HIGH SUBGROUPS OF ABELIAN TORSION GROUPS

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In what follows, all groups considered are Abelian. Let G^1 be the subgroup of elements of infinite height in an Abelian group G (see [2]). A subgroup H of G maximal with respect to disjointness from G^1 will be called a *high* subgroup of G. If N is a subgroup of G, H will be called *N-high* if and only if H is a subgroup of G maximal with respect to disjointness from N. Zorn's lemma guarantees the existence of N-high subgroups for any subgroup N of G. A group E minimal divisible among those groups containing G will be called a *divisible hull* of G. Unless otherwise specified, the notation and terminology will be essentially that of L. Fuchs in [1].

The main theorem says that high subgroups of Abelian torsion groups are pure. After proving some preparatory lemmas, we will prove the main theorem. Then we will discuss Fuchs' Problem 4 and list some of the more important properties of high subgroups. Finally we will state some generalizations.

A lemma describing N-high subgroups is

LEMMA 1. *Let G be a primary group with H an N-high subgroup of G, D a divisible hull of G, A any divisible hull of H in D (this means that $A \subset D$), and B any divisible hull of N in D.*
Then
(a) $D = A \oplus B$.
(b) $A \cap G = H$, and H and $B \cap G$ are neat in G.
(c) Any complementary direct summand of A in D containing N is a divisible hull of N.
(d) Any complementary direct summand of B in D containing H is a divisible hull of H.
(e) D is a divisible hull of any subgroup M with $(H \oplus N)[p] \subset M \subset G$.
(f) $D[p] = (H \oplus N)[p] = H[p] \oplus N[p] = G[p]$.
(g) All N-high subgroups H of G may be obtained as $E \cap G$, where E is a complementary direct summand of a divisible hull F of N in D.

Proof. When $N = 0$ there is nothing to prove, so suppose $N \neq 0$.

(a)   To see this, we first show that $A \cap B = 0$. If $0 \neq x \in A \cap B$, then by Kulikov's lemma ([1], p.66) there exist positive integers r and s such that $p^r x \neq 0 \neq p^s x$ and $p^r x \in H$, $p^s x \in N$. But $0 \neq p^{\max(r,s)} x \in H \cap N = 0$, which is impossible. The divisibility of $A \oplus B$ provides a decomposition $D = A \oplus B \oplus C$. If $C \cap G \neq 0$, H will not be N-high in G, whence $C \cap G = 0$. By Kulikov's lemma, $C = 0$, and we have $D = A \oplus B$ as stated.

(b)   That $A \cap G = H$ is clear. The neatness follows from [1], p. 92, h,

(c)   and (d) follow from (a) and the definition of H,

(e)   and (f) follow from Kulikov's lemma,

(g)   follows from (a) and (b). This concludes the proof of Lemma 1.

In what follows $\langle x \rangle$ will denote the cyclic subgroup of G generated by $x \in G$. An interesting and helpful lemma is

LEMMA 2.   *Let N be a subgroup of a primary group G, H an N-high subgroup of G, and let H contain a basic subgroup B of G. Then H is pure in G.*

Proof. The group G/B is divisible since B is basic in G. Now $H/B \subset G/B$, and by [1], p. 66, Theorem 20.2, there exists a divisible hull E/B of H/B in G/B. Suppose $E/B > H/B$. Then $E > H$, and hence $E \cap N \neq 0$. Thus there exists a nonzero element $g \in N$ with $0 \neq \langle g + B \rangle \subset E/B$. Now $\langle g + B \rangle \cap (H/B) = 0$. To see this, suppose $0 \neq m(g + B) = mg + B = h + B$. Then $mg - h = b \in B$, and $0 \neq mg = h + b \in H$, contradicting $H \cap N = 0$. Thus we have $\langle g + B \rangle \cap (H/B) = 0$. By Kulikov's lemma, $\langle g + B \rangle = 0$, and therefore $g \in B$, which implies that $g = 0$, contrary to the choice of g. Thus $E/B = H/B$ is divisible, and therefore is pure in G/B. Then the purity of B in G together with [1], p. 78, M imply that H is pure in G.

