3}$$

Moreover, by (4.2) we have

$$\begin{aligned} A &= \sum_{n=1}^{\infty} n R_n\ , \quad A^{-1} = \sum_{n=1}^{\infty} \frac{1}{n} R_n^* \\ A^{*-1} &= \sum_{n=1}^{\infty} \frac{1}{n} R_n\ , \quad A^* = \sum_{n=1}^{\infty} n R_n^*\ . \end{aligned} \tag{4.4}$$

These formulas constitute a spectral resolution of $A, A^{-1}, A^{*-1}, A^*$. It is our purpose to show that every operator $A$ can be resolved in terms of elementary operators having the properties similar to those given in (4.3).

The example just given can be modified so as to include all complex valued functions in $\mathfrak{H}$ and so that $A = i(d/dt)$. Then $A$ is a symmetric operator but is not self-adjoint. The theory for this case is not significantly different from that just described.

EXAMPLE 2. Let $\mathfrak{H}$ be the class of all real valued Lebesgue square integrable functions $x(s, t)$ on the square $0 \leqq s \leqq \pi$, $0 \leqq s \leqq \pi$. Then $\mathfrak{H}$ together with the inner product $(x, y) = \int_0^\pi \int_0^\pi x(s, t)y(s, t)dsdt$ defines a Hilbert space with the real numbers as its scalar field. Let $\mathfrak{A}$ be the class of all functions $x$ in $\mathfrak{H}$ such that

(i) $x(s, t)$ is absolutely continuous in $s$ on $0 \leqq s \leqq \pi$ for almost all $t$ on $0 \leqq t \leqq \pi$ and is absolutely continuous in $t$ on $0 \leqq t \leqq \pi$ for almost all $s$ on $0 \leqq s \leqq \pi$;

(ii) The partial derivatives $x_s, x_t$, (which exist almost everywhere) are in $\mathfrak{H}$. Let $\mathfrak{H}'$ be the Hilbert space defined by the cartesian product $\mathfrak{H} \times \mathfrak{H}$. Observe that the gradient of $x$, written grad $x$, is defined on $\mathfrak{A}$ and maps $\mathfrak{A}$ into $\mathfrak{H}'$.

We shall be concerned with the operator $Ax = \text{grad } x$ whose domain $\mathscr{D}_A$ consists of all functions $x$ in $\mathfrak{A}$ which vanish on the boundary, in the sense that $x(0, t) = x(\pi, t) = 0$ for almost all $t$ on $0 \leqq t \leqq \pi$ and $x(s, 0) = (x\, \pi) = 0$ for almost all $s$ on $0 \leqq s \leqq \pi$. It can be shown that the mapping $A$ so defined is a closed dense operator $A$ from $\mathfrak{H}$ to $\mathfrak{H}'$. In fact it is the closure of the transformation grad $x$ restricted to functions of class $C'$ that vanish on the boundary of the given square. Its adjoint $A^*$ is defined by $A^*y = -\text{div } y$, where div $y$ is the closure of the usual divergence operator defined on the class of all vectors $y$ in $\mathfrak{H}'$ of class $C'$. The ranges of $A$ and $A^*$ are closed. Consequently $A$ and $A^*$ are reciprocally bounded. The operators $A^{-1}$ and $A^{*-1}$ are bounded and can be given an integral representation but we shall not pause to do so here.

The functions

$$x_{mn}(s, t) = \frac{2}{\pi} \sin ms \sin nt \qquad (m, n = 1, 2, 3, \cdots)$$

form a complete orthonormal system in $\mathfrak{H}$. Consequently every vector $x$ in $\mathfrak{H}$ can be expressible in the form

$$x = \sum_{m,n=1}^{\infty} a_{mn} x_{mn} ,$$

where convergence is taken in the mean of order 2. The vectory $y_{mn}$ in $\mathfrak{H}'$ whose components are

$$\frac{2m}{\pi\sqrt{m^2+n^2}}\cos ms \sin nt\ , \quad \frac{2n}{\pi\sqrt{m^2+n^2}}\sin ms \cos nt$$

form an orthonormal system in $\mathfrak{H}'$ that is incomplete. However, it is complete in $\mathscr{R}_A$. Consequently every vector $y$ in $\mathfrak{H}'$ is expressible in the form

$$y = y_0 + \sum_{m,n=1}^{\infty} b_{mn} y_{mn}$$

where $y_0 \in \mathscr{R}_A^{\perp}$, that is, $A^* y_0 = 0$. If $x$ and $y$ are in the appropriate domains we have.

$$Ax = \sum \lambda_{mn} a_{mn} y_{mn}\ , \quad A^{-1} y = \sum \frac{b_{mn}}{\lambda_{mn}} x_{mn}$$

$$A^{*-1} x = \sum \frac{a_{mn}}{\lambda_{mn}} y_{mn}\ , \quad A^* y = \sum \lambda_{mn} b_{mn} x_{mn}$$

where $\lambda_{mn} = (m^2+n^2)^{1/2}$ and $m, n$ summed over the positive integers. Defining $R$ and $R_{mn}$ by the formulas.

$$Rx = \sum a_{mn} y_{mn}\ , \quad R_{mn} x = a_{mn} y_{mn}$$

it is found that $R$ and $R_{mn}$ satisfies relation analogous to (4.3) and that $R$ maps $\mathfrak{H}$ isometrically onto $\mathscr{R}_A$. Moreover,

$$A = \sum \lambda_{mn} R_{mn}\ , \quad A^{-1} = \sum \lambda_{mn}^{-1} R_{mn}^*$$
$$A^{*-1} = \sum \lambda_{mn}^{-1} R_{mn}\ , \quad A^* = \sum \lambda_{mn} R_{mn}^*\ .$$

These formulas are analogous to (4:4) and with minor modifications illustrate the spectral theory given below for an arbitrary closed operator whose reciprocal is compact.

**5. Some properties of nonnegative self-adjoint operators.** It is the purpose of this section to establish certain properties of nonnegative self-adjoint operators. The first of these is given in the following

THEOREM 5.1. *Let $A$ be a nonnegative self-adjoint operator from $\mathfrak{H}$ to $\mathfrak{H}$ and let $E$ be the projection*

(5.1) $$E = \overline{A^{-1}A} = \overline{AA^{-1}}\ .$$

*There exists a unique pair of nonnegative self-adjoint operators $C$ and $D$ such that*

(5.2) $$C + D = E\ , \quad A = CD^{-1} = D^{-1}C\ , \quad A^{-1} = C^{-1}D = DC^{-1}\ .$$

*The operators $C$ and $D$ are bounded and are given by the formulas*

$$C^{-1} = A^{-1} + E\,, \quad D^{-1} = A + E\,. \tag{5.3}$$

*They have the same null space as $A$. Moreover*

$$CD = DC\,, \quad C^{-1}D^{-1} = D^{-1}C^{-1} = C^{-1} + D^{-1}\,. \tag{5.4}$$

In order to prove this result let $C$ and $D$ be defined by the formula (5.3). Then $C$ and $D$ are bounded. In fact $\|C\| \leqq 1$, $\|D\| \leqq 1$. The set $\mathscr{D} = \mathscr{D}_A \cap \mathscr{D}_{A^{-1}}$ is the domain of each of the transformations $C^{-1}D^{-1}$, $D^{-1}C^{-1}$, $C^{-1} + D^{-1}$. In view of (5:1) we have

$$C^{-1}D^{-1} = A^{-1}A + A + A^{-1} + E = C^{-1} + D^{-1} = D^{-1}C^{-1}\,.$$

These operators are accordingly reciprocally bounded operators and are the reciprocals of $CD$ and $DC$, by Theorem 3.5. Hence (5.4) holds. In addition

$$\begin{aligned} C^{-1}C &= D^{-1}D = E = D^{-1}C^{-1}CD = (C^{-1} + D^{-1})CD = D + C\,, \\ C^{-1} &= C^{-1}(C + D) = E + C^{-1}D = (C + D)C^{-1} = E + DC^{-1}\,, \\ D^{-1} &= D^{-1}(C + D) = D^{-1}C + E = (C + D)D^{-1} = CD^{-1} + E\,. \end{aligned}$$

Comparing this result with (5.3) it is seen that (5.2) holds. On the other hand equations (5.2) imply that $A$, $C$, $D$ have the same null space and it follows from the computation just made that (5.3) holds. This proves the theorem.

THEOREM 5.2. *Let $A$ be a nonnegative self-adjoint operator from $\mathfrak{H}$ to $\mathfrak{H}$. There is a unique nonnegative self-adjoint operator $P$ from $\mathfrak{H}$ to $\mathfrak{H}$ such that $P^2 = A$. The operator $P$ will be called the square root of $A$ and will be denoted alternatively by $A^{1/2}$. The square root of $A^{-1}$ is $P^{-1}$.*

If $A$ is bounded, this result can be established by elementary means[3]. In this event every bounded self-adjoint operator that commutes with $A$ also commutes with $A^{1/2}$. The truth of the theorem for the unbounded case can be obtained from the spectral theorem. Assuming the truth of the theorem for the bounded case one can establish its truth for the unbounded case without the direct use of the spectral theorem. As a first step in the proof we shall prove the following

LEMMA 5.1. *Let $P$ and $A$ be two self-adjoint operators from $\mathfrak{H}$ to $\mathfrak{H}$ such that $P^2 = A$. Then $A$ is nonnegative and $(P^{-1})^2 = A^{-1}$.*

Clearly $A$ is nonnegative. In order to show that $C = (P^{-1})^2$ is the

[3] See, for example, F. Reisz and B. Nagy, Lecons d'Analyse Fonctionelle, p. 262.

reciprocal of $A$ observe that on $\mathscr{D}_A$ we have $CA = (P^{-1}P)^2 \subseteqq E$, where $E = \overline{A^{-1}A}$. It follows that $C \supseteqq A^{-1}$. Since $C$ and $A^{-1}$ are self-adjoint we have $A^{-1} = (A^{-1})^* \supseteqq C^* = C$. Hence $A^{-1} = C$, as was to be proved.

COROLLARY. *A reciprocally bounded nonnegative self-adjoint operator possesses a unique square root.*

We are now in position to complete the proof of Theorem 5.2. To this end let $C$ and $D$ be related to $A$ as described in Theorem 5.1. Since $C$ and $D$ are bounded and commute, their square roots $M$ and $N$ satisfy the relations

$$M^2 + N^2 = E\,, \quad MN = NM\,, \quad M^{-1}N^{-1} = N^{-1}M^{-1}\,.$$

Moreover $E = M^{-1}M = N^{-1}N$ and

$$N^{-1} = N^{-1}(M^2 + N^2) = N^{-1}M^2 + N = (M^2 + N^2)N^{-1} = M^2N^{-1} + N\,.$$

Hence $N^{-1}M^2 = M^2N^{-1}$ and

$$\begin{aligned} MN^{-1} &= EMN^{-1} = N^{-1}M^2N^{-1} = M^{-1}N^{-1}M^2 = N^{-1}M^{-1}M^2 \\ &= N^{-1}EM = N^{-1}M \end{aligned}$$

Similarly $M^{-1}N = NM^{-1}$. In addition

$$(N^{-1}M)^2 = N^{-1}MN^{-1}M = N^{-2}M^2 = D^{-1}C = A\,, \quad (M^{-1}N)^2 = A^{-1}\,.$$

Setting $y = Nx$ with $x$ in the carrier of $N$ and using the fact that $MN = NM \geqq 0$ we find that

$$(MN^{-1}y, y) = (Mx, Nx) = (MNx, x) \geqq 0$$

for all $y$ in the carrier of $MN^{-1}$. Hence $MN^{-1}$ is a nonnegative self-adjoint operator whose square is $A$. It remains to show that if $P$ is a nonnegative self-adjoint operator whose square is $A$, then $P = MN^{-1}$. To do so observe that

$$(P + P^{-1})^2 = P^2 + P^{-2} + 2E = A + A^{-1} + 2E = C^{-1}D^{-1} = (M^{-1}N^{-1})^2\,.$$

Since reciprocally bounded operators have unique square roots it follows that

$$P + P^{-1} = M^{-1}N^{-1}\,.$$

Moreover

$$\begin{aligned} &PM^{-2} = PC^{-1} = PA^{-1} + P \subseteqq P + P^{-1} = M^{-1}N^{-1} = N^{-1}M^{-1} \\ &P = PM^{-2}M^2 \subseteqq N^{-1}M^{-1}M^2 = N^{-1}M\,. \end{aligned}$$

Since $P$ and $N^{-1}M$ are self-adjoint, they are equal. This completes the proof of Theorem 5.2.

**6. Elementary operators and the polar form.** By an *elementary operator* $R$ from $\mathfrak{H}$ to $\mathfrak{H}'$ will be meant one that is its own $*$-reciprocal, or equivalently one whose adjoint is its reciprocal. It is characterized by the relation

$$RR^*R = R . \tag{6.1}$$

An elementary operator maps its carrier isometrically onto its range. If $R \neq 0$ then $\|R\| = 1$. It is easily seen that an operator $R$ is elementary if and only if $E = R^*R$ is a projection in $\mathfrak{H}$. Similarly $R$ is elementary if and only if $E' = RR^*$ is a projection in $\mathfrak{H}'$. If $\mathfrak{H} = \mathfrak{H}'$, then an elementary operator $R$ is normal if and only if $\mathrm{E} = E'$, that is, if and only if $R$ and $R^*$ have the same null spaces. A projection is a nonnegative self-adjoint elementary operator. An elementary operator $R$ is self-adjoint if and only if it is expressible as the difference $R = E_+ - E_-$ of two projections $E_+$ and $E_-$ that are orthogonal. For if $R$ is self-adjoint, then

$$E_+ = \tfrac{1}{2}(E + R) , \quad E_- = \tfrac{1}{2}(E - R)$$

satisfy the relations

$$E_+^2 = E_+ = E_+^* , \quad E_-^2 = E_- = E_-^* , \quad E_+E_- = E_-E_+ = 0$$

and hence are projections. Moreover

$$R = E_+ - E_- , \quad E = E_+ = E_- = R^2 .$$

Conversely, if $R$ is expressible in this form it is a self-adjoint elementary operator, as one readily verifies.

