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AN IMBEDDING SPACE FOR SCHWARTZ DISTRIBUTIONS

DONALD EARL MYERS

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1. Introduction. We consider here a facet of the problem of justifying the methods of the operational calculus and in particular the use of the "Dirac Delta Function". L. Schwartz's "Theorie des Distributions" [6] is the most complete exposition to date on generalized functions but the operational calculus as such is largely omitted. B. Van Der Pol [8] discusses the latter but not in the context of distributions. Ketchum and Aboudi [4] suggested using unilateral Laplace Transforms to construct a link between Schwartz's theory and the operational calculus. This paper will enlarge on the latter suggestion. Two principal results are obtained. An imbedding space is constructed and a comparison between the topologies is made.

Let S denote the strip $\sigma_1 < R(z) < \sigma_2$, in the complex plane. Consider the one parameter family of functions $\{e^{zt}\}$, where the parameter z ranges over S and $-\infty < t < \infty$. This family is not a linear space but each member possesses derivatives of all orders. In a manner analogous to Schwartz we define an L_S -Distribution to be an analytic complex-valued functional on the above family of functions, where by analytic we mean with respect to the parameter z . If α is any complex scalar and F, σ are two such functionals then we require that $F \cdot e^{zt} + \sigma \cdot e^{zt} = (F + \sigma) \cdot \sigma^{zt}$, and $(\alpha F) \cdot e^{zt} = F \cdot (\alpha e^{zt})$. The latter property then allows us to define the derivative in a manner similar to that of Schwartz, that is $F' \cdot e^{zt} = F \cdot (e^{zt})' = F \cdot z e^{zt} = z F \cdot e^{zt}$. It also follows that the Laplace Transform supplies an integral representation of some of the functionals. The other L_S -Distributions define generalized functions for similar integral representations. That is, each function analytic for $z \in S$ has for its values, the values of an L_S -Distribution acting on a function e^{zt} and the L_S -Distribution has an integral representation utilizing the symbolic inverse Laplace Transform of the analytic function. In most of this paper we deal only with analytic functions whose inverse transforms exist but the definitions and theorems will be stated without this restriction where possible. Following a practice used by other authors, we will call the inverse Laplace Transform, symbolic or not, an L_S -Distribution rather than the functional. Because of the relation between the functional and an analytic function we concentrate on the latter and utilize the already known properties of such functions. By emphasizing the integral representations rather than the functionals we utilize the

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Riesz Representation Theorem for continuous linear functionals to establish a correspondence to Schwartz Distributions.

As stated above each functional has a representation by an analytic functions, using this we will define convergence in a fashion similar to that of Schwartz. That is, a sequence of L_S -Distributions will converge if the sequence of values, when operating on an arbitrary member of the one-parameter, converges. Because of the parameterization this definition can be stated directly in terms of the representations by the analytic functions.

2. L_S -Distributions.

DEFINITION 1. If an L_S -Distribution is determined by an analytic function $f(z)$, then $f(z)$ is its bilateral Laplace Transform. Denote this L_S -Distribution by $[f(z)]_t$ or f_t . Further, abbreviate L_S -Distribution by L_S -D.

DEFINITION 2. The derivative of an L_S -D, $[f(z)]_t = f_t$ is the L_S -D, $[zf(z)]_t = f'_t$. For a fixed S , the set of all L_S -D's is metrized by a Frechet type metric on the transforms. See [7], page 137. For a pair of functions $f(z), g(z)$ analytic in S , denote the metric by $N_S(f, g)$. The following property of this metric could have been used a definition since it is the only property used in this paper.

THEOREM 3. *A sequence of functions, all analytic in S , converges with respect to the metric N_S if and only if the sequence converges uniformly on every compact subset of S .*

DEFINITION 4. $\rho_S(f_t, g_t) = N_S(f, g)$ where f_t, g_t are the L_S -D's whose transforms are $f(z), g(z)$ respectively.

DEFINITION 5. If $f(z)$, analytic in S , is the bilateral Laplace Transform of a point-function $F(t)$, then $F(t)$ is called a Point-Function L_S -D or P.F. L_S -D.