A useful lemma with a standard proof is

LEMMA 3.   *If $G = S \oplus T$, where each element of S has finite height, then $G^1 \subset T$ and $T^1 = G^1$.*

A lemma which displays an inheritance property is

LEMMA 4.   *If $G = S \oplus T$, where $S \subset H$ and H is high in G, then $H = S \oplus H \cap T$, and $H \cap T$ is high in T.* (Note: This implies that $H \cap T$ is maximal with respect to disjointness from G^1 in T by Lemma 3.)

Proof. Put $M = H \cap T$, and suppose that there exists $0 \neq t \in T \backslash H$, with $\{M, t\} \cap T^1 = 0$. But this means that $\{M, t\} \cap G^1 = 0$, and hence $[S \oplus \{M, t\}] \cap G^1 = 0$; for otherwise we would have $s + (m + kt) = g \neq 0$ with $s \in S$, $m \in M$, $g \in G^1$. Then $s = 0$ and $m + kt = g \neq 0$. But $m + kt \in \{M, t\}$, which is not possible. Thus $[S \oplus \{M, t\}] \cap G^1 = 0$ and

$[S \oplus \{M, t\}] > H$, contrary to the assumption that H is high in G.

A lemma on making new basic subgroups out of old ones is

LEMMA 5.   *Let* $B_1 \oplus \cdots \oplus B_n \oplus G_n = G$, *where* $B = \Sigma B_n$ *is basic in* G. *Let* $T_n = B'_{n+1} \oplus B'_{n+2} \oplus \cdots$ *be basic in* G_n. *Then* $C = B_1 \oplus \cdots \oplus B_n \oplus T_n$ *is basic in* G.

Proof   [1], p. 109, Exercise 9a.

An $x \in G$ will be called a *pure element* of G if and only if $\langle x \rangle$ is a pure subgroup (and therefore is a direct summand) of G.

The next lemma is the kingpin in the proof that if H is high in G, then H contains B basic in G. It is not altogether obvious that H contains nonzero pure subgroups of G. The proof of the next lemma will be carried out in several steps. We will consider special cases which are perhaps unnecessary, but which will help to clarify the method of proof.

LEMMA 6.   *Let* H *be a high subgroup of a primary group* G. *If* G *contains nonzero pure elements of order* p^n, *but not of smaller order, then* H *contains pure elements of* G *of order* p^n.

Proof.

Case 1. $n = 1$.   Let $b \in G$ be pure of order p with $b \notin H$. Then there exists $h \in H$ such that $h + b = g \neq 0$, where $g \in G^1$. Clearly this means that the orders of h and b are the same. Now h and b both have finite height, and hence their heights must be equal (since their sum is an element of infinite height in G). Here we are making use of the fact that if $o(h) = p$ and $h(h) = 0$, then $\langle h \rangle$ is pure in G. The fact that b is a pure element of order p in G necessarily means that $h(b) = 0$; whence $h(h) = 0$, and h is a pure element of G.

Case 2. $n > 1$.   Let b be a pure element of G of order p^n such that $b \notin H$. Then there exists an $h \in H$ such that $h + p^j b = g \neq 0$, where $g \in G^1$ and $0 \leq j < n$.

Case 2.1. $j = 0$.   Then we have $h + b = g$ and $p^{n-1}h + p^{n-1}b = p^{n-1}g \in G^1$. Clearly $p^{n-1}h$ has order p and height $n - 1$ in G and in $\langle h \rangle$. Thus by [1], p. 78, J, we have that h is a pure element of G.

Case 2.2. $1 \leq j < n$.   Now the equation $h + p^j b = g \neq 0$ clearly implies that the height of h in G is j. If the height of h in H were also j, that is if $h = p^j h'$ for some $h' \in H$, then h' would be a pure

element of G of order p^n. To see this simply consider the equation $p^{n-1}h' + p^{n-1}b = p^{n-j-1}g \in G^1$. $o(p^{n-1}h') = p$ and obviously $h(p^{n-1}h') = n - 1$. The height of $p^{n-1}h'$ in $\langle h' \rangle = n - 1$, so that by [1], p. 78, J, h' is a pure element of G of order p^n in H. Thus it remains to verify that the height of h in H is j.