It should be observed in passing that if $R$ is an elementary operator from $\mathfrak{H}$ to $\mathfrak{H}'$ and $F$ is a projection in $\mathfrak{H}$ that commutes with the projection $E = R^*R$, then $S = RF$ is also elementary. This follows because $S^*S = FR^*RF = FEF = FE$ is projection. Similarly if $F'$ is a projection in $\mathfrak{H}'$ that commutes with $RR^*$, then $F'R$ is elementary.

Let $R$ be an elementary operator from $\mathfrak{H}$ to $\mathfrak{H}'$. An operator $A$ from $\mathfrak{H}$ to $\mathfrak{H}'$ will be said to be *self-adjoint relative to* $R$ in case

$$A = RA^*R . \tag{6.2}$$

If $\mathfrak{H} = \mathfrak{H}'$ and $R$ is the identity, this concept reduces to the usual definition of self-adjointness. We have the following

THEOREM 6.1. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$ that is self-adjoint relative to an elementary operator $R$. Then $A^{*-1}$ is self-adjoint relative to $R$. Similarly $A^*$, $A^{-1}$ are self-adjoint relative to*

$R^*$. *Moreover* $\mathfrak{N}_R \subset \mathfrak{N}_A$ *and* $\mathfrak{N}_{R^*} \subset \mathfrak{N}_{A^*}$. *The operators* $A$ *and* $R$ *satisfy the further relations*

$$A = RR^*A = AR^*R\,, \quad RA^*A = AA^*R \tag{6.3a}$$

$$A^*R = R^*A\,, \quad AR^* = RA^* \tag{6.3b}$$

$$(A^*R)^2 = A^*A\,, \quad (AR^*)^2 = AA^*\,. \tag{6.3c}$$

It is clear from (6.2) that $\mathfrak{N}_R \subset \mathfrak{N}_A$, $\mathfrak{N}_{R^*} \subset \mathfrak{N}_{A^*}$ and $R^*AR^* = A^*$. Moreover

$$RR^*A = RR^*RA^*R = RA^*R = A = AR^*R$$
$$RA^*A = RA^*RA^*R = AA^*R\,.$$

Hence (6.3a) holds. The relation (6.3b) and (6.3c) follow from the computations

$$A^*R = R^*AR^*R = R^*A\,, \quad RA^* = RR^*AR^* = AR^*$$
$$(A^*R)^2 = A^*RR^*A = A^*A\,, \quad (AR^*)^2 = AR^*AR^* = AA^*\,.$$

In view of the corollary to Theorem 3.5 it is seen that $RA^{-1}R$ is the reciprocal of $A^* = R^*AR^*$, that is, $A^{*-1} = RA^{-1}R$. This proves the theorem.

It is easily seen from the formula (6.2) that $\mathfrak{N}_R = \mathfrak{N}_A$ if and only if $\mathfrak{N}_{R^*} = \mathfrak{N}_{A^*}$. In addition we have the following

COROLLARY. *An operator* $A$ *is self-adjoint relative to an elementary operator and only if*

$$A = RR^*A = AR^*R\,, \quad R^*A = A^*R\,, \quad RA^* = AR^*\,. \tag{6.4}$$

The existence of elementary operator $R$ relative to which $A$ is self-adjoint is established in the following

THEOREM 6.2. *Given an operator* $A$ *from* $\mathfrak{H}$ *to* $\mathfrak{H}'$ *there is a unique elementary operator* $R$ *such that* $A$ *is self-adjoint relative to* $R$, $\mathfrak{N}_A = \mathfrak{N}_R$ *and* $A^*R$ *is nonnegative. The operators* $A^{-1}R$, $AR^*$ *and* $A^{*-1}R^*$ *are also nonnegative and* $\mathfrak{N}_{A^*} = \mathfrak{N}_{R^*}$.

In order to prove this result $P$ be the square root of $A^*A$. Then $P$ is nonnegative and $\mathfrak{N}_P = \mathfrak{N}_A$. We shall show that the operator $R = (P^{-1}A^*)^*$ has the properties described theorem. Observe first that

$$R \supseteqq AP^{-1}\,, \quad R^* \supseteqq P^{-1}A^*\,, \tag{6.5}$$

and hence that

$$E = R^*R \supseteqq P^{-1}A^*AP^{-1} = P^{-1}P^2P^{-1} = (P^{-1}P)(PP^{-1})\,.$$

This is possible only in case $E$ is a projection. Hence $R$ is elementary and $\mathfrak{N}_R = \mathfrak{N}_P = \mathfrak{N}_A$, $\mathfrak{N}_{A^*} = \mathfrak{N}_{R^*}$. By Theorem 3.5, the operators $R^*A$ and $A^*R$ are closed. Moreover by (6.4).

$$E = R^*R \supseteqq R^*AP^{-1}, \quad E = R^*R \supseteqq P^{-1}A^*R .$$

It follows that $P^*R = R^*A \geqq 0$. Consequently

$$RP = RA^*R = RR^*A = A ,$$

the last inequality holding since $\mathfrak{N}_{A^*} = \mathfrak{N}_{R^*}$ and $RR^*$ is a projection on $\mathfrak{H}'$. Since $A^{-1}R = P^{-1}$, $AR^* = RPR^*$, $A^{*-1}R^* = RA^{-1}RR^* = RP^{-1}R^*$, these products are nonnegative. The uniqueness of $R$ follows from (6.3c) and the uniqueness of the square root of $A^*A$.

The elementary operator $R$ described in Theorem 6.2 will be called the *elementry operator belonging to or associated with* $A$. The projections $E = R^*R$, $E' = RR^*$ are such that $E'A = AE = A$ and will be called the *projection associated with* $A$. It should be observed that if we set $P = A^*R$, $Q = AR^*$, then, $A = RA^*R = RP = QR$. This formula is commonly called the *polar decomposition* of $A$. It was first established for an unbounded operator in Hilbert space by J. von Neumann.[4]

COROLLARY. *If $R$ is the elementary operator associated with $A$, then $R$ is the elementary operator associated with $A^{*-1}$ and $R^*$ is the elementary operator associated with $A^*$ and $A^{-1}$.*

THEOREM 6.3. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}$ and let $R$ be the associated elementary operator. Then $A$ is normal if and only if $R^*$ commutes with $A$. If $A$ is normal so also is $R$. The operator $A$ is self-adjoint if and only if $R$ is self-adjoint and commutes with $A$. Finally $A$ is self-adjoint and nonnegative if and only if $R$ is a projection.*

Since $A^*A$ and $AA^*$ are equal if and only if their square roots $R^*A$ and $AR^*$ are equal, it follows that $A$ is normal if and only if $A$ commutes with $R^*$. If $A = A^*$ then $R^*A = AR^* = RA$, by (6.3b). Hence $R = R^*$ and $R$ commutes with $A$. Conversely if $R$ commutes with $A$ and $R = R^*$, then $A$ is normal and $A^*R = RA = AR$. Hence $A^* = A$. If $R$ is a projection, $AR = RA^*R^2 = RA^*R = A = RA$. Hence $A$ is self-adjoint and nonnegative. The converse is immediate and the theorem is established.

COROLLARY 1. *If $A$ is a self-adjoint operator from $\mathfrak{H}$ to $\mathfrak{H}$, it is expressible as the difference $A = A_+ - A_-$ of orthogonal nonnegative*

[4] Neumann, J. v. *Über Adjungierte Funktionaloperatoren* Annals of Math., **33** (1932), 294-310.

*self-adjoint operators.*

This follows because its associated elementary operator $R$ is self-adjoint and hence is the difference $R = E_+ - E_-$ of two orthogonal projections. Since $R$ and $E = E_+ + E_-$ commute with $A$ so also does $E_+$ and $E_-$. Using this fact it is seen that $A_+ = AE_+$, $A_- = AE_-$ have properties described in the corollary.

COROLLARY 2. *If $A$ is self-adjoint relative to an elementary operator $T$ so also is its associated elementary operator $R$, that is $R = TR^*T$.*

7. **$*$-orthogonality and sections.** Two operators $A$ and $B$ will be said to be *$*$-orthogonal* if their carriers are orthogonal and their ranges are orthogonal. This is equivalent to the statemnt that $A^*B = 0$ (or $B^*A = 0$) on a dense set in $\mathfrak{H}$ and $AB^* = 0$ (or $BA^* = 0$) on a dense set in $\mathfrak{H}'$. It is clear that $A$ is $*$-orthogonal to $B$ if and only if $A^{*-1}$ is $*$-orthogonal to $B$. If one of the pairs $A$, $B$; $A^*$, $B^*$; $A^{-1}$, $B^{-1}$; $A^{*-1}$, $B^{*-1}$ form a $*$-orthogonal pair, then the remaining pairs form $*$-orthogonal pairs. Finally two operators $A$ and $B$ are $*$-orthogonal if and only if their associated elementary operators $R$ and $S$ are $*$-orthogonal. The following result is readily verified.

THEOREM 7.1. *Let $B$ and $C$ be $*$-orthogonal operators from $\mathfrak{H}$ to $\mathfrak{H}'$. Then $A = B + C$ is an operator and $A^{-1} = B^{-1} + C^{-1}$, $A^* = B^* + C^*$, $A^{*-1} = B^{*-1} + C^{*-1}$. Moreover $A$ is elementary if and only if $B$ and $C$ are elementary. If $S$ and $T$ are respectively the elementary operators associated with $B$ and $C$, then $R = S + T$ is the elementary operator associated with $A = B + C$.*

An operator $B$ will be called a *section of an operator $A$*, if there is an operator $C$ $*$-orthogonal to $B$ such that $A = B + C$. If $B$ is a section of $A$, its associated elementary operator $S$ is a section of the associated elementary operator $R$ of $A$. As a first result characterizing sections of $A$ we have the following.

THEOREM 7.2. *Let $E = \overline{A^{-1}A}$, $E' = \overline{AA^{-1}}$ be the projections associated with $A$. Let $F$, $F'$ be projections in $\mathfrak{H}$ and $\mathfrak{H}'$ respectively. Suppose that $AF \supseteqq F'A$. Then $EF = FE$ and $F'F' = F'E'$. Moreover $AF$ is a section of $A$ and its adjoint is $A^*F'$.*

Since the domain of $F'A$ is $\mathscr{D}_A$ it follows from the relation $AF \supseteqq F'A$ that $B = AF$ is dense. Since $B$ is closed, it is an operator. Since $AE = A$ it follows that $AFE \supseteqq F'A$. Hence $AFE - AF = A(EFE -$

$EF) = 0$ on $\mathscr{D}_A$. This possible only in case $EFE = EF$ and hence only in case $EF = FE$. Similarly, since $A^*F' \supseteqq FA^*$, it follows that $F'E' = E'F'$. Moreover $B^* = A^*F'$. The projections associated with $B$ are accordingly $G = EF$ and $G' = E'F'$. The operator $C = A(E - G)$ has $E - G$ and $E' - G'$ as its associated projections. It follows that $C$ and $B$ are $*$-orthogonal. Moreover $A = B + C$ and the theorem is established.

THEOREM 7.3. *An operator $B$ is a section of $A$ if and only if $A^*B = B^*B$ and $AB^* = BB^*$.*

If $A = B + C$, where $B$ is $*$-orthogonal to $C$, then $A^*B = (B^* + C^*)B = B^*B$ and $AB^* = B^*B$. Conversely suppose that $A^*B = B^*B$ and $AB^* = BB^*$. Let $F = \overline{B^{-1}B}$, $F' = \overline{BB^{-1}}$. Then

$$B^* = A^*BB^{-1} \subseteqq A^*F' , \quad B = AB^*B^{*-1} \subseteqq AF .$$

It follows that $F'A \subseteqq B \subseteqq AF$ and hence that $B = AF$. In view of Theorem 7.2 the operator $B$ is a section of $A$, as was to be proved.

THEOREM 7.4. *Let $R$ be an elementary operator and let $E = R^*R$. Let $F$ be a projection in $\mathfrak{H}$. Then $S = RF$ is a section of $R$ if and only if $EF = FE$. Similarly if $F'$ is projection to $\mathfrak{H}'$, then $F'R$ is a section of $R$ if and only if $E'F' = F'E'$, where $E' = RR^*$.*

If $S = RF$ is a section of $R$, then

$$S^*S = R^*S = R^*RF = EF ,$$

is a projection in $\mathfrak{H}$. Hence $EF = FE$. Conversely if $EF = FE$ then

$$R^*S = R^*RF = EF = FEF = FR^*RF = S^*S ,$$
$$SS^* = RFR^* = RS^* .$$

Consequently, $S$ is a section of $R$, by Theorem 7.3. The last statement in the theorem follows similarly.

**8. $*$-commutativity.** A bounded operator $B$ from $\mathfrak{H}$ to $\mathfrak{H}'$ will be said to *$*$-commute* with an operator $A$ from $\mathfrak{H}$ to $\mathfrak{H}'$ if

(8.1) $$A^*B \supseteqq B^*A , \quad AB^* \supseteqq BA^* .$$

It should be observed that products $A^*B$ and $AB^*$ appearing in (8.1) are closed and dense and hence are operators. In the present section we shall derive some elementary properties of $*$-commutative operators of this type. Throughout this section the operator $B$ is restricted to be bounded, while $A$ is arbitary. The associated projections will be denoted by

$$E = \overline{A^{-1}A}\,, \quad E' = \overline{AA^{-1}}\,, \quad F = \overline{B^{-1}B}\,, \quad F' = \overline{BB^{-1}}\,, \tag{8.2}$$

as a first result we have

LEMMA 8.1. *Suppose that $B$ $*$-commutes with $A$. The product $A^*B$ is self-adjoint and is the closure of $B^*A$. Similarly, the product $AB^*$ is self-adjoint and is the closure of $BA^*$.*

Suppose that $B$ $^*$-commutes with $A$ and that $A^*B$ is not the closure of $B^*A$. Then there is vector $x_0 \neq 0$ in the domain of $A^*B$ such that

$$(x_0, x) + (A^*Bx_0, B^*Ax) = 0$$

for all $x$ in $\mathscr{D}_A$. Since $(B^*A)^* = A^*B$ it follows that

$$(x_0, x) + (A^*BA^*Bx_0, x) = 0 \tag{8.3}$$

for all $x$ in $\mathscr{D}_A$, and hence for all $x$ in $\mathfrak{H}$. Choosing $x = B^*Bx_0$ and making use of (8.1) we find that

$$(x_0, B^*Bx_0) + (A^*AB^*Bx_0, B^*Bx_0) = 0.$$

Since $B^*Bx_0$ is in $\mathscr{D}_A$ we have

$$\| Bx_0 \|^2 + \| AB^*Bx_0 \|^2 = 0$$

and hence $Bx_0 = 0$. Using (8.3) we find that $x_0 = 0$, contrary to our choice of $x_0$. The closure of $B^*A$ is accordingly $A^*B$. The last statement in the lemma follows by symmetry.

