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**GAME THEORETIC PROOF THAT CHEBYSHEV
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1. **Summary.** This paper is concerned with showing that Chebyshev inequalities obtained by the standard method are sharp. The proof is based on relating the bound to the solution of a game. An optimum strategy yields a portion of the extremal distribution, and the remainder is obtained as a solution of the relevant moment problem.

2. **Introduction.** Let X be a random vector taking values in $\mathcal{X} \subset R^k$, and suppose that $Ef(X) \equiv E(f_1(X), \dots, f_r(X)) = (\varphi_1, \dots, \varphi_r) \equiv \varphi$ is given, where f_j is a real valued function on \mathcal{X} . For convenience, we suppose $f_1 \equiv 1$. An upper bound for $P\{X \in \mathcal{T}\}$, $\mathcal{T} \subset \mathcal{X}$, may be obtained as follows. If $a = (a_1, \dots, a_r) \in R^r$ and $\chi_{\mathcal{T}}$ is the indicator of \mathcal{T} then $a f' \geq \chi_{\mathcal{T}}$ on \mathcal{X} implies $P\{X \in \mathcal{T}\} \leq a \varphi'$, and if $\mathcal{A}_0 = \{a: a f' \geq \chi_{\mathcal{T}} \text{ on } \mathcal{X}\}$, a "best" bound is given by

$$(2.1) \quad P\{X \in \mathcal{T}\} \leq \inf_{a \in \mathcal{A}_0} a \varphi'.$$

In general, a bound is called sharp if it cannot be improved. For some cases, when \mathcal{T} is assumed to be closed, the bound can actually be attained by a distribution satisfying the moment hypotheses.

The main result of this paper is

THEOREM 2.1. *Inequality (2.1) is sharp in the following cases.*

- (I) $X = (X_1, \dots, X_k)$ with $EX_i X_j$ or EX_i and $EX_i X_j$ given, $i, j = 1, \dots, k$.
- (II) X has range $(-\infty, \infty)$, $[0, \infty)$, or $[0, 1]$, and EX^j is given, $j = 1, \dots, m$.
- (III) X is a random angle in $[0, 2\pi)$ and the trigonometric moments $Ee^{i\alpha X}$, $\alpha = \pm 1, \dots, \pm m$ are given.

Sharpness has been shown in (I) by Marshall and Olkin [6] when \mathcal{T} is convex, and by Isii [3, 4] in the unbounded cases of (II). Sharpness has also been proved in a number of specialized situations.

In § 3 the proof for (I) will be given in detail. The necessary alterations for each of the remaining cases will be given in § 4, 5, 6, 7. The solution of certain moment problems depend on conditions on Hankel matrices, i.e., matrices of the form $H = (h_{i+j})$, and some results concerning these matrices are given in § 8.

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The notation $A > 0 (\geq 0)$ is used to mean that the matrix A is symmetric and positive definite (p.s.d).

3. The multivariate case. The relation between inequality (2.1) and a game can be greatly simplified if we use matrix theoretic arguments. This is true in part because functions of the form af' , $a \in \mathcal{A}_0$, can be written very naturally as quadratic or bilinear forms.

Let $X = (X_1, \dots, X_k)$ be a random vector on R^k with $EX = \mu$ and moment matrix $EX'X = \Sigma$. If $u \equiv u(x) = (1, x)$ for $x \in R^k$, then $Eu'(X)u(X) = \begin{pmatrix} 1 & \mu' \\ \mu' & \Sigma \end{pmatrix} = \Pi$. We assume $\Pi > 0$, for otherwise the dimensionality of X can be reduced.

Functions of the form af' , $a \in \mathcal{A}_0$ can be written as uAu' , $A: k + 1 \times k + 1$, $A \in \mathcal{A} = \{A; A \geq 0, uAu' \geq 1 \text{ for } x \in \mathcal{T}\}$. Hence

$$(3.1) \quad P\{X \in \mathcal{T}\} \leq \inf_{a \in \mathcal{A}_0} a\varphi' = \inf_{A \in \mathcal{A}} \text{tr} A\Pi .$$

Let x_1, \dots, x_m be points (row vectors) in R^k , $u_i = u(x_i)$, p_1, \dots, p_m , $\sum p_i = 1$ be probabilities, $T = (u'_1, \dots, u'_m)$, $D_p = \text{diag}(p_1, \dots, p_m)$, and $H = TD_pT'$. By $H \sim \mathcal{T}$ we mean that all $x_i \in \mathcal{T}$. The condition $uAu' \geq 1$ for $x \in \mathcal{T}$ can then be written as $\text{tr} AH \geq 1$ for $H \sim \mathcal{T}$, so that $\mathcal{A} = \{A: A \geq 0, \text{tr} AH \geq 1 \text{ for } H \sim \mathcal{T}\}$.