From the neatness of H and the fact that $g \in G^1$, it follows that $h = ph_1$ for some $h_1 \in H$. Now if $h(h_1) > 0$, we again have by the neatness of H that $h_1 = ph_2$ for some $h_2 \in H$. Continuing in this way, we must eventually arrive at $h = p^{j-k}h_k$, $h_k \in H$ where the height of h_k in G is 0. If $k > 0$, then let m be the least positive integer such that $p^m h_k = p^{m+1}z$ for $z \in G$ (if worst comes to worst $m = j - k$ will do). Then clearly $0 < m \leq j - k < j < n$, $p^m(h_k - pz) = 0$, and $p^{m-1}(h_k - pz) \neq 0$ has height $m - 1$ in G by the choice of m and h_k. Thus since $o(p^{m-1}(h_k - pz)) = p$, and the height of $p^{m-1}(h_k - pz)$ is $m - 1$ in $\langle h_k - pz \rangle$, we have by [1], p. 78, J, that $h_k - pz$ is a pure element of G of order $p^m < p^n$. This contradicts the hypotheses on G. Hence we must have $k = 0$, $h = p^j h_k$, and h_k is a pure element of G in H of order p^n.

If $B = \Sigma B_n$ is a basic subgroup of G where B_n is a direct sum of cyclic groups of order p^n, then such a subgroup B_n which does not consist of 0 alone will be referred to as a B_n of G.

LEMMA 7. *Let G be a primary group, H a high subgroup of G, and n the least positive integer such that G contains a B_n. Then H contains a B_n of G.*

Proof. By Lemma 6, H contains pure elements of G of order p^n. The fact that the union of an ascending chain of pure subgroups is pure together with [1], p. 80, Theorem 24.5 allows us to apply Zorn's lemma to obtain a p^n-bounded direct summand H_n of G, maximal with respect to the property of being contained in H. We wish to show that H_n is a B_n for G. To see this write $G = H_n \oplus R_n$ and $H = H_n \oplus H \cap R_n$ where by Lemma 4, $H \cap R_n$ is high in R_n. Suppose that H_n is not a maximal p^n-bounded direct summand (a B_n) of G. Then there exists a B_n of G with $H_n < B_n$. Now $G = H_n \oplus R_n$, so that $B_n = H_n \oplus B_n \cap R_n$. Now the transitivity of purity tells us that $B_n \cap R_n \neq 0$ is pure in G. Thus R_n contains pure elements of order p^n since G contains no pure elements of order less than p^n. This means by Lemma 6 that $H \cap R_n$ as a high subgroup of R_n must contain a pure element h of order p^n. Then $G = H_n \oplus R_n = H \oplus \langle h \rangle \oplus R'_n$, and $(H_n \oplus \langle h \rangle) > H_n$ implies that H_n is not a maximal p^n-bounded direct summand of G contained in H, contrary to the choice of H_n. This means that H_n is a B_n of G contained in H after all, and this concludes the proof.

LEMMA 8. *Let G be a primary group, and let H be a high sub-*

group of G. Then H contains a basic subgroup of G.

Proof. By a theorem of Baer ([1], p.62), it suffices to consider the reduced case. Lemma 7 provides a start for the induction. Let B'_{n_1} be a first B_n of G. By lemma 7, H contains a B_{n_1} and $G = B_{n_1} \oplus R_1$ with $H = B_{n_1} \oplus H \cap R_1$. Let B_{n_2} be the next B_j of G. By Szele's theorem ([1], p. 99) and Lemma 5, R_1 contains a B_{n_2} but no preceding B_j. We apply Lemma 7 to $H \cap R_1$ as a high subgroup of R_1 to see that $H \cap R_1$ contains a B_{n_2} of G. By successive application of this procedure, we have by induction, Szele's theorem, Lemmas 5 and 7 that H contains a basic subgroup of G.

We are now ready to state and prove our main theorem.