LEMMA 8.2. *Suppose that $\mathfrak{N}_A = \mathfrak{N}_B$ and $\mathfrak{N}_{A^*} = \mathfrak{N}_{B^*}$. If the first of the relations*

$$A^*B \supseteqq BA\,, \quad AB^* \supseteqq BA^*\,, \tag{8.4a}$$

$$A^{-1}B \supseteqq B^*A^{*-1}\,, \quad A^{*-1}B^* \supseteqq BA^{-1}\,, \tag{8.4b}$$

$$B^{-1}A^{*-1} \supseteqq A^{-1}B^{*-1}\,, \quad B^{*-1}A^{-1} \supseteqq A^{*-1}B^{-1}\,, \tag{8.4c}$$

$$B^{-1}A \supseteqq A^*B^{*-1}\,, \quad B^{*-1}A^* \supseteqq AB^{-1}\,, \tag{8.4d}$$

*holds, so the others hold also. If* (8.4) *holds, the products appearing on the right are operators.*

The last statement in the theorem follows from Theorem 3.5. Suppose now that (8.4a) holds. Then $A^*B = B^*A$ on $\mathscr{D}_A$. Consequently, on $\mathscr{D}_{A^{-1}}$ we have

$$A^{*-1}A^*BA^{-1} = A^{*-1}B^*AA^{-1} = A^{*-1}B^*\,.$$

Hence the second relation in (8.4b) holds. The first relation follows similarly. The right and left members of (8.4c) are the reciprocals of the corresponding right and left numbers of (8.4a). Hence (8.4c) holds. Similarly (8.4d) holds.

LEMMA 8.3. *Suppose that $B$ $*$-commutes $A$. Then $A$ and $B$ are expressible uniquely as sums of sections*

$$A = A_0 + A_1\,, \quad B = B_0 + B_1 \tag{8.5}$$

*such that* $(\alpha) A_0$ *is $*$-orthogonal to $B$ and $B_0$ is $*$-orthogonal to $A$*: $(\beta)$ *$B_1$ $*$-commutes with $A_1$ and $\mathfrak{N}_{A_1} = \mathfrak{N}_{B_1}$, $\mathfrak{N}_{A_1^*} = \mathfrak{N}_{B_1^*}$. Moreover*

$$A_1 = B^{*-1}A^*B\,, \quad A_1^* = B^{-1}AB^*\,. \tag{8.6}$$

*Conversely, if $A$ and $B$ are expressible in the form* (8.5) *such that* $(\alpha)$ *and* $(\beta)$ *hold, then $B$ $*$-commutes with $A$.*

Suppose first that $B$ $*$-commutes with $A$. Using (8.1) and (8.2) it is seen that

$$EB^*A = B^*A\,, \quad E'BA^* = BA^*\,, \quad FA^*B = A^*B\,, \quad F'AB^* = AB^*\,.$$

Hence

$$EB^*E' = B^*E' : \quad E'BE = BE\,, \quad FA^*F' = A^*F' \quad F'AF = AF\,.$$

Consequently $BE = E'B$, $AF \supseteqq F'A$. In view of Theorem 7.2 it follows that $A_1 = AF$, $B_1 = BE$ are respectively sections of $A$ and $B$, each having $EF$ and $E'F'$ as their associated projections. We have accordingly $\mathfrak{N}_{A_1} = \mathfrak{N}_{B_1}$, $\mathfrak{N}_{A^*} = \mathfrak{N}_{B_1^*} = {}_B\mathfrak{N}_1^*$. Choose $A_0$ and $B_0$ so that (8.5) holds. The operator $A_0$ has $E - EF$ and $E' - E'F'$ as its associated projections and is accordingly $*$-orthogonal to $B$, $B_0$, $B_1$ and $A_1$. Similarly $B_0$ is $*$-orthogonal to $A$, $A_0$, $A_1$ and $B_1$. Using (8.1) again we see that

$$B_1^*A_1 = EB^*AF \subseteqq EA^*BF = (A_0^* + A_1^*)(B_0 + B_1) = A_1^*B_1\,.$$

Likewise $B_1A_1^* \subseteqq A_1B_1^*$. This proves the first conclusion of the lemma. The last statement is immediate.

It remains to obtain the formulas (8.6). To this end observe that

$$B^{*-1}A^*B = B_1^{*-1}A_1^*B_1 \supseteqq B_1^{*-1}B_1^*A_1 = A_1\,.$$

In view of the result we may suppose that $A = A_1$. Assume that $A \neq B^{*-1}A^*B$. Since $B^{*-1}AB^*$ and $A$ are closed, there is a vector $x_0 \neq 0$ such that

$$(x_0, x) + (B^{*-1}A^*Bx_0, Ax) = 0 \tag{8.7}$$

for all $x$ in $\mathscr{D}_A$. Consequently, by (8.4d),

$$(x_0, x) = -(A^*B^{*-1}A^*Bx_0\, x) = -(B^{-1}AA^*Bx_0, x)$$

for all $x$ in $\mathscr{D}_A$ and hence for $x$ in $\mathfrak{H}$. Choosing $x = B^*Bx_0$ we find that

$$\| Bx_0 \|^2 = -(AA^*Bx_0\, Bx_0) = -\| A^*Bx_0 \|^2 .$$

This relation together with (8.7) can hold only in case $x_0 = 0$. It follows that the first formula in (8.6) holds. The second is obtained by symmetry and the lemma is established.

COROLLARY 1. *Suppose that $B$ $*$-commutes with $A$. The associated projections* (8.2) *satisfy the relation $EF = FE$, $E'F' = F'E'$. Moreover $\mathfrak{N}_A = \mathfrak{N}_B$ if and only if $\mathfrak{N}_A^* = \mathfrak{N}_B^*$.*

As a further result we have

COROLLARY 2. *If an elementary operator $T$ $*$-commutes with $A$, then $TA^*T$ is a section of $A$.*

In view of Lemma 8.2 and 8.3 we have

COROLLARY 3. *Suppose that $B$ $*$-commutes with $A$. Then $B$ $*$-commutes with $A^{*-1}$ and with $\alpha A + \beta A^{*-1}$, where $\alpha$ and $\beta$ are positive numbers.*

The restriction that $\alpha$, $\beta$ are positive is made only to insure that $\alpha A + \beta A^{*-1}$ be closed.

LEMMA 8.4. *Let $T$ be an elementary operator such that $TA^*T = A$ and suppose that $B$ $*$-commutes with $T$. Then $B$ $*$-commutes with $A$ if and only if $AT^*B \supseteqq BT^*A$.*

If $B$ $*$-commutes with $A$ then

$$AT^*B = TA^*TT^*B = TA^*B \supseteqq TB^*A = BT^*A .$$

Conversely, if $AT^*B \supseteqq BT^*A$, then

$$A^*B = T^*AT^*B \supseteqq T^*BT^*A = B^*TT^*A = B^*A$$
$$AB^* = TA^*TB^* \supseteqq TB^*TA^* = BT^*TA^* = BA^*$$

as was to be proved.

**9. Decomposition of an operator.** As a first result we have

THEOREM 9.1. *Let $R$ be the elementary operator associated with* *an operator $A$ from $\mathfrak{H}$ to $\mathfrak{H}'$. Let $T$ be second elementary operator that $*$-commutes with $A$. Then $T$ $*$-commutes with $R$ and the operators $A$, $R$, $T$ are expressible uniquely as sums and difference*

$$(9.1)\quad A = A_0 + A_+ + A_- , \quad R = R_0 + R_+ + R_- , \quad T = T_0 + R_+ - R_-$$

*of mutually $*$-orthogonal operators such that $R_0$, $R_+$, $R_-$ are the elementary operators associated respectively with $A_0$, $A_+$, $A_-$ and $T_0$ is $*$-orthogonal to $A$. Moreover $T$ is $*$-orthogonal to $A_0$ and $R_0$ and $*$-commutes with $A_+$, $A_-$, $R_+$ and $R_-$. Conversely if $A$, $R$, $T$ are expressible in the form* (9.1) *then $T$ $*$-commutes with $A$ and $R$.*

Suppose that $T$ $*$-commutes with $A$. Then, by Lemma 8:3, they are expressible in the forms $A = A_0 + A_1$, $T = T_0 + T_1$, where $A_0$ is $*$-orthogonal to $T$ and $A_1$, $T_0$ is $*$-orthogonal to $A$ and $T$, $\mathfrak{N}_{T_1} = \mathfrak{N}_{A_1}$, $\mathfrak{N}_{T_1^*} = \mathfrak{N}_{A_1^*}$ and $T_1$ $*$-commutes with $A_1$ Moreover, by Theorem 7.1 $R = R_0 + R_1$, where $R_0$ is the elementary operator belonging to $A_0$ and $R_1$ is the elementary operator belonging to $A_1$. In view of this result we can restrict ourselves to the case in with $A_0 = 0$, $T_0 = 0$, $R_0 = 0$. Then $\mathfrak{N}_A = \mathfrak{N}_T = \mathfrak{N}_R$ and $\mathfrak{N}_{A^*} = \mathfrak{N}_{T^*} = \mathfrak{N}_{R^*}$. Since $A^*T$ is self-adjoint, its associated elementary operator $S$ is self-adjoint and hence is expressible as the difference $S = E_+ - E_-$ of two orthogonal projections $E_+$, $E_-$ whose sum is $E = R^*R$. The operator $A^*TS$ is nonnegative and self-adjoint. It follows from Theorem 6.1 that $\mathrm{R} = TS$ and $T = RS$. Setting $R_+ = RE_+$, $R_- = RE_-$ we see that

$$R = RE = R_+ + R_- , \quad \mathrm{T} = RS = R_+ - R_- .$$

Since $AR^* = AR_+^* + AR_-^*$ and $AT^* = AR_+^* - AR_-$ are self-adjoint, so also are $AR_+^*$ and $AR_-^*$. Moreover $AR_+^* \geqq 0$ and $AR_-^* \geqq 0$ since they are orthogonal and $AR^* \geqq 0$. The elementary operators $R_+$ and $R_-$ are therefore the elementary operators associated respectively with $A_+ = AE_+$ and $A_- = AE_-$. Since $R_+$ and $R_-$ are $*$-orthogonal it follows that $A_+$ and $A_-$ are $*$-orthogonal. Consequently $A$, $R$, $T$ are expressible in the form (9.1). The remaining statements in the theorem are easily established.

COROLLARY. *Two elementary operators $R$ and $T$ on $\mathfrak{H}$ to $\mathfrak{H}'$ $*$-commute if and only if there exist mutually $*$-orthogonal elementary operators $R_0$, $R_+$, $R_-$, $T_0$ such that $R = R_0 + R_+ + R_-$, $T = T_0 + R_+ - R_-$. Moreover, this representation is unique.*

THEOREM 9.2. *Let $R$ be the elementary operator associated with an operator $A$ from $\mathfrak{H}$ to $\mathfrak{H}'$. Given a positive number $\lambda$ there are unique decompositions.*

$$(9.2) \qquad A = A_{\lambda+} + A_{\lambda 0} + A_{\lambda-} , \quad R = R_{\lambda+} + R_{\lambda 0} + R_{\lambda-}$$

*of $A$ and $R$ into sections such that $R_{\lambda+}$, $R_{\lambda 0}$, $R_{\lambda-}$, $R_{\lambda+} - R_{\lambda-}$ are respectively the elementary operators associated with $A_{\lambda+}$, $A_{\lambda 0}$, $A_{\lambda-}$, $A - \lambda R$. Moreover,*

$$(9.3a) \qquad A_{\lambda+} = R_{\lambda+}A^*R_{\lambda+} , \quad A_{\lambda 0} = \lambda R_{\lambda 0} , \quad A_{\lambda-} = R_{\lambda-}A^*R_{\lambda-} .$$

*The relations*

$$(9.3b) \qquad \| A_{\lambda+}x \| > \lambda \| R_{\lambda+}x \| , \quad (A_{\lambda+}x, R_{\lambda+}x) > \lambda \| R_{\lambda+}x \|^2 ,$$

*hold for all $x$ in $\mathscr{D}_A$ such that $R_{\lambda+}x \neq 0$ and the relations*

$$(9.3c) \qquad \| A_{\lambda-}x \| < \lambda \| R_{\lambda-}x \| , \quad (A_{\lambda-}x, R_{\lambda-}x) < \lambda \| R_{\lambda-}x \|^2$$

*hold for all $x$ in $\mathfrak{H}$ such that $R_{\lambda-}x \neq 0$. If $\lambda < \mu$, then $A_{\lambda-} + A_{\lambda 0}$ is a section of $A_{\mu-}$, and $A_{\mu 0} + A_{\mu+}$ is a section of $A_{\lambda+}$. Similarly $R_{\lambda-} + R_{\lambda 0}$ is a section of $R_{\mu-}$ and $R_{\mu 0} + R_{\mu+}$ is a section of $R_{\lambda+}$.*

In order to prove this result let $C = A - \lambda R$, where $\lambda$ is a fixed positive number. Let $T$ be the elementary operator associated with $C$. Since $R$ $*$-commutes with $A$ and $R$, it follows that $R$ $*$-commutes with $C$. By virtue of Theorem 9:1 $R$ also $*$-commutes with $T$. Similarly $T$ $*$-commutes $R$ and $C$ and hence also with $A = C + \lambda R$. Applying Theorem 9.1 to $A$, $R$, $T$ and to $C$, $T$, $R$ it is seen that they aree xpressible uniquely as sums

$$A = A_0 + A_+ + A_- , \quad R = R_0 + R_+ + R_-$$
$$C = C_0 + C_+ - C_- , \quad T = T_0 + R_+ - R_-$$

of mutually $*$-orthogonal operators such that $R_+$ is the elementary operator associated with $A_+$ and $C_+$; $R_-$ is the elementary operator associated with A$_-$ and $C_-$; $R_0$ is the elementary operator associated with $A_0$. Since $\mathfrak{N}_C \supset \mathfrak{N}_A$ it follows that $C_0 = T_0 = 0$. From the relation $C = A - \lambda R$ we obtain the relations

$$A_0 = \lambda R_0 , \quad A_+ = C_+ + \lambda R_+ , \quad A_- = -C_- + \lambda R_- .$$

Moreover, if we set $E_+ = R_+^*R_+$, $E_- = R_-^*R_-$

$$R_+^* A_+ = R_+^*C_+ + \lambda E_+ , \quad R_-^*A_- = -R_-^*C_- + \lambda E_- .$$

It follows that the second relations in (9.3b) and (9.3c) hold. If $x$ is in $\mathscr{D}_A$, then

$$\| A_+x \|^2 - \lambda^2 \| R_+x \|^2 = \| C_+x \|^2 + 2\lambda(R_+^* C_+x, x) \geqq 0 .$$

Hence the first relation in (9.3c) holds. Since $P = R_-^*A_- \geqq 0$ and $Q = R_-^*C_- \geqq 0$ satisfy the relation $P + Q = \lambda E_-$, they are bounded and commute. Hence $PQ = A_-^*C_- = C_-^*A_- \geqq 0$. Using the relations

$$\| A_-x \|^2 + 2(A_-^*C_-x, x) + \| C_-x \|^2 = \lambda^2 \| R_-x \|^2$$

it is seen that the first relation in (9.3c) holds.