THEOREM 6. *If $F_i(t), i = 0, 1, 2, 3, \dots$ all possess bilateral Laplace Transforms analytic in a strip $S, \sigma_1 < R(z) < \sigma_2$, and*

$$\int_0^{\infty} |e^{-\sigma_1 t} F_i(t)|^2 dt < \infty ,$$

$$\int_{-\infty}^0 |e^{-\sigma_2 t} F_i(t)|^2 dt < \infty$$

for all i , then let

$$d(F_k, F_j) = \left[\int_0^\infty e^{-2\sigma_1 t} |F_k(t) - F_j(t)|^2 dt \right]^{1/2} + \left[\int_{-\infty}^0 e^{-2\sigma_2 t} |F_k(t) - F_j(t)|^2 dt \right]^{1/2}.$$

If $d(F_i, F_0) \rightarrow 0$ as $i \rightarrow \infty$ then

$$F_i(t) \rightarrow F_0(t) \text{ as P.F.L}_S\text{-D's.}$$

Proof. Write the transform of $F_i(t) - F_0(t)$ as

$$\int_{-\infty}^0 e^{-t(z-\sigma_2)} e^{-\sigma_2 t} [F_i(t) - F_0(t)] dt + \int_0^\infty e^{-t(z-\sigma_1)} e^{-\sigma_1 t} [F_i(t) - F_0(t)] dt.$$

By the Cauchy-Schwartz Inequality

$$|f_i(z) - f_0(z)| \leq \left[\frac{1}{2[\sigma_2 - R(z)]} \int_{-\infty}^0 e^{-2\sigma_2 t} |F_i(t) - F_0(t)|^2 dt \right]^{1/2} + \left[\frac{1}{2[R(z) - \sigma_1]} \int_0^\infty e^{-2\sigma_1 t} |F_i(t) - F_0(t)|^2 dt \right]^{1/2}.$$

If

$$g(z) = \max_{z \in S} \left[\sqrt{\frac{1}{2(R(z) - \sigma_1)}} , \sqrt{\frac{1}{2(\sigma_2 - R(z))}} \right]$$

then

$$|f_i(z)| \leq g(z)g(F_i, F_0)$$

and hence $f_i(z) \rightarrow f_0(z)$ uniformly on each compact subset in S if $d(F_i, F_0) \rightarrow 0$ as $i \rightarrow \infty$.

An interpretation of Theorem 6 might be that if $\{e^{-\sigma_1 t} F_i(t)\}$ converges in $L_S[0, \infty]$ to $e^{-\sigma_1 t} F_0(t)$ and $\{e^{-\sigma_2 t} F_i(t)\}$ converges in $L_S[\infty, 0]$ to $e^{-\sigma_2 t} F_0(t)$ and each $F_i(t)$ has a bilateral Laplace Transform then the sequence of P.F.L_S-D's converges with respect to the metric ρ_S .

THEOREM 7. *Let $f_j(z), j = 0, 1, 2, 3, \dots$ be an infinite sequence of functions analytic in a strip $S, \sigma_1 < R(z) < \sigma_2$, and further suppose there exists a C such that $|f_j(z)| < Ce^{-\eta_0 |I(z)|}$ for some $\eta_0 > 0$, in all of S . If $N_S(f_j, f_0) \rightarrow 0$ as $j \rightarrow \infty$ then $F_j(t) \rightarrow F_0(t)$ uniformly on every bounded interval in the t -line. $F_j(t)$ denotes the inverse bilateral Laplace Transform of $f_j(z)$.*

Proof. The hypothesis is sufficient to ensure the existence of the

inverse transform of each $f_j(z)$, [2]. That is,

$$F_j(t) = \int_{-\infty}^{\infty} \frac{e^{xt}}{2\pi} [e^{iy} f_j(x + iy)] dy$$

for $\sigma_1 < x < \sigma_2$.

Then

$$\begin{aligned} |F_j(t) - F_0(t)| &\leq \frac{e^{xt}}{2\pi} \left| \int_{-\infty}^{-\rho} e^{iy} [f_j(x + iy) - f_0(x + iy)] dy \right| \\ &+ \frac{e^{xt}}{2\pi} \left| \int_{-\rho}^{\rho} e^{iy} [f_j(x + iy) - f_0(x + iy)] dy \right| \\ &+ \frac{e^{xt}}{2\pi} \left| \int_{\rho}^{\infty} e^{iy} [f_j(x + iy) - f_0(x + iy)] dy \right|. \end{aligned}$$