THEOREM 1. *Let G be a primary group and H a high subgroup of G. Then H is pure in G.*

Proof. As in the proof of Lemma 8, it suffices to consider the case where G is reduced. Lemmas 2 and 8 complete the proof.

In his book [1], L. Fuchs asks the following question: "Let G be a primary group and H an infinite subgroup without elements of infinite height. Under what conditions can H be imbedded in a pure subgroup of the same power and again without elements of infinite height?" Theorem 1 allows us to give the best possible solution to this problem.

THEOREM 2. *Let G be an Abelian primary group. If S is any infinite subgroup of G with $S \cap G^1 = 0$, then S can be embedded in a pure subgroup K of G so that $K \cap G^1 = 0$ and $|K| = |S|$.*

Proof. By Zorn's lemma, there exists a subgroup H high in G with $H \supset S$. By Theorem 1, H is pure in G. Szele has shown that every infinite subgroup can be embedded in a pure subgroup of the same power ([1], p. 78). So let K be a pure subgroup of H containing S and of the same power as S. Then by the transitivity of purity, we have that K is pure in G. Since $K \subset H$, it follows that $K \cap G^1 = 0$. This concludes the proof.

The following discussion yields the solution to Fuchs' question in the torsion case. The proofs of the next two lemmas are standard and consequently will be omitted.

LEMMA 9. *Let G be a torsion group. If $G = \Sigma G_\alpha$, then $G^1 = \Sigma G_\alpha^1$.*

LEMMA 10. *Let G be a torsion group. Then an internal direct sum of pure subgroups of the direct summands of a given direct decomposition of G is a pure subgroup of G.*

Concerning the primary decomposition of a torsion group G, we have,

LEMMA 11. *If H is a high subgroup of a torsion group G, then writing G and H in terms of their primary components $G = \Sigma G_p$ and $H = \Sigma H_p = \Sigma H \cap G_p$, we have that H_p is a high subgroup of G_p for each relevant prime p in the primary decomposition of G.*

Proof. Clearly $H_p \cap G_p^1 = 0$. So suppose for some p, H_p is not high in G_p. Then there exists an $x \in G_p \backslash H_p$ with $\{H_p, x\} \cap G_p^1 = 0$. Replacing H_p by $S_p = \{H_p, x\}$ in $H = \Sigma H_p$, we obtain from Lemma 9 a subgroup $S > H$ with $S \cap G^1 = 0$. But this is contrary to H high in G.

A generalization of Theorem 1 is

THEOREM 3. *If H is a high subgroup of a torsion group G, then H is pure in G.*

Proof. Write $G = \Sigma G_p$ and $H = \Sigma H_p$ and by Lemma 11, we have that H_p is high in G_p so that by Theorem 1 we have H_p is pure in G_p. Now by Lemma 10, H is pure in G.

The generalization of the solution to Fuchs' question to torsion groups is

THEOREM 4. *Any infinite subgroup S of a torsion group G with $S \cap G^1 = 0$ can be embedded in a pure subgroup K of G so that $|K| = |S|$ and $K \cap G^1 = 0$.*

Proof. Use Theorem 3 and the proof of Theorem 2.

We mention for completeness that Lemma 8 has a suitable generalization to torsion groups.

LEMMA. 12. *Let G be a torsion group and let H be a high subgroup of G. Then H contains a basic subgroup of G.*

Proof. Use Lemma 8, the primary decomposition of H, and [1], p. 109, Exercise 9a.

Some of the more interesting properties of high subgroups are contained in

THEOREM 5. *Let G be a reduced primary group with $G^1 \neq 0$, and let H and K be high subgroups of G. Then*
   (a)   *H contains B basic in G*
   (b)   *H is pure in G*
   (c)   *G/H is a divisible hull of $(G^1 \oplus H)/H \cong G^1$*
   (d)   *$G/K \cong G/H$*
   (e)   *$p^n H$ is high in $p^n G$ for all $n \in I$ (I is the set of positive integers.)*

(f) $p^n H$ is pure in $p^n G$ for all $n \in I$

(g) $G = \{H, p^n G\}$ for all $n \in I$

(h) H is infinite

(i) H is of unbounded height in G

(j) $p^n G = \{p^n H, p^{n+k} G\}$ for all $n, k \in I$

(k) $p^n H / p^{n+k} H \cong p^n G / p^{n+k} G$ for all $n, k \in I$.