In order to prove the last statement with $\mu > \lambda$ apply the results described in the first part with $A_{\lambda+}$, $R_{\lambda+}$, $\mu$ playing role of $A$, $R$, $\lambda$. One then obtains the partitions

(9.4a) $$A_{\lambda+} = A_{\mu+} + A_{\mu 0} + A_{\lambda\mu}\ , \quad R_{\lambda+} - R_{\mu+} + R_{\mu 0} + R_{\lambda\mu}\ .$$

Setting

(9.4b) $$A_{\mu-} = A_{\lambda\mu} + A_{\lambda 0} + A_{\lambda-}\ , \quad R_{\mu-} = R_{\lambda\mu} + R_{\lambda 0} + R_{\lambda-}\ .$$

We have

$$A = A_{\mu+} + A_{\mu 0} + A_{\mu-}\ , \quad R = R_{\mu+} + R_{\mu 0} + R_{\mu-}$$

with $R_{\mu+} - R_{\mu-}$ as the elementary operator $A - \mu R$. The last statement of theorem follows from the relations (9.4). This completes the proof of Theorem 9.2.

COROLLARY 1. *Suppose that $A$ is bounded and set $M = \| A \|$. Let $m$ be the largest number such that $\| Ax \| \geqq m \| Rx \|$. If $\lambda \geqq M$, then $R_{\lambda+} = 0$. If $m > 0$ and $0 < \lambda \leqq m$, then $R_{\lambda-} = 0$. If $m < \lambda < M$, then $\| A - \lambda R \| \leqq \max\, [M - \lambda, \lambda - m]$.*

COROLLARY 2. *The operator $R_\lambda = R_{\lambda 0} + R_{\lambda-}$ $(0 < \lambda < \infty)$ is the elementary operator belonging to $A_\lambda = A_{\lambda 0} + A_{\lambda-} = R_\lambda A^* R_\lambda$. Moreover*

(1) $$\lim_{\lambda=\infty} R_\lambda = R\ , \quad \lim_{\lambda=0} R_\lambda = 0\ , \quad \lim_{\lambda=\infty} A_\lambda = A\ , \quad \lim_{\lambda=0} A_\lambda = 0\ .$$

(2) *If $\lambda < \mu$, then $R_\lambda$ is a section of $R_\mu$, $A_\lambda$ is a section of $A_\mu$, and*

$$\lambda \| R_\lambda x \| \leqq \| A_\mu x - A_\lambda x \| \leqq \mu \| R_\mu x \|\ .$$

(3) $$\lim_{\mu=\lambda+0} R_\mu = R_\lambda\ , \quad \lim_{\mu=\lambda+0} A_\mu = A_\lambda\ .$$

Let $A_\lambda (0 < \lambda < \infty)$ be the one parameter family of sections of $A$ described in the last corollary. By the *principal spectrum* $\Lambda$ of $A$ will be meant the set of all numbers $\lambda_0$ on $0 \leqq \lambda < \infty$ such that $A_\lambda$ is constant on no neighborhood of $\lambda_0$. The principal spectrum of $A^*$ is also $\Lambda$. The spectrum of $A^{-1}$ and $A^{*-1}$ is the closure of the reciprocals $1/\lambda$ of the points $\lambda \neq 0$ in $\Lambda$.

If $R_\lambda$ is the elementary operator of $A_\lambda$ described in the last corollary, we have the representations

$$A = \int_0^\infty \lambda dR_\lambda , \quad A^* = \int_0^\infty \lambda dR_\lambda^* ,$$
$$A^{*-1} = \int_0^\infty \lambda^{-1} dR_\lambda , \quad A^{-1} = \int_0^\infty \lambda^{-1} dR_\lambda^* ,$$

where the integrals are defined in the usual manner. It should be observed that $E_\lambda = R_\lambda^* R_\lambda$ and $E'_\lambda = R_\lambda R_\lambda^*$ are resolutions of $E = R^*R$ and $E' = RR^*$, respectively. Since $R_\lambda = RE_\lambda = E'_\lambda R$ we have, from the polar form of $A$,

$$A = R\int_0^\infty \lambda dE_\lambda = \left(\int_0^\infty \lambda dE'_\lambda\right) R .$$

It follows that the results given above can be derived from the self-adjoint case, if one so desires.

An extension of the results given above is found in the following

THEOREM 9.3. *Let $A$ be an operator and $T$ be an elementary operator such that $A = TA^*T$. Given a real number $\lambda$ there exist a unique decomposition*

$$(9.6) \qquad A = A_{\lambda+} + A_{\lambda 0} + A_{\lambda -} , \quad T = T_{\lambda+} + T_{\lambda 0} + T_{\lambda -} \quad (-\infty < \lambda < \infty)$$

*of $A$ and $T$ into sections such that*

$$(9.7a) \qquad A_{\lambda+} = T_{\lambda+} A^* T_{\lambda+} , \quad A_{\lambda 0} = \lambda T_{\lambda 0} , \quad A_{\lambda -} = T_{\lambda -} A^* T_{\lambda -}$$

$$(9.7b) \qquad (A_{\lambda+}x, T_\lambda x) > \lambda \| T_{\lambda+}x \|^2 \textit{ for all } x \textit{ in } \mathscr{D}_A \textit{ having } T_{\lambda+}x \neq 0 ,$$

$$(9.7c) \qquad (A_{\lambda-}x, T_\lambda x) < \lambda \| T_{\lambda-}x \|^2 \textit{ for all } x \textit{ in } \mathscr{D}_A \textit{ having } T_{\lambda-}x \neq 0 .$$

*If $\mu > \lambda$, then $A_{\lambda 0} + A_{\nu -}$ is a section of $A_{\mu -}$, $T_{\lambda 0} + T_{\lambda -}$ is a section of $T_{\mu -}$, $A_{\mu +} + A_{\mu 0}$ is a section of $A_{\lambda +}$ and $T_\mu + T_{\mu 0}$ is a section of $T_{\lambda +}$.*

In order to prove this result observe first that by Theorem 9.1 the operators $A$, $R$, $T$ have unique decompositions

$$A = A_1 + A_2 , \quad R = R_1 + R_2 , \quad T = T_0 + R_1 - R_2$$

where $R_1$, $R_2$ are the elementary operators associated with $A_1$, $A_2$ respectively and $T_0$ is $*$-orthogonal to $A$. The terms $A_0$ and $R_0$ described in Theorem 9.1 are zero since $A = TA^*T$. If $\lambda$ is positive let

$$A_1 = A_{1\lambda+} + A_{1\lambda 0} + A_{1\lambda -} , \quad R_1 = R_{1\lambda+} + R_{1\lambda 0} + R_{1\lambda -} ,$$

be the decompositions of $A_1$ and $R_1$ described in Theorem 9.2. Then

$$A_{\lambda+} = A_{1\lambda+} , \quad A_{\lambda 0} = A_{1\lambda 0} , \quad A_{\lambda -} = A_{1\lambda -} + A_2$$
$$T_{\lambda+} = R_{1\lambda+} , \quad T_{\lambda 0} = R_{1\lambda 0} , \quad T_{\lambda -} = R_{1\lambda -} - R_2 + T_0$$

have the properties described in Theorem 9.3. If $\lambda = 0$ set

$$A_{\lambda+} = A_1 , \quad A_{\lambda 0} = 0 , \quad A_{\lambda-} = A_2$$
$$T_{\lambda+} = R_1 , \quad T_{\lambda 0} = T_0 , \quad T_{\lambda-} = -R_2 .$$

If $\lambda = -\mu < 0$ let

$$A_2 = A_{2\mu+} + A_{2\mu 0} + A_{2\mu-} , \quad R_2 = R_{2\mu+} + R_{2\mu 0} + R_{2\mu-}$$

be the decomposition of $A_2$, $R_2$ described in Theorem 9.2. Then

$$A_{\lambda+} = A_1 + A_{2\mu-} , \quad A_{\lambda 0} = A_{2\mu 0} , \quad A_{\lambda-} = A_{2\mu+} \qquad (\mu = -\lambda)$$
$$T_{\lambda+} = R_1 - R_{2\mu-} , \quad T_{\lambda 0} = -R_{2\mu 0} , \quad T_{\lambda-} = -R_{2\mu+}$$

have the properties described in Theorem 9.3. The uniqueness of the decomposition follows from (9.7) and the connections between $T$ and $R$.

COROLLARY. *The operators* $T_\lambda = T_{\lambda 0} + T_{\lambda-}$, $A_\lambda = A_{\lambda 0} + A_{\lambda-} = T_\lambda = A^* T_\lambda$ *have the following properties*:

(1) $\lim_{\lambda=+\infty} T_\lambda = T , \quad \lim_{\lambda=-\infty} T_\lambda = 0 , \quad \lim_{\lambda=+\infty} A_\lambda = A , \quad \lim_{\lambda=-\infty} A_\lambda = 0 .$

(2) *If* $\lambda < \mu$, $T_\lambda$ *is a section of* $T_\mu$, $A_\lambda$ *is a section of* $A_\mu$

(3) $\lim_{\mu=\lambda+0} T_\mu = T_\lambda , \quad \lim_{\mu=\lambda+0} A_\mu = A_\lambda .$

(4) $(T_\lambda x, A_\lambda x) \leqq \lambda \| T_\lambda x \|^2$ *for all* $x$ *in* $\mathscr{D}_A$ .

In view of the results obtained in the last corollary we shall define the *spectrum* $\Lambda$ *of* A *relative to* $T$ to be the set of all real numbers $\lambda_0$ such that the operators $A_\lambda$ described in the last corollary is constant on no neighborhood of $\lambda_0$. The spectrum of $A^*$ relative to $T^*$ is also $\Lambda$. Similarly the spectrum of $A^{*-1}$ relative to $T$ and $A^{-1}$ relative to $T^*$ is the closure of the reciprocal $1/\lambda$ of the points $\lambda \neq 0$ in $\Lambda$. Moreover $A$ and $A^*$ are representable

$$A = \int_{-\infty}^{\infty} \lambda d T_\lambda , \quad A^* = \int_{-\infty}^{\infty} \lambda d T_\lambda^* .$$

If $\mathfrak{N}_A = \mathfrak{N}_T$, then

$$A^{*-1} = \int_{-\infty}^{\infty} \lambda^{-1} d T_\lambda , \quad A^{-1} = \int_{-\infty}^{\infty} \lambda^{-1} d T_\lambda .$$

When $\mathfrak{H} = \mathfrak{H}'$ and $T$ is the identity one obtains the usual spectral resolution for self-adjoint operators.

**10. Spectrum of the gradient operator.** Let $\mathfrak{H}$ be the class of all complex valued Lebesgue square integrable functions $x(t) = x(t_1, \cdots, t_m)$

of points $t = (t_1, \cdots, t_m)$ in an $m$-dimensional Euclidean space. It is convenient to normalize a function in $\mathfrak{H}$ to be equal to the limit of its integral mean whenever these limits exist and setting $x(t) = 0$ elsewhere. The class so normalized forms a Hilbert space over the field of complex numbers with

$$(x_1, x_2) = \int_{-\infty}^{\infty} x_1(t)\overline{x_2(t)}dt$$

as the inner product, where $\bar{x}(t)$ denotes the conjugate of $x(t)$. As is well known the Fourier transform

$$\hat{x}(s) = c\int_{-\infty}^{\infty} e^{-ist}x(t)dt \ , \quad st = s_1t_1 + \cdots + s_mt_m \ , \tag{10.1}$$

where $c = (2/\pi)^{m/2}$, defines an isometry on $\mathfrak{H}$ onto $\mathfrak{H}$ and hence is an elementary operator, whose inverse is given by

$$x(t) = c\int_{-\infty}^{\infty} e^{ist}\hat{x}(s)ds \ . \tag{10.2}$$

Let $\mathscr{D}$ be the class of all functions $x$ in $\mathfrak{H}$ that are linearly absolutely continuous[5] and whose partial derivatives are in $\mathfrak{H}$. A function $x$ in $\mathscr{D}$ is characterized by the condition that $s_1\hat{x}(s), \cdots, s_m\hat{x}(s)$ are square integrable, where $\hat{x}(s)$ is the Fourier transform of $x$. In fact one has

$$-i\frac{\partial x(t)}{\partial t_\alpha} = c\int_{-\infty}^{\infty} e^{ist}s_\alpha x(s)ds \ .$$

The gradient operator $A$ defined by $-i(\partial/\partial t_1), \cdots, -i(\partial/\partial t_m)$ is a closed operator from $\mathfrak{H}$ to the cartesian product $\mathfrak{H}'$ of $\mathfrak{H}$ by itself $m$ times. The domain $A$ is $\mathscr{D}$. It is not difficult to see that $A$ is the closure of the restriction of $A$ to the class of functions of class $C^\infty$ with compact support.