For $\varepsilon > 0$, and $a < t < b$, let ρ be such that $\frac{2e^{\sigma_2 b} C' e^{-\eta_0 \rho}}{\pi \eta_0} < \varepsilon/2$ and J such and J such that $\left[\frac{e^{\sigma_2 b}}{\pi} \right] N_s(f_j, f_0) < \varepsilon/2$ for $j > J$, then

$$\begin{aligned} |F_j(t) - F_0(t)| &< \varepsilon \text{ for } j > \text{ and} \\ &a \leq t \leq b. \end{aligned}$$

THEOREM 7.1. *If in Theorem 7, $\sigma_1 < 0 < \sigma_2$ then $F_j(t) \rightarrow F_0(t)$ uniformly for $-\infty < t < \infty$.*

DEFINITION 8. For each L_S -D, f_t , define f_{t+h} to be $[e^{hz} f(z)]_t$.

THEOREM 9. *If f_t is an arbitrary L_S -D then*

$$\rho\left(\frac{f_{t+h} - f_t}{h}, f'_t\right) \rightarrow 0 \text{ as } h \rightarrow 0.$$

Proof. By definition

$$\begin{aligned} f'_t &= [z f(z)]_t \\ f_{t+h} &= [e^{hz} f(z)]_t \end{aligned}$$

so that

$$\begin{aligned} &\frac{f_{t+h} - f_t}{h} - f'_t \\ &= \left[\frac{e^{hz} f(z) - f(z) - z f(z)}{h} \right]_t \\ &= \left[f(z) \left[\frac{hz^2}{2!} + \frac{h^2 z^3}{3!} + \dots \right] \right]_t \end{aligned}$$

and since

$$\frac{hz^2}{2!} + \frac{h^2z^3}{3!} + \dots \rightarrow 0$$

uniformly on each compact set in S as $h \rightarrow 0$, the theorem is proved.

DEFINITION 10. An L_S -D f_t is said to have point-values $F(t)$ for $c < t < d$ if there exists a $\sigma(t)$ such that for some k , $f(z) = z^k g(z)$, $g(z)$ being the bilateral Laplace Transform of $\sigma(t)$ and finally that $\sigma^{(k)}(t) = F(t)$ for $(c < t < d)$.

For example $[1]_t$ has zero-point values in every open interval, in the t -line, that does not contain the point $t = 0$. Since

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

its transform $1/z$ and $[1]_t = [z \frac{1}{z}]_t$ finally $H'(t) = 0$ for all $t \neq 0$. $[1]_t$ is the "Dirac Delta Function".

THEOREM 11. If $\{{}_n f_t\}$ is a sequence of L_S -D's converging to an L_S -D ${}_0 f_t$ then $\{{}_n f_t^{(k)}\}$ converges to ${}_0 f_t^{(k)}$ for all $k = 0, 1, 2, \dots$

Proof. By definition $\{{}_n f_t\}$ converges to

$${}_0 f_t \Leftrightarrow \max_{z \in K} |{}_n f(z) - {}_0 f(z)| \rightarrow 0$$

as $n \rightarrow \infty$ for all compact $K \subset S$. Since in the complex plane, a set is compact \Leftrightarrow if it is closed and bounded, there exists an M_K for each K $|z| \leq M_K$ for $z \in K$.

$$\text{Then } \max_{z \in K} |{}_n f(z) - {}_0 f(z)| \rightarrow 0 \Leftrightarrow$$

$$|M_K|^k |{}_n f(z) - {}_0 f(z)| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for each fixed}$$

positive integer k apply Definition 2.

EXAMPLE. The following will be used as a counter-example in the last section. Consider the Taylor-expansion for

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots + \frac{(-1)^n z^n}{n!} + \dots$$

$[e^{-z}]_t$ is the "Delta Dirac Function" translated so that L_S -D has zero point-values for all t except for

$$t = 1. \quad [(-1)^n \frac{z^n}{n!}]_t = \frac{(-1)^n}{n!} [z^n]_t = \frac{(-1)^n}{n!} [1]_t^{(n)}.$$

The latter L_S -D has zero point-values for all $t \neq 0$. Since the Taylor Series converges for all z and hence uniformly for compact sets the series of L_S -D's converges

3. General L_S -Distribution. The set of all L_S -D's for any fixed S does not contain a subset isomorphic with the set point-functions having pointvalues a.e. For example, the function $F(t) = 1$ does not have a transform even though it is continuous for $-\infty < t < \infty$. However each member of the sequence of functions

$$\begin{aligned} F_i(t) &= 1, & (-i \leq t \leq i) \\ &= 0, & (t > i, \quad t < -i) \\ i &= 0, & (0, 1, 2, 3, 4, \dots) \end{aligned}$$

does possess a transform. Further for each open interval (c, d) only a finite number of the elements of the sequence have different point-values than $F(t)$ in (c, d) . The sequence represents $F(t)$.