With this notation, we can rewrite the bound (3.1) in a form which is suggestive of a game.

$$(3.2) \quad \begin{aligned} \inf_{A \in \mathcal{A}} \text{tr} A\Pi &= \inf_{\substack{A: \inf_{H \sim \mathcal{T}} \text{tr} AH \geq 1, A \geq 0}} \text{tr} A\Pi \\ &= \inf_{S \geq 0} \left(\frac{\text{tr} S\Pi}{\inf_{H \sim \mathcal{T}} \text{tr} SH} \right) = \left(\sup_{S \geq 0} \inf_{H \sim \mathcal{T}} \frac{\text{tr} SH}{\text{tr} S\Pi} \right)^{-1} \\ &= \left(\sup_{\{S: S \geq 0, \text{tr} SH \leq 1\}} \inf_{H \sim \mathcal{T}} \text{tr} SH \right)^{-1} \equiv \nu^{-1} . \end{aligned}$$

In view of (3.2) it is natural to consider the game $G = (\mathcal{S}, \mathcal{H}, g)$, where $\mathcal{S} = \{S: S \geq 0, \text{tr} S\Pi \leq 1\}$ and $\mathcal{H} = \{H: H \sim \mathcal{T}\}$ are the strategy spaces for players I and II, respectively, and $g(S, H) = \text{tr} SH$ is the payoff to player I.

Clearly \mathcal{S} and \mathcal{H} are closed and convex. Further, \mathcal{S} is bounded since

$$\|S\|^2 \equiv (\text{tr} SS') \leq (\text{tr} S)c_m(S) \leq (\text{tr} S)^2 \leq (\text{tr} S\Pi)^2/c_m^2(\Pi) \leq 1/c_m^2(\Pi) ,$$

where $c_m(A)$, $c_M(A)$ are the minimum and maximum characteristic roots of A . For the present we assume that \mathcal{H} is bounded, then by [2, Section 2.5], G has a value and there exist optimal strategies $S_0 \in \mathcal{S}$, $H_0 \in \mathcal{H}$, such that

$$(3.3) \quad \text{tr} SH_0 \leq \text{tr} S_0 H_0 = \nu \leq \text{tr} S_0 H, \quad \text{for all } S \in \mathcal{S}, H \in \mathcal{H}.$$

The optimal strategy S_0 has the property that $\inf_{A \in \mathcal{A}} \text{tr} A \Pi = \text{tr} A_0 \Pi$, where $A_0 = S_0/\nu$.

To prove sharpness of (3.1), we must show that there exists a distribution for X such that $P\{X \in \mathcal{S}\} = 1/\nu$, and $Eu'u = \Pi$. H_0 is the moment matrix of a distribution F_1 on points in \mathcal{S} . If we can prove the existence of a probability distribution F for X of the form $F = F_1/\nu + F_2$, and with moment matrix Π , then this distribution attains equality in (3.1). To see this, note that F assigns at least probability ν to \mathcal{S} , and by (3.1) it can assign at most probability ν to \mathcal{S} .

To show the above, we need only show that a distribution F_2 exists with total variation $1 - 1/\nu$ and moment matrix $\Psi = \Pi - H_0/\nu$. The following Lemma yields this result.

LEMMA 3.1. *Let $\Pi > 0$, $\mathcal{S} = \{S: S \geq 0, \text{tr} S \Pi \leq 1\}$.*

- (i) *If $\text{tr} SH \leq \nu$ for all $S \in \mathcal{S}$, then $\Psi = \Pi - H/\nu \geq 0$.*
- (ii) *If $\text{tr} SH = \nu$ for some $S_0 \in \mathcal{S}$, then Ψ is not strictly > 0 .*
- (iii) *If $\text{tr} SH < \nu$ for all $S \in \mathcal{S}$, then $\Psi > 0$.*

Proof. There exists a representation $\Pi = WW'$, $H = WD_\theta W'$, $|W| \neq 0$, $D_\theta = \text{diag}(\theta_0, \dots, \theta_k)$, and hence $\Psi \geq 0$ if and only if $\theta_i \leq \nu$, $i = 0, \dots, k$. If $W'SW = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $S \in \mathcal{S}$, and from $\text{tr} SH = \text{tr} W'SWD_\theta \leq \nu$, we obtain $\theta_0 \leq \nu$. Part (i) follows using permutations. If $\text{tr} SH < \nu$, then in the above argument, each $\theta_i < \nu$. If $\text{tr} S_0 H = \text{tr}(W'S_0 W)D_\theta = \nu$ and $\text{tr} W'S_0 W \leq 1$, then at least one of the θ_i is equal to ν .