Let $y(t) = [y_1(t), \cdots, y_m(t)]$ be a function in $\mathfrak{H}'$. If $y(t)$ is of class $C^\infty$ and has compact support, then the divergence

$$\operatorname{div} y = i\frac{\partial y_1}{\partial t_1} + \cdots + i\frac{\partial y_m}{\partial t_m}$$

is in $\mathfrak{H}$. This operator from $\mathfrak{H}'$ to $\mathfrak{H}$ is preclosed and its closure is the adjoint $A^*$ of $A$. If $\hat{y}(s)$ is the Fourier transform of $y(t)$ then $y$ is in $\mathscr{D}_{A^*}$ if and only if the sum $s_\alpha\hat{y}_\alpha(s)$ is square integrable. Moreover

$$A^*y = c\int_{-\infty}^{\infty} e^{ist}s_\alpha\hat{y}_\alpha(s)ds \ , \quad (\alpha \text{ summed}) \ .$$

[5] Calkin J. W. "Functions of several variables and absolute continuity I," *Duke Math J.* Vol 6 (1940) pp. 170-186. See also Morrey Jr. C. B. "Functions of several variables and absolute continuity II, Duke Math J. **6** (1940), 187-215.

The elementary operator $R$ associated with $A$ is given by the formulas.

$$(Rx)_\alpha = c\int_{-\infty}^{\infty} e^{ist}\frac{s_\alpha \hat{x}(s)}{|s|}ds \qquad (\alpha = 1, \cdots m),$$

$$R^*y = c\int_{-\infty}^{\infty} e^{ist}\frac{s_\alpha \hat{y}_\alpha(s)}{|s|}ds$$

where $|s|$ is the distance from $s$ to the origin. The carrier of $R^*$ is the set of all functions $y$ in $\mathfrak{H}'$ whose Fourier transforms $\hat{y}_\alpha(s)$ are of the form $s_\alpha\hat{x}(s)/|s|$ such that $\hat{x}(s)$ is in $\mathfrak{H}$. Similarly the carrier of $A^*$ consists of all functions $y$ in $\mathfrak{H}'$ whose Fourier transform is of the form $\hat{y}_\alpha(s) = s_\alpha\hat{x}(s).$, such that $|s|^2\hat{x}(s)$ is in $\mathfrak{H}$. It is easily seen that

$$(A^{*-1}x)_\alpha = c\int_{-\infty}^{\infty} e^{ist}\frac{s_\alpha \hat{x}(s)}{|s|^2}ds$$

$$A^{-1}y = c\int_{-\infty}^{\infty} e^{ist}\frac{s_\alpha \hat{y}_\alpha(s)}{|s|^2}ds .$$

The operator $A^*A$ is, of course, the Laplacian.

The operators $A_\lambda$, $R_\lambda$ described in Corollary 2 to Theorem 9.2 are defined by the formulas

$$(A_\lambda x)_\alpha = c\int_{-\infty}^{\infty} e^{ist}\varphi_\lambda(s)s_\alpha\hat{x}(s)ds$$

$$(R_\lambda x)_\alpha = c\int_{-\infty}^{\infty} e^{ist}\varphi_\lambda(s)\frac{s_\alpha \hat{x}(s)}{|s|}ds$$

where $\varphi_\lambda(s)$ is the characteristic function of the sphere $|s| \leqq \lambda$. The principal spectrum of $A$ is accordingly point set $0 \leqq \lambda < \infty$.

**11. Principal values and principal vectors.** In the present section we shall be interested in certain special points of the principal spectrum of $A$ which we shall call principal values of $A$. Before defining this concept it will be convenient to introduce the concept of the rank of an operator. By the *rank* of an operator $A$ will be meant the dimension of its carrier, or equivalently the dimension of its range. It is clear that the ranks of $A$, $A^*$, $A^{-1}$, $A^{*-1}$, $A^*A$, $AA^*$ are the same. If the rank of $A$ is finite, then $A$ is bounded and reciprocally bounded.

A number $\lambda$ on $0 < \lambda < \infty$ will be said to be a *principal value* of an operator $A$ if the rank of the section $A_{\lambda 0} = \lambda R_{\lambda 0}$ of $A$ described in Theorem 9.2 is not zero. The rank of $A_{\lambda 0}$ will be called the *order of* $\lambda$ as a principal value of $A$ and $A_{\lambda 0}$ will be called the corrersponding *principal section* of $A$. The non-null vectors in the carrier of $A_{\lambda 0}$ will be called the *principal vectors* of $A$ corresponding to $\lambda$. The non-null vectors in range of $A_{\lambda_0}$ will be called the *principal reciprocal vectors* of $A$ corresponding to $\lambda$. The latter are the principal vectors of $A^{-1}$ corresponding to $1/\lambda$. The order of $1/\lambda$ as principal value of $A^{-1}$ is equal to the order of $\lambda$ as a principal value of $A$. A number $\lambda$ is a principal value of $A$ if and only if it is a principal value of $A^*$ and its order as a principal value of $A$ is equal to its order as a principal value of $A^*$. A positive number $\lambda$ is a principal value of $A$ if and only if $\lambda^2$ is a principal value of $A^*A$. Again the order of corresponding principal values are the same. The principal values of $A^*A$ are the nonzero eigenvalues of $A^*A$. The eigenvectors $A^*A$ corresponding to nonzero eigenvalues are the principal vectors of $A$. Similarly the eigenvectors of $AA^*$ corresponding to nonzero eigenvalues are the principal reciprocal vectors of $A$. Principal values of $A$ belong to the principal spectrum of $A$. Isolated points of the principal spectrum of $A$ are principal values of $A$.

A principal value $\lambda$ of $A$ can be characterized in another way. A value $\lambda$ is a principal value of $A$ if and only if there is a non-null vector $x$ in its carrier such that $Ax = \lambda Rx$, where $R$ is its associated elementary operator of $A$. The vector $y = Rx$ is a principal reciprocal vector of $A$ and satisfies the relation $A^*y = \lambda R^*y$. Consequently,

$$Ax = \lambda y \ , \quad A^*y = \lambda x \ . \tag{11.1}$$

Conversely if $\lambda$ is a positive number such that there exist a vector $x \neq 0$ on $\mathscr{D}_A$ and a vector $y \neq 0$ in $\mathscr{D}_A$ such that (11.1) holds, then $\lambda$ is a principal value of $A$, $x$ is a principal vector and $y$ is a principal reciprocal vector. From these remarks, it follows that the principal values of $A$ are the positive eigenvalues of the self-adjoint operator

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

from the cartesian product $\mathfrak{H} \times \mathfrak{H}'$ to $\mathfrak{H} \times \mathfrak{H}'$. It is clear that the foregoing results could have been obtained from the study of this self-adjoint operator. However, the author prefers the more direct approach here given.

THEOREM 11.1. *Suppose the principal spectrum of $A$ apart from $\lambda = 0$ consists of a set of isolated points $\lambda_1, \lambda_2, \cdots$. Then $\lambda_i$ is a principal value of $A$ and has associated with it a unique elementary operator $R_i$ as described in Theorem* 9.2. *The elementary operators $R_1, R_2, \cdots$, are mutually $*$-orthogonal and*

$$A = \Sigma\lambda_i R_i \ , \quad A^* = \Sigma\lambda_i R_i^* \ , \quad A^{*-1} = \Sigma\frac{1}{\lambda_i}R_i \ , \quad A^{-1} = \Sigma\frac{1}{\lambda_i}R_i^* \ .$$

**12. Further results on $*$-commutativity.** Throught this section we shall be concerned with a closed operator $A$ and bounded operator $B$ from $\mathfrak{H}$ to $\mathfrak{H}'$. As a first results we have the following converse of a statement in Lemma 8:4.

LEMMA 12.1. *If $B$ $*$-commutes with $\alpha A + \beta A^{*-1}$ for every pair of positive numbers $\alpha$ and $\beta$, then $B$ $*$-commutes with $A$.*

Suppose that $B$ $*$-commutes with $C = \alpha_1 A + \beta_1 A^{*-1}$ and $D = \alpha_2 A + \beta_2 A^{*-1}$ where $\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2$ are positive numbers such that $\alpha_1\beta_2 - \beta_2\alpha_1 = 1$. These operators have the common domain $\mathscr{D} = \mathscr{D}_A \cap \mathscr{D}_{A^{*-1}}$. The operator $\beta_2 C - \alpha_1 D$ is the restriction of $A$ to $\mathscr{D}$. Since $C^*B = B^*C$ and $D^*B = B^*D$ on $\mathscr{D}$ it follows that $A^*B = B^*A$ on $\mathscr{D}$. In order to show that $A^*B = B^*A$ on $\mathscr{D}_A$ consider a vector $x$ in $\mathscr{D}_A$. Let $x_n = E_n x - E_{1/n}x$, where $E_\lambda = R_\lambda^* R_\lambda$ and $R_\lambda$ is the section of $R$ described in Corollary 2 to Theorem 9.2. The vector $x_n$ is in $\mathscr{D}$ and $x_n \Rightarrow x$, $Ax_n \Rightarrow Ax$. Consequently $A^*Bx_n = B^*Ax_n \Rightarrow B^*Ax$. Since $A^*B$ is closed we have $A^*Bx = B^*Ax$. We have accordingly $A^*B \supseteqq B^*A$. Similarly $AB^* \supseteqq B^*A$ and the lemma is proved.

COROLLARY. *If $\mathfrak{H}' = \mathfrak{H}$ and $\alpha A + \beta A^{*-1}$ is self-adjoint for all pairs of positive numbers $\alpha$ and $\beta$, then $A$ is also self-adjoint.*

This result is obtained from lemma by selecting $B = I$, the identity.

LEMMA 12.2. *Suppose that $\mathfrak{H}' = \mathfrak{H}$ and that $B$ $*$-commutes with $A$. If one of $A$ or $B$ is self-adjoint and positive, the other is self-adjoint.*

Consider first the case in which $A$ is bounded and $A = A^* > 0$. Since $AB = B^*A$ and $AB^* = BA$, the difference $C = B - B^*$ satisfies the relation $AC = -CA$. Hence $A^nC = (-1)^n CA^n$ and $e^{tA}C = Ce^{-tA}$. The inverse $e^{tA}$ is $e^{-tA}$. We have accordingly

$$C = e^{-tA}Ce^{-tA} .$$

Since $A > 0$ it follows from the spectral theorem for $A$ that $\lim_{t=+\infty} e^{-tA}x = 0$ for each $x$ in $\mathfrak{H}$. Consequently $C = 0$, that is, $B = B^*$ for the case here considered.

If $A$ is reciprocally bounded, then $A^{*-1}$ is bounded and $B$ $*$-commutes with $A^{*-1}$. If $A$ is self-adjoint and positive, so also is $A^{*-1} = A^{-1}$ and $B$ $*$-commutes with $A^{-1}$. Hence $B = B^*$ by virture of the result just obtained. If $B = B^* > 0$ then $A^{*-1}$ is self-adjoint and hence $A$ is also self-adjoint.

In the general case if $A = A^* > 0$, then $A + A^{-1}$ is self-adjoint, reciprocally bounded and positive. Since $B$ $*$-commutes with $A + A^{-1}$, it follows that $B = B^*$. If on the other hand $B = B^* > 0$, then $C = \alpha A + \beta A^{*-1}$ is self-adjoint whenever $\alpha$ and $\beta$ are positive. This follows because $C$ is reciprocally bounded and $*$-commutes with $B$. By virtue of the last corollary the operator $A$ is self-adjoint and the lemma is established.

LEMMA 12.3. *Let $R$ and $S$ be the elementary operators associated with $A$ and $B$ respectively. If $B$ $*$-commutes with $A$, then $B$ $*$-commutes with $R$, and $S$ $*$-commutes with $A$.*

In order to prove this result we may suppose, by Lemma 8.3 that $\mathfrak{N}_A = \mathfrak{N}_B$ and $\mathfrak{N}_{A^*} = \mathfrak{N}_{B^*}$. In fact, we may assume that $\mathfrak{N}_A = \mathfrak{N}_B = 0$, $\mathfrak{N}_{A^*} = \mathfrak{N}_{B^*} = 0$. Under these assumptions $P = A^*R = R^*A$ is positive and self-adjoint. Setting $Q = R^*B$ we obtain

$$PQ = A^*RR^*B = A^*B \supseteqq B^*A = B^*RR^*A = Q^*P$$
$$P^*Q = R^*AB^*R \supseteqq R^*BA^*R = QP\,.$$

Hence $Q$ $*$-commutes with $P$ and $Q = R^*B = B^*R$ by the last lemma. Similarly $RB^* = BR^*$. Consequently $B$ $*$-commutes with $R$.

In order to prove that $S$ $*$-commutes with $A$ it is sufficient, by Lemma 12.1, to show that $S$ $*$-commutes with $C = \alpha A + \beta A^{*-1}$, where $\alpha$ and $\beta$ are positive numbers. The operator $C$ is reciprocally bounded and $*$-commutes with $B$. The operator $C^{*-1}$ is bounded and $*$-commutes with $B$. Hence $S$ $*$-commutes $C^{*-1}$ and also with $C$. This completes the proof of the lemma.

LEMMA 12.4. *If an elementary operator $T$ $*$-commutes with $A$, then $T$ $*$-commutes with a section $A_1$ of $A$ if and only if it $*$-commutes with the elementary operator $R_1$ associated with $A_1$.*

Let $A_1$ be a section of $A$ and let $A_0$ be the section of $A$ such that $A = A_0 + A_1$. Let $R_0$, $R_1$ be the elementary operator assococated with $A_0$, $A_1$ respectively. Suppose that $T$ $*$-commutes with $R_1$. Since $T$ $*$-commutes with $R = R_0 + R_1$, it follows that $T$ $*$-commutes with $R_0$. Consequently $T_0 = R_0T^*R_0$ and $T_1 = R_1T^*R_1$ are $*$-orthogonal sections of $T$. The section $T_2 = T - T_0 - T_1$ is $*$-orthogonal to $A$. Consequently the operators $A^*T$ and $AT^*$ are expressible as sums

$$A^*T = A_0^*T_0 + A_1^*T_1\,, \quad AT^* = A_0T_0^* + A_1T_1^*$$

of orthogonal operators. Hence $A_1^*T_1$ and $A_1T_1^*$ are self-adjoint and $T_1$ $*$-commutes with $A_1$. Since $A_1^*T = A_1^*T_1$ and $A_1T^* = A_1T_1^*$, it follows that $T$ $*$-commutes with $A_1$ as was to be proved.