DEFINITION 12. A sequence $\{{}_n f_i\}$ of L_S -D's is called Fundamental if for each open interval (c, d) there exists an integer N such that for $n > N$ ${}_n f_i - {}_{n+p} f_i$, $p = 0, 1, 2, 3, \dots$ is an L_S -D with zero point-values in (c, d) . Fundamental sequence of L_S -D in abbreviated by F.S.S.

DEFINITION 13. Two F.S.S.'s, $\{{}_n g_i\}$ and $\{{}_n f_i\}$ are said to be Similar if for each open interval (c, d) there exists an integer N such that for $n > N$ ${}_n g_i - {}_n f_i$ is an L_S -D with zero pointvalues in (c, d) .

LEMMA 14. *The Similarity defined in Definition 13 for pairs of F.S.S.'s is an Equivalence relation and is invariant under addition and differentiation.*

THEOREM 15. *The equivalence classes under the Similarity relation are called G. L_S -D's or General L_S -D's. They form an Abelian group, closed with respect to scalar multiplication and differentiation.*

The Representation Theorem.

THEOREM 16. *Let A denote the entire complex plane, then there is a subset, D , of the set of all G . L_A -D's that is isomorphic with the set of all Schwartz Distributions. The isomorphism is invariant with respect to addition, scalar multiplication and differentiation.*

(a) By definition, a Schwartz Distribution is a linear functional on the space of infinity differentiable point-functions with compact supports and is continuous when restricted to the set. Each Schwartz Distribution has an integral representation when restricted to a bounded closed

interval, [3]. This representation has the form

$$D_t(\phi) = (-1)^r \int_a^b F(t) \phi^{(r)}(t) dt$$

where $F(t)$ is continuous on $[a, b]$ and r is an integer dependent on $[a, b]$ and the distribution D_t . $\phi(t)$ is any function with support the closed interval $[a, b]$. Let $[a_n, b_n]$ be a sequence of intervals where $-\infty \leftarrow a_{n+1} \leq a_n \leq b_n \leq b_{n+1} \rightarrow \infty$ as $n \rightarrow \infty$. For each n there is an $F_n(t)$ and an r_n . Let

$$\begin{aligned} G_n(t) &= (-1)^{r_n} F_n(t), & (a_n \leq t \leq b_n) \\ &= 0, & (t > b_n, t < a_n). \end{aligned}$$

Then let

$$f_n(z) = \int_{a_n}^{b_n} e^{-zt} z^{r_n} G_n(t) dt .$$

It remains to be shown that the sequence $\{ {}_n f_t \}$ is an F.S.S. and that the equivalence class is independent of the sequence of covering intervals. The G.L_S-D determined is the representative of D_t .

(b) Let I be an arbitrary open interval in the t -line, denoted (c, d) . There exists an N then such that for $n > N [a_n, b_n] \supset (c, d)$. Let $F_n(t)$, $F_{n+p}(t)$, r_n, r_{n+p} be the continuous functions and integers given for the representation of the distribution D_t on the intervals $[a_n, b_n]$ and $[a_{n+p}, b_{n+p}]$ respectively. Using Halperin's notation, let $S[a_n, b_n]$ denote the class of testing functions associated with the interval $[a_n, b_n]$ that is, if $\phi \in S[a_n, b_n]$ then $\phi^{(k)}(t)$ is zero for $t \notin [a_n, b_n]$, and $\phi^{(k)}(t)$ exists for all $t \in [a_n, b_n]$ for $k = 0, 1, 2, 3, \dots$. It is seen that $S[a_n, b_n] \subset S[a_{n+p}, b_{n+p}]$. If $\phi \in S[a_n, b_n] \subset S[a_{n+p}, b_{n+p}]$ then

$$\begin{aligned} D_t(\phi) &= (-1)^{r_n} \int_{a_n}^{b_n} F_n(t) \phi^{(r_n)}(t) dt \\ &= (-1)^{r_{n+p}} \int_{a_{n+p}}^{b_{n+p}} F_{n+p}(t) \phi^{(r_{n+p})}(t) dt \end{aligned}$$