The condition that \mathcal{H} be bounded now can be removed, since $\|H_0\|^2 \leq (\text{tr} H_0)^2 \leq [\nu \text{tr} \Pi]^2$, by Lemma 3.1.

REMARK 3.1. We note that $\text{tr} S_0 \Pi = 1$, for if not, αS_0 for $\alpha > 1$ would violate (3.3).

S_0 and H_0 are related by $\nu S_0 \Pi = S_0 H_0$. This follows from the fact that $\text{tr} S_0 \Psi = \text{tr} S_0 (\Pi - H_0/\nu) = 0$ and $\Psi \geq 0$ implies that $S_0^{1/2} \Psi S_0^{1/2} = 0$, or equivalently that $S_0^{1/2} \Psi^{1/2} = 0$, which yields the result.

REMARK 3.2. In the above development we assumed that $EX = \mu$ was given. If this is not the case, then choose $\mathcal{S} = \{S = \begin{pmatrix} \alpha & 0 \\ 0 & S_1 \end{pmatrix}: S > 0, \text{tr} S \Pi \leq 1, S_1: k \times k, \text{ and the entire development remains unchanged with } S_1 \text{ replacing } S, \text{ since } S \geq 0 \text{ if and only if } \alpha > 0, S_1 \geq 0 \text{ and } \text{tr} S \Pi = \alpha + \text{tr} S_1 \Sigma.$

We now summarize the essential points of the proof which are appropriately modified in each of the remaining cases.

- (1) Introduce vectors $u(x)$ and $v(x)$ ($u = v$ in the above) such that
 - (i) $Ev'(X)u(X) = \Pi$ is a matrix of given moments,
 - (ii) $af', a \in \mathcal{A}_0$ can be written as uAv' with $A \in \mathcal{A}$.

To define \mathcal{A} we first must characterize \mathcal{A}_0 .

- (2) Define \mathcal{H} , a set of moment matrices of the same kind as Π , but corresponding to distributions on \mathcal{T} .
- (3) Define \mathcal{S} and show that \mathcal{S} is bounded.
- (4) Use the game to assert that H_0 exists, and show that the moment problem with moments defined by $\Psi = \Pi - H_0/\nu$ has a solution with $\psi_{11} = 1 - 1/\nu$.

4. Univariate distributions on $(-\infty, \infty)$. Let $u(x) = v(x) = (1, x, \dots, x^n)$. Then polynomials $af'(x)$ of degree $\leq 2n$ which are nonnegative in $(-\infty, \infty)$ can be expressed as uAu' , $A \geq 0$, [7, p. 82]. Hence $\mathcal{A} = \{A: A \geq 0, uAu' \geq 1 \text{ for } x \in \mathcal{T}\}$, and (3.1) holds. Note that $\Pi = (\pi_{i+j-2}) = (EX^{i+j-2})$, $i, j = 1, \dots, n + 1$. Let $-\infty < t_i < \infty$, $u_i = u(t_i)$, $i = 1, \dots, m$, $T = (u'_1, \dots, u'_m)$, $D_p = \text{diag}(p_1, \dots, p_m) \geq 0$, $\text{tr } D_p = 1$, $H = TD_pT' = (h_{i+j-2})$, $i, j = 1, \dots, n + 1$. Define $\mathcal{H} = \{H: t_i \in \mathcal{T}, i = 1, \dots, m\}$, $\mathcal{S} = \{S: S \geq 0, \text{tr } S\Pi \leq 1\}$. We assume that the moment problem corresponding to the given moments $\{\pi_0, \dots, \pi_{2n}\}$ is not determined so that $\Pi > 0$, [8, Th. 3.3], and the previous argument that \mathcal{S} is bounded holds. Assuming that \mathcal{H} is bounded, there exists an S_0 and $H_0 = (h_{i+j-2}^0)$ satisfying (3.3), and with Lemma 3.1 we conclude as before that the boundedness condition on \mathcal{H} can be removed.