**13. Representations of operators as products.** The present section will be devoted to an extension of Lemma 5.2 and some of its consequences.

THEOREM 13.1. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$ and let $R$ be its associated elementary operator. There is a unique pair of operators $C$ and $D$ from $\mathfrak{H}$ to $\mathfrak{H}'$ such that*

$$C + D = R \,, \quad A = D^{*-1}R^*C = CR^*D^{*-1} \,. \tag{13.1}$$

*The operators $C$ and $D$ are determined by the formulas*

$$C^{-1} = A^{-1} + R^* \,, \quad D^{-1} = A^* + R^* \tag{13.2}$$

*and have $R$ as their associated elementary operator. The operators $C$ and $D$ are bounded and $*$-commute. In addition*

$$C^{-1}RD^{-1} = D^{-1}RC^{-1} = C^{-1} + D^{-1} \tag{13.3a}$$

$$\begin{aligned} A^{-1} &= C^{-1}RD^* = D^*RC^{-1} \,, \quad A^* = C^*RD^{-1} = D^{-1}RC^* \,, \\ A^{*-1} &= C^{*-1}R^*D = DR^*C^{*-1} \,. \end{aligned} \tag{13.3b}$$

This result is an easy consequence of Lemma 5.2. The operator $A_1 = R^*A$ is self-adjoint and nonnegative. Let $C_1$ and $D_1$ be the bounded nonnegative self-adjoint operators related to $A_1$ as described in Lemma 5.2. The operators $C = RC_1$, $D = RD_1$ have the properties described in the theorem, as one readily verifies. An alternate proof can be made by defining $C$ and $D$ by (13.2) and making computations analongous to those made in the proof of Lemma 5.2. Finally, a proof can be made by the use of the integral representation (9.5) of $A$. In this case $C$ and are defined by the formulas.

$$C = \int_0^\infty \frac{\lambda}{1+\lambda} dR_\lambda \,, \quad D = \int_0^\infty \frac{1}{1+\lambda} dR_\lambda \,.$$

THEOREM 13.2. *Let $C$ be the operator related to $A$ as described in Theorem* 13.1. *A bounded operator $B$ $*$-commutes with $A$ if and only if $B$ $*$-commutes with $C$.*

If $B$ $*$-commutes with $A$, then $B$ $*$-commutes with $R$ and $A^{*-1} = c^{*-1} - R$. Consequently $B$ $*$-commutes with $C^{*-1}$ and hence with $C$. Conversely if $B$ $*$-commutes with, C, then $B$ $*$-commutes with $R$, $C^{*-1} = A^{*-1} + R$, $A^{*-1}$ and $A$. This proves the theorem. It is clear that the results described in the theorem hold equally well with $C$ replaced by $D = R - C$.

We state without proof the following

THEOREM 13.3. *Let $C$ be the operator related to $A$ as described in Theorem 13.1 and let*

$$C_t = \left(\frac{1-t}{2}\right)R + tC\ , \quad D_t = R - C_t\,(-1 \leqq t \leqq 1)\ .$$

*The one-parameter family of operators*

$$A_t = D_t^{-1}R^*C_t = C_tR^*D_t^{-1} \quad (-1 \leqq t \leqq 1)$$

*contains $A$ for $t = 1$, $A^{*-1}$ for $t = -1$, $R$ for $t = 0$ and is such that $A_t$ $(-1 < t < 1)$ is bounded and reciprocally bounded.*

As a further result we have

THEOREM 13.4. *Let $C$ and $C_1$ be the bounded operators related respectively to two operators $A$ and $A_1$ as described in Theorem 13.1. Then $A_1$ is a section of $A$ if and only if $C_1$ is a section of $C$.*

Let $R_1$ and $R$ be the elementary operator associated with $A_1$ and $A$ hence also with $C_1$ and $C$. If $A_1$ is a section of $A$, then

$$R_1C^{-1}R_1 = R_1A^{-1}R_1 + R_1R^*R_1 = A_1^{-1} + R_1 = C_1^{-1}\ .$$

Since $R_1$ $*$-commutes with $C$ it follows that $C_1$ is a section of $C$. The converse is readily verified.

The result given in Theorem 13.2 enables us to extend the definition of $*$-commutativity to two unbounded operators $A_1$ and $A_2$. To this end let $C_1$ and $C_2$ be the bounded operators related respectively to $A_1$ and $A_2$ as described in Theorem 13.1. The operators $A_1$ and $A_2$ will be said to *$*$-commute* if the operators $C_1$ and $C_2$ $*$-commute. This definition is consistent with the one given heretofore for the case in which one of the operators is bounded. The result described in Lemma 12.3 is valid without the assumption that $B$ is bounded.

**14. Further decomposition of operators.** In this section we assume that $A$ and $B$ are arbitrary operators from $\mathfrak{H}$ to $\mathfrak{H}'$. As an extension of Theorem 9.1 we have

THEOREM 14.1. *Let $R$ and $S$ be the elementary operators associated with $A$ and $B$ respectively. If $B$ $*$-commutes with $A$, then $A$, $R$, $B$ $S$ are expressible uniquely as sums and differences*

$$(14.1)\qquad \begin{aligned} A &= A_0 + A_+ + A_-\ , & R &= R_0 + R_+ + R_- \\ B &= B_0 + B_+ - B_-\ , & S &= S_0 + R_+ - R_- \end{aligned}$$

*of mutually $*$-orthogonal operators such that* ($\alpha$) $R_0$, $R_+$, $R_-$ *are the elementary operators beloging to* $A_0$, $A_+$, $A_-$ *respectively;* ($\beta$) $S_0$, $R_+$, $R_-$ *are the elementary operators belonging to* $B_0$, $B_+$, $B_-$ *respectively;* ($\gamma$) $A_0$ *and* $R_0$ *are $*$-orthogonal to* $B_0$ *and* $S_0$; ($\delta$) $B_+$ *$*$-commutes with* $A_+$ *and* $B_-$ *$*$-commutes with* $A_-$. *Conversely if* $A, R, B, S$ *are so expressible then* $B$ *and* $S$ *$*$-commute with* $A$ *and* $R$.

In view of the results given in the last section we may assume that $A$ and $B$ are bounded. Suppose that $B$ $*$-commutes with $A$, then $B$ $*$-commutes with $R$, and $S$ $*$-commutes with $A$, by Lemma 12.3. By virtue of Theorem 9.1 applied to $A, R, S$, it is seen that $A, R, S$ have the decomposition (14.1) such that condition ($\alpha$) holds and $S_0$ is $*$-orthogonal to $R_0$. Applying Theorem 9.1 to the operators $B, S, R$ it is seen that $B, S, R$ have the decomposition (14.1) such that condition ($\beta$) holds and $S_0$ is $*$-orthogonal to $R_0$. Since the decomposition of R and $S$ are unique, the decomposition (14.1) holds such that ($\alpha$), ($\beta$), and ($\gamma$) hold. Since

$$A^*B = A_+^*B_+ - A_-^*B_- , \quad AB^* = A_+B_+^* - A_-B_-^*$$

are self-adjoint it follows that each of the operators on the right are also self-adjoint. Consequently $B_+$ $*$-commutes with $A_+$ and $B_-$ $*$-commutes with $A_-$. The converse is immediate and the lemma is established.

COROLLARY. *If* $B$ *$*$-commutes with* $A$ *there is elementary operator* $T$ *such that* $A = TA^*T$ *and* $B = TB^*T$.

The operator $T = S_0 + R$ has this property.

THEOREM 14.2. *Suppose that* $A$ *and* $B$ *$*$-commute and are self-adjoint relative to an elementary operator* $T$. *Then the operators* $A_0$, $A_+, A_-$ $B_0, B_+, B_-$ *described in Theorem* 14.1 *are also self-adjoint relative to* $T$.

Since the elementary operators $R$ and $S$ belonging to $A$ and $B$ are self-adjoint relative to $T$ it follows that

$$R_+ = \tfrac{1}{2}(SR^*S + RS^*R) , \quad R_- = \tfrac{1}{2}(SR^*S - RS^*R) ,$$

are self-adjoint relative to $T$. The same is true for $R_0$ and $S_0$. The theorem follows readily with the help of Lemma 12.4.

The result just given can be extended as described in the following

THEOEM 14.3. *Suppose that* $B$ *$*$-commutes with* $A$ *and* $T$ *is an elementary operator such that* $A = TA^*T$ *and* $B = TB^*T$. *Then* $T$, *A and B can be decomposed uniquely in the form*

$$
\begin{aligned}
T = T_0 &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 \\
A &= A_1 - A_2 + A_3 + A_4 + A_5 - A_6 \qquad (14.2) \\
B &= B_3 - B_4 - B_5 + B_6 + B_7 - B_8
\end{aligned}
$$

*into mutually $*$-orthogonal operators such that ($\alpha$) $T_j$ is the elementary operator associated with $A_j$ ($j = 1, 2, \cdots, 6$) ($\beta$) $T_k$ is the elementary operator associated with $B_k$ ($k = 3, 4, \cdots, 8$) ($\gamma$). The operators $T_i$, $A_j$, $B_k$ $*$-commute.*

In order to prove this result let $A, B, R, S$ have the decomposition (14.1). By virtue of the last theorem the operator $T$ $*$-commutes with each of the operators given in (14.1). Applying Theorem 14.1 to $A_0$ and $T$ we see that $A_0$ can be expressed as the difference $A_0 = A_1 - A_2$ of $*$-orthogonal operators $A_1$ and $A_2$ whose associated elementary operators $T_1$ and $T_2$ are sections of $T$. Similarly $B_0 = B_7 - B_8$, where $B_7$ and $B_8$ are $*$-orthogonal operators whose associated elementary operators $T_7$ and $T_8$ are section of $T$. Applying Theorem 14.1 to $A_+$, $T$; $B_+$, $T$; $A_-$, $T$ and $B_-$, $T$ we obtain differences $A_+ = A_3 - A_4$, $B_+ = B_3 - B_4$, $A_- = A_5 - A_6$, $B_- = B_5 - B_6$ of $*$-orthogonal operators such that $A_i$, $B_i$ have the same associated elementary operator $T_i$, a section of $T$.

From these relations one obtains the decomposition (14.2), the section $T_0$ of $T$ being $*$-orthogonal to $A$ and $B$. In view of Theorem 14.1, the operator $T_i$, $A_j$, $B_k$ $*$-commute. This proves the theorem.

THEOREM 14.4. *Let $A_{\lambda+}, A_{\lambda0}, A_{\lambda-}$ ($0 < \lambda < \infty$) be the sections of $A$ described in Theorem 9.2 and let $B_{\mu+}, B_{\mu0}, B_{\mu-}$ ($0 < \mu < \infty$) be the corresponding sections of $B$. Suppose that $B$ $*$-commutes $A$. Then the operators $B, B_{\mu+}, B_{\mu0}, B_{\mu-}, B_\mu = B_{\mu0} + B_{\mu-}$ $*$-commute with each of the operators $A, A_{\lambda+}, A_{\lambda0}, A_{\lambda-}, A_\lambda = A_{\lambda0} + A_{\lambda-}$.*

In order to prove this result recall, by Theorem 9.2, that $T = R_{\lambda+} - R_{\lambda-}$ is the elementary operator associated with $C = A - \lambda R$. Since $B$ $*$-commutes with $A$ it $*$-commutes with $R, C, T, RC^*T$ and hence also with

$$C_+ = C + RC^*T = 2(A_{\lambda+} - \lambda R_{\lambda+}) , \quad C_- = C - RC^*T = 2(A_{\lambda-} - \lambda R_{\lambda-}) .$$

The elementary operators of $C_+$ and $C_-$ are $R_{\lambda+}$ and $R_{\lambda-}$ respectively. It follows that $B$ $*$-commutes with $A_{\lambda+}, A_{\lambda-}$ and hence also with $A_{\lambda0}$. Similarly $B_{\mu+}, B_{\mu0}, B_{\mu-}$ $*$-commutes with $A$. The operators therefore $*$-commute, as described in the theorem.

THEOREM 14.5. *Suppose that $B$ $*$-commutes with $A$ and that $T$ is* *an elementary operator such that* $A = TA^*T$, $B = TB^*T$. *Let* $A_\lambda$, $T_\lambda$ $(-\infty < \lambda < \infty)$ *be the sections* $A$ *and* $T$ *described in the corollary to Theorem* 9.3. *Let* $\tilde{B}_\mu$, $\tilde{T}_\mu$ $(-\infty < \mu < \infty)$ *be the sections of* $B$ *and* $T$ *obtained by having* $B$ *playing the role of* $A$ *in this corollary. Then* $A, A_\lambda, T_\lambda, B, \tilde{B}_\mu, \tilde{T}_\mu, T_{\lambda\mu} = T_\lambda T_* \tilde{T}_\mu$ *$*$-commute with each other. Moreover*

$$A = \int_{-\infty}^{\infty} \lambda d T_\lambda = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lambda d T_{\lambda\mu}\ , \quad B = \int_{-\infty}^{\infty} \mu d \tilde{T}_\mu = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu d T_{\lambda\mu}\ .$$

*If the scalars are the complex numbers, then*

$$A + iB = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\lambda + i\mu) d T_{\lambda\mu}\ .$$

**15. Bounded normal operators relative to** $T$. In the present section it will be assumed that the scalars are the complex numbers. Let $T$ be an elementary operator from $\mathfrak{H}$ to $\mathfrak{H}'$ and let $\mathfrak{A}(T)$ be the class of all bounded operators $A$ such that the relation

$$AT^*T = TT^*A = A \tag{15.1}$$

holds. Let $\mathscr{B}(T)$ be the class of all operators $A$ in $\mathfrak{A}(T)$ that $*$-commute with $T$. These are the operators $A$ in $\mathfrak{A}(T)$ that satisfy the relation $A = TA^*T$, that is, the operators in $\mathscr{B}(T)$ that are self-adjoint relative to $T$. Every operator $A$ in $\mathfrak{A}(T)$ is expressible uniquely in the form $A = A_1 + iA_2$ where $A_1$ and $A_2$ are in $\mathfrak{B}(T)$. The operators $A_1$ and $A_2$ are given by the formulas

$$A_1 = \tfrac{1}{2}(A + TA^*T)\ , \quad A_2 = \tfrac{1}{2i}(A - TA^*T)\ . \tag{15.2}$$

It should be observed that, by virtue of Lemma 8.4, two operators $A$ and $B$ in $\mathscr{B}(T)$ $*$-commute if and only if $AT^*B = BT^*A$.