or

$$\int_{a_n}^{b_n} [F_n(t) \phi^{(r_n)}(t) - (-1)^{r_{n+p}-r_n} F_{n+p}(t) \phi^{(r_{n+p})}(t)] dt = 0$$

since

$$\phi^{(k)}(t) = 0 \text{ for } t \notin [a_n, b_n] .$$

Let

$$T \cdot F_n(t) = \int_{a_n}^t F_n(x) dx$$

$$T^2 \cdot F_n(t) = T \cdot [T \cdot F_n(t)] .$$

Then if $r_{n+p} \geq r_n$

$$0 = \int_{a_n}^{b_n} [(-1)^{r_{n+p}-r_n} T^{r_{n+p}-r_n} \cdot F_n(t) \phi(t)^{(r_{n+p})} - F_{n+p}(t) \phi^{(r_{n+p})}(t)] dt .$$

It follows then that

$$(-1)^{r_{n+p}-r_n} T^{r_{n+p}-r_n} \cdot F_n(t) - F_{n+p}(t)$$

is a polynomial $Q_m(t)$ of degree $m \leq r_{n+p} - 1$ for $a_n \leq t \leq b_n$. Similar results are obtained if $r_n \geq r_{n+p}$.

$$\begin{aligned} {}_{n+p}f(z) - {}_nf(z) &= \int_{a_{n+p}}^{b_{n+p}} e^{-zt} z^{r_{n+p}} G_{n+p}(t) dt \\ &\quad - \int_{a_n}^{b_n} e^{-zt} z^{r_n} G_n(t) dt . \end{aligned}$$

Using $Q_m(t)$ we have

$$\begin{aligned} {}_{n+p}f(z) - {}_nf(z) &= \int_{a_{n+p}}^{a_n} e^{-zt} z^{r_{n+p}} G_{n+p}(t) dt \\ &\quad + \int_{a_n}^{b_n} e^{-zt} z^{r_{n+p}} Q_m(t) dt \\ &\quad + \int_{b_n}^{b_{n+p}} e^{-zt} z^{r_{n+p}} G_{n+p}(t) dt . \end{aligned}$$

The first integral can be considered as the transform of the r_{n+p} th derivative of a function with zero point-values exterior to the interval $[a_{n+p}, a_n]$ and hence interior to the interval (c, d) . The second integral can be considered as the transform of the r_{n+p} th derivative of a polynomial of degree less than or equal to $r_{n+p} - 1$. Hence the L_S -D determined has zero pointvalues on the interior and exterior of the interval $[a_n, b_n]$ and hence on the interior of (c, d) . This L_S -D may not have zero point-values at $t = a_n$ or $t = b_n$. Finally then the third integral considered as a transform determines an $L_S - D$ with zero pointvalues exterior to the interval $[b_n, b_{n+p}]$ and hence on the interior of (c, d) . ${}_{n+p}f_t - {}_nf_t$ is an L_S -D with zero pointvalues on the interior of (c, d) , if $n > N$. The sequence constructed in part (a) is an F.S.S.

(c) Suppose $[a_n, b_n]$ and $[c_n, d_n]$ are two expanding sequence of closed intervals covering the real line. Let $\{ {}_nf_t \}$ and $\{ {}_ng_t \}$ be the F.S.S.'s obtained from the construction of part (a) using the former sequences. Let I be an arbitrary open interval in the t -line. Then there exists integers N_1^1, N_1^2 such ${}_{n+p}f_t - {}_nf_t$ for $n < N_1^1$ and ${}_{n+p}g_t - {}_ng_t$ for $n \leq N_1^2$ L_S -D's with zero point-values for $t \in I$. Further there exists an integer $M \ni [a_n, b_n] \subset [c_N, d_N]$ for $n \geq M$ and $N = N_1^2$. Consider

$${}_n f_J - {}_n g_t = {}_n f_t - Kf_t + Kf_t - Kg_t + Kg_t - {}_n g_t$$

where K is the largest of N_I^1, N_I^2, M . For $n > K$ then the first difference on the right is an L_S -D with zero point-values for $t \in I$ since $\{{}_n f_i\}$ is an F.S.A. The second difference can be shown to be an L_S -D with zero point-values for $t \in I$ by the method of part (b). Finally the third difference is an L_S -D with zero point-values for $t \in I$ since $\{{}_n g_t\}$ is F.S.A. The two F.S.A.'s are similar and hence determine the same G.L_A-D. The correspondence between the Schwartz Distribution and the G.L_A-D. is one-to-one. The invariance of this isomorphism with respect to addition, differentiation and scalar multiplication follows from Lemma 14.