Since $\pi_0 = h_0^0 = 1$, $\psi_0 = 1 - 1/\nu$. Define $\Delta_r = |\psi_{i+j-2}|_{i,j=1}^{r+1}$; then since $\Psi \geq 0$, by Theorem 8.1 it follows that $\Delta_1 > 0, \dots, \Delta_{r-1} > 0, \Delta_r = 0, \dots, \Delta_n = 0$, for some r . The reduced (Hamburger) moment problem has a solution if and only if $\Psi \geq 0$, in which case there exists a (unique) representation $\psi_j = \sum_{i=1}^r p_i \xi_i^j$, $j = 0, 1, \dots, 2n - 1$, and $\psi_{2n} = \sum_{i=1}^r p_i \xi_i^{2n} + c$, $c \geq 0$, and $c = 0$ if $r = n$, [8, p. 85].

In the event $c > 0$, by using an ε -good strategy for player II to guarantee Ψ strictly > 0 , we obtain a distribution with moments $\{\pi_0, \dots, \pi_{2n}\}$, which assigns probability $1/(\nu + \varepsilon)$ to \mathcal{T} .

REMARK 4.1. The representation obtained from [7, p. 82] is of the form $(\sum u_i c_i)^2 + (\sum u_i d_i)^2$, which is expressible as uAu' , where $A = c'c + d'd$. However, the same class of polynomials is obtained if we include all $A \geq 0$.

REMARK 4.2. If \mathcal{S} is bounded, there exists an extremal distribution with a spectrum consisting of at most $2(n + 1)$ points. This follows from the fact that the least number of points contributing to H_0 is at most $(n + 1)$, [2, § 2.5], and to Ψ is at most $(n + 1)$ points by the previous argument.

5. Univariate Case on $[0, \infty)$. Consider first the case $m = 2n - 1$, and let $u(x) = (1, x, \dots, x^{n-1})$, $v(x) = (1, x, \dots, x^n)$. Then polynomials $af'(x)$ of degree $\leq 2n - 1$ can be expressed as $u[(B, 0) + (0, C)]v' \equiv uAv'$, where $B \geq 0, C \geq 0$ are $n \times n$ matrices (See [7, p. 82] and Remark 4.1). Hence $\mathcal{N} = \{A: B \geq 0, C \geq 0, uAv' \geq 1 \text{ for } x \in \mathcal{S}\}$, and (3.1) holds. Now $\Pi = (\pi_{i+j-2}) = (EX^{i+j-2})$, $i = 1, \dots, n + 1; j = 1, \dots, n$. Let $0 < t_i < \infty$, $u_i = u(t_i)$, $v_i = v(t_i)$, $i = 1, \dots, m$, $T_1 = (u'_1, \dots, u'_m)$, $T_2 = (v'_1, \dots, v'_m)$, $D_p = \text{diag}(p_1, \dots, p_m) \geq 0$, $\text{tr} D_p = 1$, $H = T_1 D_p T'_2 = (h_{i+j-2})$, $i = 1, \dots, n + 1; j = 1, \dots, n$. Define $\mathcal{H} = \{H: t_i \in \mathcal{S}, i = 1, \dots, m\}$, $\mathcal{S} = \{S = (S_1, S_2): S_1 \geq 0, S_2 \geq 0, \text{tr}[(S_1, 0) + (0, S_2)]\Pi \leq 1\}$, $S_1, S_2: n \times n, 0: n \times 1$. Assuming that the moment problem corresponding to Π is not determined, i.e., $\Pi_{(1)} = (\pi_{i+j-2})$, $i, j = 1, \dots, n$, $\Pi^{(1)} = (\pi_{i+j-1})$, $i, j = 1, \dots, n$, are positive definite, [8, p. 6], the argument of § 3 that \mathcal{S} is bounded holds, with $\|S\| \equiv \|(S_1, 0) + (0, S_2)\|$.

Assuming that \mathcal{H} is bounded, there exists an $S_0 = (S_{10}, 0) + (0, S_{20})$ and $H_0 = (h_{i+j-2})$, $i = 1, \dots, n + 1; j = 1, \dots, n$, satisfying (3.3). Define $H_{0(1)}$ and $H_0^{(1)}$ in the same manner as $\Pi_{(1)}$ and $\Pi^{(1)}$. An application of Lemma 3.1 yields $\Psi_{(1)} = \Pi_{(1)} - H_{0(1)}/\nu \geq 0$ and $\Psi^{(1)} = \Pi^{(1)} - H_0^{(1)}/\nu \geq 0$. The boundedness condition of \mathcal{H} can now be removed since $\|H_0\|^2 \leq \|H_{01}\|^2 + \|H_0^{(1)}\|^2 \leq \nu \text{tr}(\Pi_{(1)} + \Pi^{(1)})$. Also $\psi_0 = \pi_0 - h_0/\nu = 1 - 1/\nu$.