Let $\mathscr{C}(T)$ be the class of all operators $A$ in $\mathfrak{A}(T)$ such that $TA^*A = AA^*T$. An operator $A$ in $\mathscr{C}(T)$ will be said to be *normal with respect to* $T$. It is clear that an operator that is self-adjoint relative to $T$ is also normal relative to $T$. An operator $A$ in $\mathfrak{A}(T)$ is in $\mathscr{C}(T)$ if and only if the operators $A_1$ and $A_2$ defined by (15.2) $*$-commute. In order to prove this fact observe that

$$B = A_1T^*A_1 + A_2T^*A_2\ , \quad C = A_1T^*A_2 - A_2T^*A_1$$

are in $\mathscr{B}(T)$ and

$$TA^*A = B + iC\ , \quad AA^*T = B - iC\ .$$

Consequently $TA^*A = AA^*T$ if and only if $C = 0$, that is, if and only if $A_1$ $*$-commutes with $A_2$. If $A$ is in $\mathscr{C}(T)$, there is by virtue of Theorem 14.5, a section $T_\alpha$ corresponding to each complex number $\alpha$

such that

$$A = \int \alpha d T_\alpha$$

where the integral is taken over the complex plane.

Given an operator $A$ in $\mathscr{C}(T)$ let $\mathscr{C}(A, T)$ be the class of all operators $B$ in $\mathscr{C}(T)$ such that $TA^*B = BA^*T$ and $AT^*B = BT^*A$. If $B$ is in $\mathscr{C}(A, T)$, then $TB^*A = AB^*T$ also. Moreover, $TB^*T$ is in $\mathscr{C}(A, T)$. Let $\mathfrak{M}(A, T)$ be all operators $B$ in $\mathscr{C}(T)$ such that $\mathscr{C}(A, T) \subset \mathscr{C}(B, T)$. If $B$ and $C$ are in $\mathfrak{M}(A, T)$, so also are $\alpha B + \beta C$ and $BT^*C$, where $\alpha$ and $\beta$ are complex mumbers. Moreover

$$\| BT^*C \| \leqq \|B\| \| C \| .$$

It follows that if we define $BT^*C$ to be the product of $B$ and $C$, the class $\mathfrak{M}(A, T)$ is a Banach algebra with the operator $T$ as a unit element and $TB^*T$ as an involution. The subclass $\mathscr{L}(A, T)$ of all operators $B$ in $\mathfrak{M}(A, T)$ such that $B = TB^*T$ form a Banach algebra over the reals.

**16. Compact and reciprocally compact operators.** An operator $A$ from $\mathfrak{H}$ to $\mathfrak{H}'$ will be said to be *compact* if given a bounded sequence $\{x_n\}$ in $\mathscr{D}_A$, the sequence $\{Ax_n\}$ has a strongly convergent subsequence. An operator $A$ will be said to be *reciprocally compact* if its reciprocal is compact. Since compact operators are bounded, it follows that reciprocally compact operators are reciprocally bounded. It should be observed that an operator $A$ is compact if an only if given a weakly convergent sequence $\{x_n\}$ in $\mathscr{D}_A$, the sequence $\{Ax_n\}$ converges strongly.

THEOREM 16.1 *An operator $A$ is of finite rank if and only if it is compact and reciprocally bounded. An operator $A$ is of finite rank if and only if it is bounded and reciprocally compact An operator $A$ is of finite rank if and only if it is compact and reciprocally compact.*

Suppose that $A$ is compact and reciprocally bounded. Then $\mathscr{C}_A$ and $\mathscr{R}_A$ are closed. Let $\{x_n\}$ be a sequence in $\mathscr{C}_A$ converging weakly to a point $x_0$. Since $A$ is compact $y_n = Ax_n$ converges strongly to $y_0 = Ax_0$. it follows that $x = A^{-1}y_n$ converges strongly to $x_0 = A^{-1}y$. Consequently weak convergence on $\mathscr{C}_A$ implies strong convergence. It follows that $\mathscr{C}_A$ is of finite dimension. Hence $A$ is of finite rank. Conversely if $A$ is of finite rank, then $A$ is compact and reciprocally bounded. The remaining statements follow readily.

THEOREM 16.2 *Let $A$ be the sum $A = B + C$ of two $*$-orthogonal* *operators B and C from $\mathfrak{H}$ to $\mathfrak{H}'$. Then A is compact, reciprocally compact, bounded, or reciprocally bounded if and only if B and C have the same property. If C is of finite rank, then A is compact, reciprocally compact, bounded, or reciprocally bounded if and only if B has the same property.*

The first conclusion is immediate from the definitions of the terms involved. The second follows from the first. In view of the second statement sections of finite rank can be disregarded in determining the properties of compactness, reciprocal compactness, boundedness and reciprocal boundedness.

THEOREM 16.3. *An operator A is compact if and only if its reciprocally bounded sections are of finite rank. Similarly, an operator A is reciprocally compact if and only if its bounded sections are of finite rank.*

The second statement follows from the first. If $A$ is compact, its sections are compact and hence its reciprocally bounded sections are of finite rank, by Theorem 16.1. Suppose now that $A$ is an operator whose reciprocally bounded sections are finite rank. Then as was seen in § 9, given a number $\lambda > 0$, the operator $A$ can be written as the sum $A = A_{\lambda+} + A_\lambda$ of two $*$-orthogonal operators such that $A_{\lambda+}$ is reciprocally bounded and $A_\lambda$ is of norm at most $\lambda$. In view of our hypothese $A_{\lambda+}$ is of finite rank and hence is compact. It follows that $A$ is bounded and that $\mathscr{D}_A = \mathfrak{H}$. Let $\{x_n\}$ be a sequence in $\mathscr{D}_A$ converging weakly to zero. Then,

$$|| Ax_n || \leqq || A_{\lambda+}x_n || + || A_\lambda x_n || \leqq || A_{\lambda+}x_n || + \lambda || x_n || .$$

Since $A_{\lambda+}$ is compact we have $\lim_{n=\infty} || A_{\lambda+}x_n || = 0$. Consequently $\lim_{n=\infty}\sup || Ax_n || \leqq \lambda M$ where $M$ is a bound for the sequence $|| x_n ||$. Since $\lambda$ is arbitrary it follows that $Ax_n \Rightarrow 0$ and hence that $A$ is compact, as was to be proved.

THEOREM 16.4. *An operator A is compact if and only if its spectrum (apart from $\lambda = 0$) consists of a bounded set of isolated principal values of finite order. It is reciprocally compact if and only if its spectrum consists of isolated principal values of finite order bounded away from zero.*

Again, the second statement follows from the first. In order to prove the first statement we use the decomposition $A = A_{\lambda+} + A_\lambda$ of $A$ into the $*$-orthogonal sections described in § 9, where $\lambda$ is an arbitrary positive number. The points of the spectrum of $A$ that exceed $\lambda$ comprise the spectrum of $A_{\lambda +}$. The remaining points of the spectrum of $A$ comprise the spectrum of $A_\lambda$. If $A$ is compact, then $A_{\lambda +}$ is of finite rank. Consequently the points of the spectrum of $A$ that exceed $\lambda$ consist of a finite number of principal values of $A_{\lambda +}$, each being of finite order. Since $\lambda$ is arbitrary it follows that the spectrum of $A$ consists of a bounded set of isolated principal values of finite order. Conversely if the spectrum of $A$ consists of a bounded set of isolated principal values of finite order, then $A_{\lambda +}$ is of finite rank for every value of $\lambda$. Consequently $A$ is compact, as was to be proved.

The following corollary is immediate.

COROLLARY. *If one of the operators $A, A^*, A^*A, AA^*$ is compact, then the others are compact. Similarly, if one of the operators $A$, $A^*, A^*A, AA^*$ is reciprocally compact so also are the others.*

**17. Operators of finite character.** By the *nullity* of an operator will be meant the dimension of its null space. An operator $A$ will be said to be of *finite character* if it is of finite nullity and if its bounded sections have finite rank, or equivalently by, if it is of finite nullity and is reciprocally compact. Operators of this type play an important role in the calculus of variations and in existence theorems for elliptic partial differential equations. In fact the condition of ellipticity is equivalent to the condition that an operator be of finite character relative to a suitably chosen norm, provided the domain of the independent variable is bounded. The operators described in § 4 are of finite character.

THEOREM 17.1. *An operator $A$ is of finite character if and only if given a bounded sequence $\{x_n\}$ in $\mathscr{D}_A$ such that $\{Ax_n\}$ is also bounded, then $\{x_n\}$ has a strongly convergent subsequence. An operator $A$ is of finite character if and only if it is of finite nullity and given a sequence $\{x_n\}$ in the carrier $\mathscr{C}_A$ of $A$ such that $\{Ax_n\}$ is bounded, then $\{x_n\}$ has a strongly convergent subsequence.*

Suppose that $A$ is of finite character. Then the nullity of $A$ is finite, and $A^{-1}$ is compact. Let $\{x_n\}$ be a sequence in $\mathscr{C}_A$ such that $\{Ax_n\}$ is bounded. Setting $y_n = Ax_n$ we have $x_n = A^{-1}y_n$. Since $\{y_n\}$ is in the carrier of $A^{-1}$ and $A^{-1}$ is compact it follows that $\{x_n\}$ has a strongly convergent subsequence. Suppose next that $\{x_n\}$ is a bounded sequence in $\mathscr{D}_A$ such that $\{Ax_n\}$ is bouneded. Then $x$ is expressible in the form $x_n = x_{n0} + x_{n1}$, where $x_{n0} \in \mathfrak{N}_A$ and $x_{n1} \in \mathscr{C}_A$. Since $\mathfrak{N}_A$ is of finite dimension and $Ax_n = Ax_{n1}$, the boundedness conditions imposed imply that $\{x_n\}$ has a strongly convergent subsequence. The criteria given in the theorem are accordingly necessary conditions for $A$ to be of character.

Suppose conversely that every bounded sequence $\{x_n\}$ in $\mathscr{D}_A$ for which $\{Ax_n\}$ is bounded has a strongly convergent subsequence. Then the nullity of $A$ is finite, since otherwise there would exist a orthonormal sequence $\{x_n\}$ in $\mathfrak{N}_A$. Such a sequence would have $Ax_n = 0$ and would possess no strongly convergent subsequence. The reciprocal $A^{-1}$ is bounded. If this were not so we could select a sequence $\{x_n\}$ in $\mathscr{C}_A$ such that $\| x_n \| = 1$ and $\| Ax_n \| \leqq 1/n$. In view of the last inequality the sequence could be chosen so as to converge strongly to a vector $x_0$. Since $A$ is closed it would follow that $x_0$ would be in $\mathscr{C}_A$, $\| x_0 \| = 1$ and $Ax_0 = 0$. This is impossible. Hence $A^{-1}$ is bounded. Consider next a bounded sequence $\{y_n\}$ in $\mathscr{C}_{A-1}$. Set $x_n = A^{-1}y_n$. Since $A^{-1}$ is bounded the sequence $\{x_n\}$ is also bounded and hence, by our criterion, has a strongly convergent subsequence. The operator $A^{-1}$ is therefore compact. Hence $A$ is of finite character, as was to be proved.

COROLLARY 1. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$ and let $B$ be the operator that maps a point $x$ in $\mathscr{D}_A$ into the pair $\{x, Ax\}$ in $\mathfrak{H} \times \mathfrak{H}'$. The nullity of $B$ is zero. Moreover $A$ is of finite character if and only if $B$ is of finite character.*

COROLLARY 2. *If $B$ and $C$ are $*$-orthogonal operators and $C$ is finite rank, then $A = B + C$ is of finite character if and only if $B$ is of finite character.*

Let $T$ be an elementary operator such that $A = TA^*T$ and let $R$ be the elementary operator associated with $A$. By Theorem 9.1, T is expressible uniquely in the form $\mathrm{T} = T_0 + R_+ - R_-$ where $T_0, R_+, R_-$ are $*$-orthogonal and $\mathrm{R} = R_+ + R_-$. The operator $T$ will be said to be *of finite index relative* to $A$ in case one of the operators $R_+$ and $R_-$ is of finite rank. The minimum of the ranks of $R_+$ and $R_-$ will be called the *index* of $T$. Clearly the index of $T$ is the minimum of the ranks of the sections $A_+ = R_+A^*R_+$ and $A_- = R_-A^*R_-$ of $A$. In the self-adjoint case with $T = I$, the identity, this index is the smaller of the ranks of the orthogonal nonnegative operators $A_1, A_2$ such that $A = A_1 - A_2$. In this event this index is frequently called *the index* of $A$ or of the quadratic form $(Ax, x)$.

THEOREM 17.2. *Let $T$ be an elementary operator such that $TA^*T = A$. Every bounded sequence $\{x_n\}$ such that $\{(Ax_n, Tx_n)\}$ is bounded has a strongly convergent subsequence if and only if $A$ is of finite character and $T$ is of finite index relative to $A$.*

This criterion, stated in a somewhat different form, is the basis for a large class of existence theorems for weak solutions of partial differential equations.