4. A Topology for G.L-D's

DEFINITION 16. An F.S.S. $\{{}_n f_i\}$ is said to have point-values $F(t)$ for $t \in (c, d)$, an open interval, if there exist an integer $N_{(c,d)}$ such that for $n > N_{(c,d)}$, ${}_n f_i$ is an L_S -D possessing pointvalues $F(t)$ for $t \in (c, d)$. A G.L_S-D is said to have pointvalues $F(t)$ for $t \in (c, d)$ if there is an F.S.S. unit equivalence class possessing that property.

DEFINITION 17. Let $\{{}_n f_{i1}\}, \dots, \{{}_n f_{ij}\}, \dots$ be a sequence of F.S.S.'s. Denote the n th element of the j th F.S.S. by $({}_n f_{ij})$. Then sequence is said to converge to the sequence of L_S -D's $\{{}_n f_{i0}\}$, if for $\epsilon > 0$ there exist integers N_ϵ, J_ϵ such that $\rho [({}_n f_{ij}), ({}_n f_{i0})] < \epsilon$ when $n > N_\epsilon, j > J_\epsilon$.

DEFINITION 18. Let D_1, D_2, \dots be a sequence of G.L_S-D's. Further suppose L_1, L_2, \dots is a sequence of F.S.S.'s each having support $[a, b]$ and that for each $j = 1, 2, \dots$ L_j represents D_j is (a, b) . That is, for some F.S.S. in D_j , the difference of L_j and the F.S.S. has zero point-values in (a, b) . Then if L_1, L_2, \dots is convergent in the sense of Definition 17, D_1, D_2, \dots is said to converge to D_0 where D_0 is the G.L_S-D. determined by L_0 .

THEOREM 19. *If a sequence of Schwartz Distributions is convergent in an open interval (a, b) in Schwartz's sense then the sequence of G.L_S-D's isomorphic to the respective Schwartz Distributions is convergent in the interior of every closed interval contained in (a, b) .*

Proof. Let D_1, D_2, \dots be a sequence of Schwartz Distributions convergent in Schwartz's sense in (a, b) . For any closed interval $[c, d]$ contained in (a, b) there exists a sequence of representation

$$D_i(\phi) = (-1)^{r_c} \int_c^d F_i(t) \phi(t)^{(r_c)} dt$$

for $\phi \in S[c, d]$. Since the sequence D_1, D_2, \dots is convergent there exists one integer r_c which can be used in all the representations and also the limit representation.

For each j , construct the F.S.A. $\{ {}_n f_t \}_j$ where $\{ {}_n f_t \}_j = (f_t)_j$ and

$$(f)_j = \left[\int_c^d e^{-zt} G_j(t) z^{r_c} dt \right]_t$$

where

$$\begin{aligned} G_j(t) &= (-1)^{r_c} F_j(t) , & c \leq t \leq d \\ &= 0 , & \text{otherwise .} \end{aligned}$$

Since the sequence of Distributions is convergent

$$\begin{aligned} \lim T^{r_c} F_j(t) &= F_0(t) \text{ [uniformly } [c, d] \text{]} j \rightarrow \infty \\ K &= \left| \int_c^d z^{r_c} e^{-zt} \left[T^{r_c} G_j(t) - G_0(t) \right] dt \right| \\ &\leq |d - c| e^{-c\sigma_1} | T^{r_c} G_j(t) - G_0(t) | |d^{r_0}| . \end{aligned}$$

It follows that

$$\begin{aligned} K &= |(f(z))_j - (f(z))_0| \leq M\varepsilon \\ \text{for } j &> J_\varepsilon, M = |d - c| e^{-c\sigma_1} | \sigma_2^{r_c} | . \end{aligned}$$

Then $(f(z))_j \rightarrow (f(z))_0$ uniformly on every compact set in the strip $\sigma_1 \leq R(z) \leq \sigma_2$ and hence in the metric ρ_s . By definition then $(f_t)_j \rightarrow (f_t)_0$ and hence

$\{ {}_n f_t \}_1, \dots, \{ {}_n f_t \}_j, \dots$ converges to $\{ {}_n f_t \}_0$ in the interior of (c, d) . The sequence of G. L_s-D's converges for $t \in (c, d)$.