In order for the reduced (Stieltjes) moment problem to have a solution, it is necessary that both $\Psi_{(1)}$ and $\Psi^{(1)}$ be ≥ 0 .¹

Recall from § 4 that $\Delta_r = |\psi_{i+j-2}|$, $i, j = 1, \dots, r + 1$. Now define $\Delta_r^{(1)} = |\psi_{i+j-1}|$, $i, j = 1, \dots, r + 1$. From Theorem 8.1 it follows that either

- (i) $\Delta_0 > 0, \dots, \Delta_r > 0, \Delta_{r+1} = \dots = \Delta_n = 0$ and $\Delta_0^{(1)} > 0, \dots, \Delta_r^{(1)} > 0, \Delta_{r+1}^{(1)} = \dots = \Delta_n^{(1)} = 0$, or
- (ii) $\Delta_0 > 0, \dots, \Delta_r > 0, \Delta_{r+1} = \dots = \Delta_n = 0$ and $\Delta_0^{(1)} > 0, \dots, \Delta_{r-1}^{(1)} > 0, \Delta_r^{(1)} = \dots = \Delta_n^{(1)} = 0$, for some r . But these are the conditions that there exist a distribution whose spectrum consists of $r + 1$ points distinct from 0 in case (i) and including 0 in (ii).

If $m = 2n$, let $u(x) = v(x) = (1, x, \dots, x^n)$. Then polynomials $af'(x)$ of degree $\leq 2n$ can be expressed as $v\left[B + \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}\right]v'$, where $B: n + 1 \times n + 1, C: n \times n, B \geq 0, C \geq 0$, [7, p. 82]. The remainder of the proof is essentially the same as for the case $m = 2n - 1$ above.

6. Univariate distribution on $[0, 1]$. We first deal with the case when an odd number of moments is given. Let $u(x) = (1, x, \dots, x^{n-1})$, $v(x) = (1, x, \dots, x^n)$. Now $\Pi = (\pi_{i+j-2}) = (EX^{i+j-2})$, $i = 1, \dots, n + 1; j = 1, \dots, n$. Then polynomials $af'(x)$ of degree $\leq 2n - 1$ which are

¹ This result was communicated to the authors by S. Karlin. The proof is similar to that for the reduced Hausdorff moment problem given in [5].

nonnegative in $[0, 1]$ can be expressed as $u[(B, 0) + (0, C - B)]v' \equiv uAv'$, where B and C are $n \times n$ matrices, $B \geq 0, C \geq 0$, (See [7, p. 82] and Remark 4.1). Hence $\mathcal{A} = \{A : B \geq 0, C \geq 0, uAv' \geq 1 \text{ for } x \in \mathcal{S}\}$, and (3.1) holds. We assume that the moment problem corresponding to the given moments $\{\pi_0, \dots, \pi_{2n-1}\}$ is not determined. This means that $\Pi^{(1)} = (\pi_{i+j-1}), i, j = 1, \dots, n$, and $\Pi^{(2)} = (\pi_{i+j-2} - \pi_{i+j-1}), i, j = 1, \dots, n$, are both positive definite, [5, p. 55] or [8, p. 77]. (In the latter reference the conditions are presented for the interval $[-1, 1]$.)

Let $0 \leq t_i \leq 1, u_i = u(t_i), v_i = v(t_i), i = 1, \dots, m, T_1 = (u'_1, \dots, u'_m), T_2 = (v'_1, \dots, v'_m), D_p = \text{diag}(p_1, \dots, p_m) \geq 0, \text{tr} D_p = 1, H = T_1 D_p T_2' = (h_{i+j-2}), i = 1, \dots, n + 1; j = 1, \dots, n$. Define $\mathcal{H} = \{H : t_i \in \mathcal{S}, \tilde{i} = 1, \dots, m\}, \mathcal{S} = \{(S_1, S_2) : S_1 \geq 0, S_2 \geq 0, \text{tr}[(S_1 0) + (0, S_2 - S_1)]\Pi \leq 1\}$. We first show that \mathcal{S} is bounded:

$$\|S\|^2 \equiv \|(S_1, 0) + (0, S_2 - S_1)\| \leq 2 \text{tr} S_1^2 + \text{tr} S_2^2 \leq 2(\text{tr} S_1)^2 + (\text{tr} S_2)^2.$$

But $\text{tr} S\Pi = \text{tr} S_1\Pi^{(2)} + \text{tr} S_2\Pi^{(1)} \leq 1$, and $\Pi^{(2)} > 0, \Pi^{(1)} > 0$, so that $\text{tr} S_1 \leq 1/c_m(\Pi^{(2)}), \text{tr} S_2 \leq 1/c_m(\Pi^{(1)})$, and \mathcal{S} is bounded.