(l) $p^n G / p^n H \cong G / H$ for all $n \in I$.

(m) G is minimal pure containing $H \oplus G^1$

(n) $H \oplus G^1 < G$

(o) $|H \oplus G^1| = |G|$

(p) $|G^1| < |G|$ implies $|H| = |G|$

(q) $|K| = |H|$ (This also holds for N-high subgroups of infinite rank.)

(r) $|G| \leq |H|^{\aleph_0}$

(s) $G / p^n H = H / p^n H \oplus p^n G / p^n H$ for all $n \in I$.

(t) H is not always basic

(u) If H is countable, then H is basic in G, and $H \cong K$.

Proof. (a) and (b) have already been proved.

(c) Is easy.

(d) Follows from (c) and the fact that isomorphic groups have isomorphic divisible hulls (see [1], p. 66, Theorem 20.2).

(e) To see that $p^n H$ is high in $p^n G$, suppose that there exists $x \in G$ with $\{p^n H, p^n x\} \cap G^1 = 0$ and $p^n x \notin p^n H$. (Here we are using the fact that $(p^n G)^1 = G^1$.). Now by purity of H, $p^n x \notin p^n H$ implies $p^n x \notin H$. Thus we have some $h \in H$ with $h + m p^n x = g \neq 0, g \in G^1$. But then h must be in $p^n H$ contrary to $\{p^n H, p^n x\} \cap G^1 = 0$.

(f) The purity of $p^n H$ in $p^n G$ follows from (e), and Theorem 1 applied to $p^n G$.

(g) This is an immedite consequence of (c).

(h) And (i) both follow from (g) and the fact that a high subgroup of a reduced group is not a direct summand.

(j) Follows from (e) and (g).

(k) Follows from (j), the second isomorphism theorem, and (f).

(l) Is an immediate consequence of the fact that both quotient groups are divisible hulls of G^1. This is also a straightforward application of (g).

(m) This follows from Lemma 1 (f) and [1], p. 78, K.

(n) Follows from the fact that (c) holds and hence H is not a direct summand of G.

(o) Follows from Lemma 1 (f) and an easy set theoretic argument.

(p) Is an easy consequence of (o).

(q) Here some cases are taken care of by (d), but a proof for the general case is not difficult. To show that $|H| = |K|$, it suffices (by an

easy set theoretic argument) to verify that $H[p] \cong K[p]$. For this purpose let D be a divisible hull of G, and C be a divisible hull of G^1 in D. By Lemma 1, if A and B are divisible hulls in D of H and K respectively, then A and B are complementary direct summands of C in D. Finally $A \cong D/C \cong B$ and $H[p] = A[p] \cong B[p] = K[p]$. The same argument shows the result for N-high subgroups of infinite rank.

(r) Follows trivially from (a) and [1], p. 102, Theorem 30.1.

(s) To see this, use (g) and the purity of H.

(t) Let G be the direct sum of an unbounded closed primary group and any primary group with nonzero elements of infinite height.

(u) This follows from (b), (c), (q), the fact that a countable H is a direct sum of cyclic groups, and that any two basic subgroups of G are isomorphic.

For a comparison with the properties of basic subgroups see [1], p. 101. The reader will notice that (d) is an interesting property of high subgroups which basic subgroups do not possess.

We are now ready to discuss the question of whether or not any two high subgroups of a reduced primary group are isomorphic. Let A be a subgroup of G, and let $\tilde{A}$ be the image under the natural homomorphism from G onto G/G^1. It is a simple matter to verify that $\tilde{G}$ is a reduced primary group without elements of infinite height. Thus if H is a high subgroup of G, we have that $H \cong \tilde{H}$. This provides us a natural way to study the properties of high subgroups without actually looking at these subgroups themselves.