The example given earlier for a series representation of "DeIta" Distribution with a discontinuous at $t = 1$ converges in the sense defined herein but not in Schwartz's sense. The L_s-D $[e^{-z}]_t$ and its series representation furnish a solution to the differential equation

$$f(x - 1) = f(x) + f'(x) + \frac{f''(x)}{2!} + \dots + \frac{f^{(n)}(x)}{n!} + \dots$$

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Pacific Journal of Mathematics

Vol. 11, No. 4

, 1961

A. V. Balakrishnan, <i>Prediction theory for Markoff processes</i>	1171
Dallas O. Banks, <i>Upper bounds for the eigenvalues of some vibrating systems</i>	1183
A. Białynicki-Birula, <i>On the field of rational functions of algebraic groups</i>	1205
Thomas Andrew Brown, <i>Simple paths on convex polyhedra</i>	1211
L. Carlitz, <i>Some congruences for the Bell polynomials</i>	1215
Paul Civin, <i>Extensions of homomorphisms</i>	1223
Paul Joseph Cohen and Milton Lees, <i>Asymptotic decay of solutions of differential inequalities</i>	1235
István Fáry, <i>Self-intersection of a sphere on a complex quadric</i>	1251
Walter Feit and John Griggs Thompson, <i>Groups which have a faithful representation of degree less than $(p - 1/2)$</i>	1257
William James Firey, <i>Mean cross-section measures of harmonic means of convex bodies</i>	1263
Avner Friedman, <i>The wave equation for differential forms</i>	1267
Bernard Russel Gelbaum and Jesus Gil De Lamadrid, <i>Bases of tensor products of Banach spaces</i>	1281
Ronald Kay Gettoor, <i>Infinitely divisible probabilities on the hyperbolic plane</i>	1287
Basil Gordon, <i>Sequences in groups with distinct partial products</i>	1309
Magnus R. Hestenes, <i>Relative self-adjoint operators in Hilbert space</i>	1315
Fu Cheng Hsiang, <i>On a theorem of Fejér</i>	1359
John McCormick Irwin and Elbert A. Walker, <i>On N-high subgroups of Abelian groups</i>	1363
John McCormick Irwin, <i>High subgroups of Abelian torsion groups</i>	1375
R. E. Johnson, <i>Quotient rings of rings with zero singular ideal</i>	1385
David G. Kendall and John Leonard Mott, <i>The asymptotic distribution of the time-to-escape for comets strongly bound to the solar system</i>	1393
Kurt Kreith, <i>The spectrum of singular self-adjoint elliptic operators</i>	1401
Lionello Lombardi, <i>The semicontinuity of the most general integral of the calculus of variations in non-parametric form</i>	1407
Albert W. Marshall and Ingram Olkin, <i>Game theoretic proof that Chebyshev inequalities are sharp</i>	1421
Wallace Smith Martindale, III, <i>Primitive algebras with involution</i>	1431
William H. Mills, <i>Decomposition of holomorphs</i>	1443
James Donald Monk, <i>On the representation theory for cylindric algebras</i>	1447
Shu-Teh Chen Moy, <i>A note on generalizations of Shannon-McMillan theorem</i>	1459
Donald Earl Myers, <i>An imbedding space for Schwartz distributions</i>	1467
John R. Myhill, <i>Category methods in recursion theory</i>	1479
Paul Adrian Nickel, <i>On extremal properties for annular radial and circular slit mappings of bordered Riemann surfaces</i>	1487
Edward Scott O'Keefe, <i>Primal clusters of two-element algebras</i>	1505
Nelson Onuchic, <i>Applications of the topological method of Ważewski to certain problems of asymptotic behavior in ordinary differential equations</i>	1511
Peter Perkins, <i>A theorem on regular matrices</i>	1529
Clinton M. Petty, <i>Centroid surfaces</i>	1535
Charles Andrew Swanson, <i>Asymptotic estimates for limit circle problems</i>	1549
Robert James Thompson, <i>On essential absolute continuity</i>	1561
Harold H. Johnson, <i>Correction to "Terminating prolongation procedures"</i>	1571