Assuming that \mathcal{H} is bounded, there exists an $S_0 = (S_{10}, 0) + (0, S_{20} - S_{10})$ and $H_0 = (h_{i+j-2}^0), i = 1, \dots, n + 1; j = 1, \dots, n$, satisfying (3.3). Define $H_{0(2)}$ and $H_0^{(1)}$ as for $\Pi^{(2)}$ and $\Pi^{(1)}$; then an application of Lemma 3.1 yields

$$\Psi_{(2)} = \Pi_{(2)} - H_{0(2)}/\nu \geq 0, \Psi^{(1)} = \Pi^{(1)} - H_0^{(1)}/\nu \geq 0.$$

The boundedness condition on \mathcal{H} can now be removed since $\|H_0\|^2 \leq 2\|H_{0(2)}\|^2 + 2\|H_0^{(1)}\|^2 \leq \nu \text{tr}(\Pi_{(2)} + \Pi^{(1)})$. Also $\psi_0 = \pi_0 - h_0/\nu = 1 - 1/\nu$.

In order for the reduced (Hausdorff) moment problem to have a solution, it is necessary that both $\Psi_{(2)}$ and $\Psi^{(1)}$ be ≥ 0 , [5, p. 55].

If an even number of moments is given, we let $u(x) = v(x) = (1, x, \dots, x^n)$. Now $\Pi = (\pi_{i+j-2}), i, j = 1, \dots, n + 1$. Polynomials $af'(x)$ of degree $\leq 2n$ which are nonnegative in $[0, 1]$ can be expressed as $u\left[\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -C \end{pmatrix}\right]u' \equiv uAu'$, where B and C are $n \times n$ matrices, $B \geq 0, C \geq 0$, (See [7, p. 82] and Remark 4.1). Hence $\mathcal{A} = \{A : B \geq 0, C \geq 0, uAu' \geq 1 \text{ for } x \in \mathcal{S}\}$, and (3.1) holds. We assume that the moment problem corresponding to the given moments $\{\pi_0, \dots, \pi_{2n}\}$ is not determined. This means that Π and $\Pi_{(3)} = (\pi_{i+j-1} - \pi_{i+j}), i, j = 1, \dots, n$, are positive definite, [5, p. 55] or [8, p. 77].

The remainder of the argument is analogous to the odd moment case.

REMARK 6.1. As in Remark 4.1, if \mathcal{S} is bounded, there exists an extremal distribution with a spectrum consisting of at most $2(n + 1)$ points. This follows from [2, § 2.5] and [5, § 17].

REMARK 6.2. A condition for the solution of the Hausdorff moment problem with an infinite number of moments is the condition that

$$\Delta^k \mu_j = \mu_j - \binom{k}{1} \mu_{j+1} + \binom{k}{2} \mu_{j+2} + \dots + (-1)^k \mu_{j+k} \geq 0, \quad k, j = 0, 1, \dots$$

However, this condition with $k, j = 0, 1, \dots, n$ is not sufficient for a solution of the reduced moment problem. It is interesting to note that this condition enters naturally using an alternative formulation. Polynomials $af'(x)$ which are nonnegative in $[0, 1]$ may be represented as $\sum a_{ij}(1-x)^i x^j$, where $a_{ij} \geq 0$. If we let $u(x) = (1, (1-x), \dots, (1-x)^n)$, $v(x) = (1, x, \dots, x^n)$, then the representation is uAv' , $a_{ij} \geq 0$. Now $\Pi = (E(1-X)^{i-1} X^{j-1}) = (\Delta^{i-1} \mu_{j-1})$, $i, j = 1, \dots, n+1$. Using a similar development as before, $\mathcal{S} = \{S : s_{ij} \geq 0, \text{tr} S \Pi \leq 1\}$, and from Lemma 3.1, $\Psi = \Pi - H_0/\nu = (\Delta^{i-1} \mu_{j-1}) - (\Delta^{i-1} \mu_{j-1}/\nu) \geq 0$. Let $\psi_j = \mu_j - h_j/\nu$, $\Psi = (\Delta^{i-1} \psi_{j-1})$; we wish to show that $\Delta^{i-1} \psi_{j-1} \geq 0$. By choosing S to have all zeros except $s_{ij} = 1/\Delta^{i-1} \pi_{j-1}$, $\text{tr} S \Pi = 1$. The result follows after using (3.3).