A result concerning Ulm invariants as defined by Kaplansky in [2], and providing another proof that two countable high subgroups of a group G are isomorphic is the following

THEOREM 6. *Let H and K be high subgroups of a primary group* G. *Then* $(p^n\tilde{H})[p]/(p^{n+1}\tilde{H})[p] = (p^n\tilde{K})[p]/(p^{n+1}\tilde{K})[p]$. *In particular, H and K have the same Ulm invariants. Moreover, their nth Ulm invariants are the same as the nth Ulm invariant of G.*

Proof. Consider $\tilde{H}$ and $\tilde{K}$. First we notice

(i) $\tilde{H}[p] = \tilde{K}[p]$.

To see this we observe that $o(h) = o(h + G^1)$. Suppose $h \in H[p] \backslash H \cap K$. Then there exists $k \in K$ with $h - k = g \neq 0$ where $g \in G^1$. Clearly $o(k) = p$ and we have $h = k + g$. This proves that $\tilde{H}[p] \subset \tilde{K}[p]$. Thus by symmetery $\tilde{H}[p] = \tilde{K}[p]$. Next we have

(ii) $p^n\tilde{H}[p] = p^n\tilde{K}[p]$ for $n \in I$.

To see this use Theorem 5 (e), and the foregoing (i).

Now from (ii) we have that $(p^n\tilde{H})[p]\backslash(p^{n+1}\tilde{H})[p] = (p^n\tilde{K})[p]/(p^{n+1}\tilde{K})[p]$ since the numerators are equal and the denominators are equal, and hence the Ulm invariants of $\tilde{H}$ and $\tilde{K}$ are equal. Finally the fact that

$H \cong \tilde{H}$ gives us that H and K have the same nth Ulm invariants. The last part of the theorem follows from $(p^n G)[p]/(p^{m+1}G)[p] \cong (p^n \tilde{H})[p]/(p^{n+1}\tilde{H})[p]$ which is obtained with the help of Lemma 1 (f), Theorem 5 (e), and the second isomorphism theorem.

We will now mention a few generalizations to modules. In what follows, R will denote a principal ideal ring. This means that R is an integral domain (commutative ring with an identity and no divisors of zero) in which every ideal is principal. By an R-module we mean a unitary left R-module, and by submodule of an R-module we mean a sub-R-module. An R-module M is called *primary* if and only if the order ideal of every element of M is generated by a power of the same prime element p of R. We shall be content with a generalization to primary modules of our main results for primary groups. We rely heavily on the generalizations of Theorems 1 to 14 in [2].

We make a blanket assertion: All of our lemmas and theorems for primary groups are true for primary modules. Only minor, straightforward modifications of the definitions and proofs are necessary, and these can be easily carried out by imitating all the previous definitions and proofs. When referring to orders of elements in a primary module, we say that $o(x)$ is smaller than $o(y)$ if and only if the generator of the order ideal of x divides the generator of the order ideal of y.

In conclusion we state without proof the most worthwhile lemmas and theorems.

LEMMA 13. *Let M be a primary R-module. Let L, N be submodules of M with L containing a basic submodule B of M, and L maximal with respect to disjointness from N. Then L is pure in M.*

THEOREM 7. *Let H be a high submodule of a primary R-module M. Then H is pure in M.*

The solution of Fuchs' question for primary modules is

THEOREM 8. *Let S be an infinitely generated submodule of the primary R-module M with R countable and $S \cap M^1 = 0$. Then S can be embedded in a pure submodule K of M such that $K \cap M^1 = 0$ and $|K| = |S|$.*

The only essential difference between this theorem and Theorem 2 is that the word infinite has been replaced by the words infinitely generated to make $|K| = |S|$ true in all cases. The proof is the same as before. The countability assumption on R makes the proof of [1] p. 78 N easy.

The author conjectures that all high subgroups of a given primary group are isomorphic, and also wishes to pose the questions:

For what subgroups N of a primary group G is it true that

(a) all N-high subgroups are pure

(b) all N-high subgroups are isomorphic

(c) all N-high subgroups are endomorphic images of G

(d) G/N divisible implies N contains B basic in G?

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