Since $\| T^*Ax \| = \| Ax \|$ it follows from Theorem 17.1 that $A$ is finite character if and only if $T^*A$ is of finite character. Moreover $T^*A$ is self-adjoint. It follows that it is sufficient to consider the case $A = A^*$ and $T = I$. Let $\{x_n\}$ be a bounded sequence such that $\{Ax_n\}$ is bounded. From the inequality

$$|(Ax, x)| \leqq \| x \| \| Ax \|$$

it follows that $\{(Ax_n, x_n)\}$ is bounded. Consequently if the criterion described in the theorem holds, then $\{x_n\}$ has a strongly convergent subsequence. By virtue of Theorem 17.1 the operator $A$ is of finite character. It remains to show that if $A$ is expressed as the difference $A = B - C$ of two orthogonal nonnegative self-adjoint operators, then either $B$ or $C$ is of finite rank. If this were not the case one could select an orthogonal sequence $\{y_n\}$ in $\mathscr{C}_B$ and $\{z_n\}$ in $\mathscr{C}_C$ such that $(By_m, y_n) = (Cz_m z_n) = \delta_{mn}$. The vectors $x_n = \alpha_n(y_n + z_n)$ then satisfy the relation

$$(Ax_m, x_n) = \alpha_m\alpha_n[(By_m, y_n) - (Cy_m, y_n)] = 0 \ \ (m, n = 1, 2, 3 \cdots) .$$

Choosing $\alpha_n$ such that $\| x_n \| = 1$, we obtain an orthogonal sequence $\{x_n\}$ such that $(Ax_n, x_n) = 0$. This sequence cannot have a strongly convergent subsequence. Consequently either $B$ or $C$ is of finite rank, as was to be proved.

Conversely suppose that $B$ or $C$ is of finite rank and $A = B - C$ is of finite character. For definiteness suppose that $C$ is of finite rank. Then $B$ is of finite character. Consider now a bounded sequence of vectors $\{x_n\}$ in $\mathscr{C}_A$, such that $\{(Ax_n, x_n)\}$ is bounded. Select $y_n$ in $\mathscr{C}_B$ and $z_n$ in $\mathscr{C}_C$ such that $x_n = y_n + z_n$. Then

$$(Ax_n, x_n) = (By_n, y_n) - (Cz_n, z) .$$

It follows that $\{(By_n, y_n)\}$ is bounded. Consequently, $\{y_n\}$ has a convergent subsequence. Since $\{z_n\}$ is restricted to a finite dimensional subspace of $\mathscr{D}_A$, it follows that $\{x_n\}$ has a strongly convergent subsequence. This completes the proof of the theorem.

THEOREM 17.3. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$ of finite character and let $B$ be an operator from $\mathfrak{H}$ to a Hilbert space $\mathfrak{H}''$. If $\mathscr{D}_B \subset \mathscr{D}_A$, then $B$ is of finite character.*

Since $\mathscr{D}_B \subset \mathscr{D}_A$ there is a constant $\alpha$ such that if $x$ is in $\mathscr{D}_B$ then

$$\| Ax \| \leqq \alpha \{\| Bx \| + \| x \|\} .$$

If $\{x_n\}$ is a sequence in $\mathscr{D}_B$ such that $\| x_n \|$, $\| Bx_n \|$ are bounded, then $\| Ax_n \|$ is bounded also. It follows from Theorem 17.1 that $\{x_n\}$ converges strongly in subsequence. Consequently $B$ is of finite character, by virtue of Theorem 17.1.

A linear transformation $K$ from $\mathfrak{H}$ to $\mathfrak{H}''$ will be said to be *compact relative* to $A$ if $\mathscr{D}_K \supset \mathscr{D}_A$ and if for every bounded sequence $\{x_n\}$ in $\mathscr{D}_A$ such that $\{Ax_n\}$ is bounded, the sequence $\{Kx_n\}$ has a strongly convergent subsequence.

THEOREM 17.4. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$ of finite character. Let $K$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}''$ such that $\mathscr{D}_K \supset \mathscr{D}_A$. Then $K$ is compact relative to $A$ if and only if given a positive number $\alpha$ there is a number $\beta$ such that the inequality*

$$\| Kx \| \leqq \alpha \| Ax \| + \beta \| x \| \tag{17.1}$$

*holds on $\mathscr{D}_A$.*

Suppose that $K$ is compact relative to $A$. Suppose further there is an $\alpha > 0$ such that (17.1) holds on $\mathscr{D}_A$ for no constant $\beta$. We can select a non-null sequence $\{x_n\}$ such that

$$\| Kx_n \| \geqq \alpha \| Ax_n \| + n \| x_n \| .$$

We can suppose that $\| Kx_n \| = 1$. Then $\| Ax_n \|$ is bounded and $x_n \Rightarrow 0$. Since $K$ is compact relative to $A$ it follows that $Kx_n \Rightarrow 0$, in subsequence, contrary to the fact that $\| Kx_n \| = 1$.

Suppose that (17.1) holds as stated. Let $\{x_n\}$ be a bounded sequence such that $\{Ax_n\}$ is bounded. A subsequence, rename it $\{x_n\}$, converges strongly to a vector $x_0$. The point $x_0$ is in $\mathscr{D}_A$ since $A$ is closed. Given $\alpha > 0$ choose $\beta$ so that (17.1) holds. Then

$$\| K(x_n - x_0) \| \leqq \alpha \| A(x_n - x_0) \| + \beta \| x_n - x_0 \|$$

and

$$\limsup_{n=\infty} \| Kx_n - Kx_0 \| \leqq \alpha \limsup \| A(x_n - x_0) \| .$$

Since $\alpha$ is arbitrary it follows that $\{Kx_n\}$ converges strongly to $Kx_0$. The operator $K$ is therefore compact relative to $A$, as was to be proved.

THEOREM 17.5. *Let $A$ and $K$ be operators from $\mathfrak{H}$ to $\mathfrak{H}'$ such that $K$ is compact relative to $A$. The operator $A$ is of finite character if and only if $B = A + K$ is an operator of finite character.*

In order to see that $B$ is closed when $A$ is of finite character let $\{x_n\}$ be a sequence such that $x_n \Rightarrow x_0$, $Bx_n \Rightarrow y_n$. In view of (17.1) with $\alpha < 1$

$$\| Kx_n \| \leqq \alpha \| Ax_n \| + \beta \| x_n \| \leqq \alpha \| Bx_n \| + \alpha \| Kx_n \| + \beta \| x_n \| .$$

We see that $\{Kx_n\}$ is bounded. Consequently $\{Ax_n\}$ is bounded also. It follows that $\{Kx_n\}$ converges to $Kx_0$ and that $Ax_n \Rightarrow y_0 - Kx_0$. Since $A$ is closed $y_0 - Kx_0 = Ax_0$, that is, $y_0 = Bx_0$. The operator $B$ is accordingly closed. Since $B$ and $A$ has the same domain, $B$ is of finite character. Conversely if $B$ is an operator of finite character, so also is $A$ since $\mathscr{D}_B = \mathscr{D}_A$.

In a similar manner we obtain

THEOREM 17.6. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}'$ and let $K$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}''$ that is compact relative to $A$. Let $B$ be the operator that maps a point $x$ in $\mathscr{D}_A$ into the point $\{Ax, Kx\}$ in $\mathfrak{H}' \times \mathfrak{H}''$. Then $A$ is of finite character if and only if $B$ is an operator of finite character.*

THEOREM 17.7. *Let $A$ be an operator from $\mathfrak{H}$ to $\mathfrak{H}$ and suppose that every bounded sequence $\{x_n\}$ in $\mathscr{D}_A$ for which $\{(Ax_n, x_n)\}$ is bounded has a strongly convergent subsequence. Then $A$ is of finite character. Moreover, a linear subclass $\mathscr{B}$ of $\mathscr{D}_A$ on which $(Ax, x) = 0$ is of finite dimension.*

The proof of this result is like that of Theorem 17.2 and is equivalent to the result given in Theorem 17.2 is $A = A^*$. In this theorem the role of $(Ax, x)$ may be replaced by $(Ax, x) + (x, Ax)$.

**18. Elliptic partial differential equations.** The purpose of the present section is to indicate the connections between the results described in the preceding pages with the theory of elliptic partial differential equations. To this end let $\Omega$ be a bounded region in an $m$-dimensional Euclidean space of points $t = (t_1, \cdots, t_m)$. The boundary of $\Omega$ will be assumed to be nonsingular and to be of class $C^\infty$. The results given below are valid under much weaker assumptions but we shall not consider them at this time.

The symbol $\alpha$ will be used to designate an $m$-tuple $\alpha = (\alpha_1, \cdots, \alpha_m)$ of nonnegative integers. Let $|\alpha| = \alpha_1 + \cdots + \alpha_m$. The symbol $D_\alpha$. will be used to denote the differential operator

$$D_\alpha = (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha 1} \cdots \partial t_m^{\alpha m}} .$$

Let $\mathfrak{H}_{nk}$ be the class of all Lebesgue square integrable complex valued functions $x_\alpha^j(t)$ $(t \in \Omega; j = 1, \cdots, n; |\alpha| = k)$, normalized so as to be equal to the limit of their integral means whenever this limit exists and to be zero elsewhere. The class $\mathfrak{H}_{nk}$ with

$$(x, y)_k = \int_\Omega x_\alpha^j(t)\overline{y_\alpha^j(t)}dt \ (j \leqq n, |\alpha| = k) ,$$

($j$ and $\alpha$ summed) as its inner product forms a Hilbert space over the field of complex number. The symbol $x_k$ will be used to denote an element in $\mathfrak{H}_{nk}$. The cartesian product of $\mathfrak{H}_{n0}, \mathfrak{H}_{n1}, \cdots, \mathfrak{H}_{nk}$ will be denoted by $\mathfrak{H}_n^k$. Its elements $x$ are of the form $x = (x_0, x_1, \cdots, x_k)$. An element $x$ in $\mathfrak{H}_n^k$ such that $x_0$: $x^j(t)$ is of class $C^k$ and such that $x_r$ is the set of derivatives $x_\alpha^j = D_\alpha x^j \, |\alpha| = r$ of order $r$ will be denoted by $\mathscr{C}_n^k$. The closure of $\mathscr{C}_n^k$ will be denoted by $\mathscr{D}_n^k$. In view of our normalization of the functions in $\mathfrak{H}_{nr}$, it can be shown the formula $x_\alpha^j(t) = D_\alpha x^j(t)$ $|\alpha| = r \leqq k$ holds almost everywhere on $\Omega$, where $x^j(t)$ are the functions defining $x_0$ in $(x_0, x_1, \cdots, x_k)$. The projection of $\mathscr{D}_k^n$ in $\mathfrak{H}_{nk}$ will be denoted by $\mathscr{D}_{nk}$. The class $\mathscr{D}_{nk}$ is a closed subspace of $\mathfrak{H}_{nk}$.

Since an element $(x_0, x_1, \cdots, x_k)$ in $\mathscr{D}_n^k$ is uniquely determined by its inital element $x_0$, a function $G_k$ on $\mathfrak{H}_n^0 = \mathfrak{H}_{no}$ to $\mathfrak{H}_n^k$ is defined. The range of the function is $\mathscr{D}_n^k$. Its domain $\mathscr{D}_{G_k}$ is the projection of $\mathscr{D}_n^k$ on $\mathfrak{H}_{n0}$. The functions $G_k$ $(k = 1, 2, 3, \cdots)$ have the following properties:

(1) The function $G_k$ is a closed and dense linear transformation from $\mathfrak{H} = \mathfrak{H}_{n0}$ to $\mathfrak{H}_n^k$.

(2) The operator $G_k$ $(k > 0)$ is of finite character and zero nullity.

(3) The operator $G_j$ $(j < k)$ is compact relative to $G_k$.

These results follow from well known connections between partial derivatives and can be found in papers on this subject.

Let $C$ be a bounded operator from $\mathscr{D}_n^k$ $(k > 0)$ to a Hilbert space $\mathfrak{H}_{q0}$. Given a restriction $B_k$ of $G_k$ that is closed and dense in $\mathfrak{H} = \mathfrak{H}_{n0}$, the product $A_k = CB_k$ defines a dense linear transformation. Such an operator will be said to be *elliptic* in case it is closed. This definition of ellipticity is an extension of the one usually given. An elliptic operator of this type is necessarily of finite character by Theorem 17.3 since $B_k$ has this property. It is clear that $A_k$ is elliptic if and only if there is a constant $h > 0$ such that

$$\| B_k x \| \leqq h \, [\| A_k x \| + \| x \|] \tag{18.1}$$

for all $x$ in $\mathscr{D}_{B_k}$. It should be observed that if $A_k$ $(k \geqq 1)$ is elliptic, then the equation $A_k x = y$ has a solution $x$ for all $y$ orthogonal to the solutions $z$ of $A^*{}_k z = 0$. The existence of strong solutions is thereby established.

In order to illustrate these ideas consider the case in which the operator C is defined by a formula of the form

$$c_\alpha^{\delta j}(t)x_\alpha^j(t) \quad (\delta = 1, \cdots, q; j = 1, \cdots, n: |\alpha| \leqq k) \tag{18.2}$$

where $j$ and $\alpha$ are summed and the coefficients are continuous on the closure of $\Omega$. Select $B_k = G_k$. Then $A_k = CB_k$ is elliptic, that is, an inequality of the form (17.1) holds if and only if the following two conditions are met:

(1) Given a point $t$ in $\Omega$ there is no non-null set of real numbers $\xi = (\xi_1, \cdots, \xi_m)$ and no non-null set of complex numbers $\zeta = (\zeta^1, \cdots, \zeta^n)$ such that the relations

$$C_\alpha^{\sigma j}\xi^\alpha\zeta^j = 0 \quad (\sigma = 1, \cdots, q, |\alpha| = k) \tag{18.3}$$

holds, where $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2} \cdots \xi_m^{\alpha_m}$.

(2) Given a point $t$ on the boundary of $\Omega$ the relations (18.3) cannot be satisfied by non-null complex numbers $\zeta = (\zeta^1, \cdots, \zeta^n)$ and by non-null numbers $\xi = (\xi_1, \cdots, \xi_m)$ whose normal component is complex and whose tangential components are real.

If the first of these conditions is met then $A_k = CB_k$ is elliptic, where $B_k$ is the restriction of $G_k$ defined by the closure of the subclass of $\mathscr{D}_n^k$ whose elements are continuous and have $x_\alpha^j(t) = 0$ ($|\alpha| < k$) on the boundary of $\Omega$.

These and related results can be found in the recent papers[6] on partial differential equations by Aronszajn Browder, Friedrichs, Gaarding, Hormander, Morrey, Nirenberg, Schechter and the author.

For a list of references see Hestenes, Magnus R. Quadratic Variational-theory and linear elliptic partial differential equations to be published soon in the Transactions of the American Mathematical Society.

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