7. Random angle in $[0, 2\pi)$. If $u(x) = v(x) = (1, e^{inx}, \dots, e^{inx})$, then polynomials $af'(x)$ which are nonnegative in $[0, 2\pi)$ can be expressed as uAu' , $A \geq 0$, (See [7, p. 82] and Remark 4.1). Hence $\mathcal{A} = \{A : A \geq 0, uAu' \geq 1 \text{ for } x \in \mathcal{S}\}$, and (3.1) holds. Now $\Pi = (\pi_{j-k}) = (Ee^{i(j-k)x})$, $j, k = 1, \dots, n+1$.

The proof is virtually that of § 4, noting only that the reduced trigonometric (Herglotz) moment problem has a solution if the Toeplitz matrix $\Pi > 0$. (See footnote, § 5.)

7.1. An example. The authors are unaware of any Chebyshev inequalities when trigonometric moments are available, and we present a simple illustration.

THEOREM 7.1. If X is a random angle in $[0, 2\pi)$ and $E \sin X = \alpha$, $E \cos X = \beta$, then

$$(7.1) \quad P\{2\theta < X < 2\varphi\} \geq 1 - \frac{1 - \alpha \sin(\theta + \varphi) - \beta \cos(\theta + \varphi)}{1 - \cos(\varphi - \theta)},$$

$$(7.2) \quad P\{2\theta \leq X \leq 2\varphi\} \leq \frac{1 + \alpha \sin(\theta + \varphi) + \beta \cos(\theta + \varphi)}{1 + \cos(\varphi - \theta)},$$

$$0 \leq \theta \leq \varphi \leq \pi.$$

Proof. Choose $f(x) = c_1 + c_2 \sin x + c_3 \cos x$. The conditions $f(\theta + \varphi) = 0$, $f(2\theta) = f(2\varphi) = 1$ lead to (7.1), and the conditions $f(\theta + \varphi + \pi) = 0$, $f(2\theta) = f(2\varphi) = 1$ lead to (7.2).

8. Properties of Hankel matrices. In this section we obtain several properties of Hankel matrices which were required in §§ 4 and 5. These properties are known as a consequence of the solution of moment problems, but it may be of interest to present matrix theoretic proofs. We need the following preliminaries.

A matrix $U = (u_{i+j-2})$, $i, j = 1, \dots, n$ is called a Hankel matrix. By the r th compound, $A^{(r)}$, of a matrix $A: n \times n$ we mean the matrix whose elements are the r th order minors of A arranged in lexicographic order; thus $A^{(r)}: \binom{n}{r} \times \binom{n}{r}$. The following properties of compound matrices are well-known, e.g., [1].

(8.1) Let A be symmetric. The characteristic roots of $A^{(r)}$ are the $\binom{n}{r}$ products of r characteristic roots of A . Thus, $A^{(r)} \geq 0$ if and only if $A \geq 0$.

$$(8.2) \quad |A^{(r)}| = |A|^{\binom{n-1}{r-1}}.$$

THEOREM 8.1. *If the Hankel matrix $U = (u_{i+j-2})$, $i, j = 1, \dots, r+1$, is ≥ 0 , and if $\Delta_r = |u_{i+j-2}|_{i,j=1}^r = 0$, then $\Delta_{r+1} = 0$.*

Proof. Suppose $u_0 = 0$, then by nonnegativity of each 2×2 principal minor, it follows that $u_0 = u_1 = \dots = u_{2n-1} = 0$, $u_{2n} \geq 0$. But $U^{(r)} \geq 0$ has first element 0, and hence its first row is 0, so that $\Delta_r = 0$.

THEOREM 8.2. *Let $U = (u_{i+j-2})$, $i, j = 1, \dots, r+1$, $V = (v_{i+j-1})$, $i, j = 1, \dots, r+1$, $U \geq 0$, $V \geq 0$. Then $\Delta_r = 0 \Rightarrow \Delta_r^{(1)} = 0 \Rightarrow \Delta_{r+1} = 0$, where $\Delta_m = |u_{i+j-2}|$, $i, j = 1, \dots, m$; $\Delta_m^{(1)} = |v_{i+j-1}|$, $i, j = 1, \dots, m$.*

Proof. In the r th compound $U^{(r)}$, $\Delta_r = u_{11}^{(r)} = 0$ implies that $u_{12}^{(r)} = \Delta_r^{(1)} = 0$. In the r th compound $V^{(r)}$, $\Delta_r^{(1)} = v_{11}^{(r)} = 0$, and hence all $v_{ij}^{(r)} = 0$, except possibly the last diagonal element, which is a function of u_{2r+1} . In $U^{(r+1)}$, the last column does not depend on u_{2r+1} , and its elements are the $v_{ij}^{(r)}$ which are zero. Hence $|U^{(r+1)}| = 0$, so that $\Delta_{r+1} = 0$.

9. Acknowledgment. We are grateful to Herman Rubin for some valuable discussions. He also pointed out that sharpness of Chebyshev inequalities can be proved quite generally without knowledge of moment problem solutions by an application of the Hahn-Banach extension theorem. However, the present proof provides considerable information concerning extremal distributions.

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A. V. Balakrishnan, <i>Prediction theory for Markoff processes</i>	1171
Dallas O. Banks, <i>Upper bounds for the eigenvalues of some vibrating systems</i>	1183
A. Białyński-Birula, <i>On the field of rational functions of algebraic groups</i>	1205
Thomas Andrew Brown, <i>Simple paths on convex polyhedra</i>	1211
L. Carlitz, <i>Some congruences for the Bell polynomials</i>	1215
Paul Civin, <i>Extensions of homomorphisms</i>	1223
Paul Joseph Cohen and Milton Lees, <i>Asymptotic decay of solutions of differential inequalities</i>	1235
István Fáry, <i>Self-intersection of a sphere on a complex quadric</i>	1251
Walter Feit and John Griggs Thompson, <i>Groups which have a faithful representation of degree less than $(p - 1/2)$</i>	1257
William James Firey, <i>Mean cross-section measures of harmonic means of convex bodies</i>	1263
Avner Friedman, <i>The wave equation for differential forms</i>	1267
Bernard Russel Gelbaum and Jesus Gil De Lamadrid, <i>Bases of tensor products of Banach spaces</i>	1281
Ronald Kay Getoor, <i>Infinitely divisible probabilities on the hyperbolic plane</i>	1287
Basil Gordon, <i>Sequences in groups with distinct partial products</i>	1309
Magnus R. Hestenes, <i>Relative self-adjoint operators in Hilbert space</i>	1315
Fu Cheng Hsiang, <i>On a theorem of Fejér</i>	1359
John McCormick Irwin and Elbert A. Walker, <i>On N-high subgroups of Abelian groups</i>	1363
John McCormick Irwin, <i>High subgroups of Abelian torsion groups</i>	1375
R. E. Johnson, <i>Quotient rings of rings with zero singular ideal</i>	1385
David G. Kendall and John Leonard Mott, <i>The asymptotic distribution of the time-to-escape for comets strongly bound to the solar system</i>	1393
Kurt Kreith, <i>The spectrum of singular self-adjoint elliptic operators</i>	1401
Lionello Lombardi, <i>The semicontinuity of the most general integral of the calculus of variations in non-parametric form</i>	1407
Albert W. Marshall and Ingram Olkin, <i>Game theoretic proof that Chebyshev inequalities are sharp</i>	1421
Wallace Smith Martindale, III, <i>Primitive algebras with involution</i>	1431
William H. Mills, <i>Decomposition of holomorphs</i>	1443
James Donald Monk, <i>On the representation theory for cylindric algebras</i>	1447
Shu-Teh Chen Moy, <i>A note on generalizations of Shannon-McMillan theorem</i>	1459
Donald Earl Myers, <i>An imbedding space for Schwartz distributions</i>	1467
John R. Myhill, <i>Category methods in recursion theory</i>	1479
Paul Adrian Nickel, <i>On extremal properties for annular radial and circular slit mappings of bordered Riemann surfaces</i>	1487
Edward Scott O'Keefe, <i>Primal clusters of two-element algebras</i>	1505
Nelson Onuchic, <i>Applications of the topological method of Ważewski to certain problems of asymptotic behavior in ordinary differential equations</i>	1511
Peter Perkins, <i>A theorem on regular matrices</i>	1529
Clinton M. Petty, <i>Centroid surfaces</i>	1535
Charles Andrew Swanson, <i>Asymptotic estimates for limit circle problems</i>	1549
Robert James Thompson, <i>On essential absolute continuity</i>	1561
Harold H. Johnson, <i>Correction to "Terminating prolongation procedures"</i>	1571